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ON HYPERBOLIC MIXED PROBLEMS WITH DYNAMIC AND WENTZELL BOUNDARY CONDITIONS

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# On hyperbolic mixed problems with dynamic and Wentzell boundary conditions

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## Abstract

We study mixed hyperbolic systems with dynamic and Wentzell boundary conditions. The boundary condition contains a tangential operator which is strongly elliptic on the boundary. We prove results of generation of strongly continuous groups and well-posedness.

**Keywords:** Hyperbolic problems, dynamic boundary conditions, Wentzell boundary conditions  
**2010 MSC:** 35L53, 47D06.

## 1 Introduction

The aim of this paper is the study of a problem in the form

$$\left\{ \begin{array}{l} D_t^2 u(t, x) = A(x, D_x)u(t, x) + f(t, x), \quad (t, x) \in (0, T) \times \Omega, \\ D_t^2 \gamma u(t, x') = \nabla_\tau \cdot (B(x') \nabla_\tau \gamma u)(t, x') + F(x', D_x)u(t, x') + h(t, x'), \\ (t, x') \in (0, T) \times \partial\Omega, \\ u(0, x) = u_0(x), \quad x \in \Omega, \\ D_t u(0, x) = u_1(x), \quad x \in \Omega. \end{array} \right. \quad (1.1) \quad \boxed{\text{eq3.1}}$$

Roughly speaking (precise assumptions will be given in the following),  $A(x, D_x)$  is a strongly elliptic differential operator in divergence form in the bounded domain  $\Omega$  with smooth boundary  $\partial\Omega$ ;  $\gamma f$  is the trace of  $f$  on  $\partial\Omega$ ;  $\nabla_\tau$  is the tangential gradient in  $\partial\Omega$ ,  $\nabla_\tau \cdot$  is the divergence operator in  $\partial\Omega$ ,  $B(x')$  is a positive definite symmetric operator in the tangent space  $T_{x'}(\partial\Omega)$ , with  $x' \in \partial\Omega$ ,  $F(x', D_x)$  is a linear differential operator of order not exceeding one (not necessarily tangential) and coefficients defined in  $\partial\Omega$ .

(1.1) is strictly connected with the problem

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$$\left\{ \begin{array}{l} D_t^2 u(t, x) = A(x, D_x)u(t, x) + f(t, x), \quad (t, x) \in (0, T) \times \Omega, \\ A(x', D_x)u(t, x') = \nabla_\tau \cdot (B(x')\nabla_\tau u)(t, x') + F(x', D_x)u(t, x') + h(t, x'), \quad (t, x') \in (0, T) \times \partial\Omega, \\ u(0, x) = u_0(x), \quad x \in \Omega, \\ D_t u(0, x) = u_1(x), \quad x \in \Omega. \end{array} \right. \quad (1.2) \quad \text{eq0.2}$$

formally obtained replacing in (1.1)  $D_t^2 \gamma u(t, x')$  in the second equation with the trace of the second term in the first equation. In case (1.2) one usually speaks of Wentzell boundary conditions.

A physical interpretation of (1.2) is given in [10], Chapter 6.

In our knowledge, problems (1.1) and (1.2) have been always considered in the particular case that

$$F(x', D_x) = -\beta(x') \frac{\partial}{\partial \nu_A} - c(x'), \quad (1.3) \quad \text{eq0.3}$$

where we indicate with  $\frac{\partial}{\partial \nu_A}$  the conormal derivative associated with  $A(x, D_x)$ . See, for example, [1], [9], [12], often connected with problems of control.

The most general results are contained in [2], where  $F(x', D_x)$  is in the form (1.3) with  $\beta(x') > 0$  which is allowed (to some extent) to be unbounded and with infimum equal to 0. The authors do not even assume that the coefficients of  $A(x, D_x)$  and  $B(x')$  are continuous; they need to be just measurable and bounded. They work in the basic space  $L^2(\bar{\Omega}, d\mu) := L^2(\Omega) \times L^2(\partial\Omega, dS/\beta)$  with  $F(x', D_x)$  as in (1.3). They show that a certain operator connected with (1.1) and (1.2) is self-adjoint and upper bounded. This allows to formulate theorems of well-posedness in a certain generalized sense. They consider also the case when  $D_t^2$  is replaced by  $D_t^2 + aD_t$  (this is the telegraph equation).

Roughly speaking, in this paper we want to show that, at least in case of "regular coefficients" for  $A(x, D_x)$  and  $B(x')$ , (1.1) and (1.2) are well posed whenever the operator  $F(x', D_x)$  has bounded and measurable coefficients in  $\partial\Omega$ .

This is the plan of this paper: Section 2 is dedicated to the proof of Theorem 2.1. We begin by considering a particular case, with  $F(x', D_x) = -\frac{\partial}{\partial \nu_A} - \gamma$ . In this situation the result is essentially known (see for this also [3]), but we have decided to give a complete proof in order to make the paper more or less self-contained. The general statement is obtained by combining an estimate of the conormal derivative of the solution to a hyperbolic Cauchy-Dirichlet system (see Theorem 2.10) with a perturbation theorem of Miyadera type (Theorem 2.12). The estimate is inspired by a nice result due to I. Lasiecka, J. L. Lions, R. Triggiani (see [8]).

The final Section 3 contains developments and applications of Theorem 2.1 to a generalization of (1.1), and to (1.2).

To conclude this preliminary section, we describe some notations we are going to use.

If  $\Omega$  is a domain with smooth boundary and  $x' \in \partial\Omega$ , we shall indicate with  $\nu(x')$  the unit normal vector to  $\partial\Omega$  in  $x'$ , pointing outside  $\Omega$ , with  $\frac{\partial}{\partial \nu}$  the corresponding normal derivative.  $T_{x'}(\partial\Omega)$  will be the tangent space to  $\partial\Omega$  in  $x'$  and  $T(\partial\Omega)$  the tangent bundle. If  $A$  is the differential operator

$$\sum_{i=1}^n \sum_{j=1}^n D_{x_i} (a_{ij}(x) D_{x_j} \cdot) + \sum_{j=1}^n a_j(x) D_{x_j} + a_0(x),$$

and  $x' \in \partial\Omega$ , we set

$$D_{\nu_A} u(x') = \frac{\partial u}{\partial \nu_A}(x') = \sum_{i=1}^n \sum_{j=1}^n a_{ij}(x') D_{x_j} u(x') \nu_i(x').$$

$C$  will indicate a positive constant we are not interested to precise. In a sequence of formulas we shall write  $C_1, C_2, \dots$ . If the constants depend on  $T$ , we shall write  $C(T), C_1(T), \dots$ .

If  $X$  and  $Y$  are normed spaces, we shall indicate with  $\mathcal{L}(X, Y)$  the space of linear bounded operators from  $X$  to  $Y$ . If  $X = Y$ , we shall write  $\mathcal{L}(X)$ . If  $V$  is a Hilbert space, we shall indicate with  $V^*$  the space of antilinear bounded functionals in  $V$ .

## 2 The main theorem

se2

As we said, in this section we shall study a simplified version of (1.1). We begin by stating our assumptions.

(A1)  $\Omega$  is an open bounded subset of  $\mathbb{R}^n$  lying on one side of its boundary  $\partial\Omega$ , which is a submanifold of  $\mathbb{R}^n$  of dimension  $n-1$  and class  $C^2$ .

(A2)  $A_0(x, D_x) = \sum_{i=1}^n \sum_{j=1}^n D_{x_i} [a_{ij}(x) D_{x_j} \cdot]$ ,  
with  $a_{ij} \in C^1(\overline{\Omega})$  ( $1 \leq i, j \leq n$ ), real valued,  $a_{ij} = a_{ji}$ ,

$$\sum_{i=1}^n \sum_{j=1}^n a_{ij}(x) \xi_i \xi_j \geq \nu |\xi|^2,$$

for any  $x \in \overline{\Omega}$ ,  $\xi \in \mathbb{R}^n$ , for some  $\nu$  positive.

(A3)  $\forall x' \in \partial\Omega$   $B(x')$  is symmetric and positive definite element of  $\mathcal{L}(T_{x'}(\partial\Omega))$ .

(A4)  $B(x')$  depends smoothly on  $x'$ , in the sense that, if  $u$  is a  $C^1$  section with values in  $T(\partial\Omega)$ ,  $B(\cdot)u(\cdot)$  is a  $C^1$  section.

(A5)  $F(x', D_x)u(x') = \sum_{j=1}^n f_j(x') D_{x_j} u(x') + f_0(x') u(x')$ , with  $f_j \in L^\infty(\partial\Omega)$  ( $0 \leq j \leq n$ ).

We set

$$H = L^2(\Omega) \times L^2(\partial\Omega), \quad (2.1)$$

Of course,  $H$  is a Hilbert space with the usual scalar product

$$((f_0, h_0), (f_1, h_1))_H = \int_{\Omega} f_0(x) \overline{f_1(x)} dx + \int_{\partial\Omega} h_0(x') \overline{h_1(x')} d\sigma,$$

where  $\sigma$  is the standard Riemannian measure in  $\partial\Omega$ . We set also

$$V = \{(\phi, \psi) \in H^1(\Omega) \times H^1(\partial\Omega) : \gamma\phi = \psi\}. \quad (2.2)$$

We equip  $V$  with the scalar product

$$\begin{aligned} & ((\phi_0, \psi_0), (\phi_1, \psi_1))_V \\ &:= \int_{\Omega} \sum_{i=1}^n \sum_{j=1}^n a_{ij}(x) D_{x_j} \phi_0(x) \overline{D_{x_i} \phi_1(x)} dx + \int_{\partial\Omega} [B(x') \nabla_{\tau} \psi_0(x') \cdot \overline{\nabla_{\tau} \psi_1(x')} + \psi_0(x) \overline{\psi_1(x')}] d\sigma. \end{aligned} \quad (2.3)$$

We introduce the following operator  $A_2$  in  $H \times H$ :

$$\begin{cases} D(A_2) = W := \{(\phi, \psi) \in H^2(\Omega) \times H^2(\partial\Omega) : \psi = \gamma\phi\}, \\ A_2(\phi, \psi) = (A_0(\cdot, D_x)\phi, \nabla_{\tau} \cdot (B(\cdot) \nabla_{\tau} \psi) + F(\cdot, D_x)\phi). \end{cases}$$

The main result of this section is the following

th0.1

**Theorem 2.1.** Suppose that (A1)-(A5) are fulfilled. We introduce the following operator  $M$ :

$$\begin{cases} D(M) = W \times V, \\ M((\phi, \psi), (f, h)) = ((f, h), A_2(\phi, \psi)) \end{cases}$$

Then  $M$  is the infinitesimal generator of a strongly continuous group in  $V \times H$ .

We begin the proof of Theorem 2.1 by recalling the well known procedure of identifying the element  $(f, h)$  of  $H$  with the element  $J(f, h)$  of  $V^*$  defined as

$$(J(f, h), (\phi, \psi)) = ((f, h), (\phi, \psi))_H = \int_{\Omega} f(x) \overline{\phi(x)} dx + \int_{\partial\Omega} h(x') \overline{\psi(x')} d\sigma, \quad (\phi, \psi) \in V.$$

From Poincaré inequality, we deduce

$$|(J(f, h), (\phi, \psi))| \leq \|(f, h)\|_H \|(\phi, \psi)\|_H \leq C_0 \|(f, h)\|_H \|(\phi, \psi)\|_V,$$

for any  $(\phi, \psi)$  in  $V$ . We deduce that  $\|J(f, h)\|_{V^*} \leq C_0 \|(f, h)\|_H$ . So the identification of  $(f, h)$  with  $J(f, h)$  carries to  $\|(f, h)\|_{V^*} \leq C_0 \|(f, h)\|_H$  and  $H \hookrightarrow V^*$ . We introduce the operator  $A_0$ , defined as follows:

$$\begin{cases} D(A_0) = \{(u, v) \in V : \exists (f, h) \in H : ((u, v), (\psi, \psi))_V = ((f, h), (\phi, \psi))_H \ \forall (\phi, \psi) \in V\}, \\ A_0(u, v) = (f, h). \end{cases} \quad (2.4) \quad \boxed{\text{eq0.9}}$$

The following result is well known (for a proof, see [11], Chapter 2.2).

**1e2.2** **Lemma 2.2.** *If  $A_0$  is the linear operator defined in (2.4),  $D(A_0)$  is dense in  $H$ ,  $A_0$  is self-adjoint and positive and  $D(A_0^{1/2}) = V$ .*

Concerning  $D(A_0)$ , we have:

**1e0.2** **Lemma 2.3.** *Suppose that (A1)-(A4) hold. Then*

$$D(A_0) = W$$

and  $\forall (u, v) \in W$

$$A_0(u, v) = (-A_0(\cdot, D_x)u, -\nabla_\tau \cdot (B(\cdot)\nabla_\tau v) + \frac{\partial u}{\partial \nu_A} + v).$$

*Proof.* We consider the operator  $A_1 : W \rightarrow H$ ,

$$A_1(u, v) = (-A_0(\cdot, D_x)u, -\nabla_\tau \cdot (B(\cdot)\nabla_\tau v) + \frac{\partial u}{\partial \nu_A} + v).$$

It is immediately seen, employing Green's formula, that

$$A_1 \subseteq A_0. \quad (2.5) \quad \boxed{\text{eq0.10}}$$

On the other hand, as  $A_0$  is self adjoint and positive,  $-A_0$  is the infinitesimal generator of an analytic semigroup in  $H$ . By Theorem 4.1 in [6], the same happens for  $A_1$ . So  $\rho(A_0) \cap \rho(A_1) \neq \emptyset$ . This, together with (2.5), implies the conclusion.  $\square$

**Remark 2.4.** By Lemma 2.3, in case  $F(\cdot, D_x) = -\frac{\partial}{\partial \nu_A} - \gamma$ ,  $A_2 = -A_0$ .

Now we are able to employ the following result (see [7], Theorem 7.2):

**th0.3** **Theorem 2.5.** *Let  $B$  be the infinitesimal generator of a strongly continuous group in the Banach space  $X$ . Assume that  $0 \in \rho(B)$ . Define*

$$\begin{cases} D(M_0) = D(B^2) \times D(B), \\ M_0(u, v) = (v, B^2 u). \end{cases}$$

*Then  $M_0$  is the infinitesimal generator of a strongly continuous group in the Banach space  $D(B) \times X$ .*

**co0.4** **Corollary 2.6.** *Suppose that (A1)-(A4) hold. We introduce the following operator  $M_0$ :*

$$\begin{cases} D(M_0) = W \times V, \\ M_0((\phi, \psi), (f, h)) = ((f, h), -A_0(\phi, \psi)). \end{cases} \quad (2.6) \quad \boxed{\text{eq0.11A}}$$

*Then  $M_0$  is the infinitesimal generator of a strongly continuous group in  $V \times H$ .*

*Proof.* We set  $B := iA_0^{1/2}$ . Then  $B$  is skew-adjoint and  $D(B) = V$ . By Stone's theorem,  $B$  is the infinitesimal generator of a strongly continuous group of isometries in  $H$ . We have  $B^2 = -A_0$ . So the conclusion follows from Theorem 2.5.  $\square$

We shall indicate with  $(e^{tM_0})_{t \in \mathbb{R}}$  the group generated by  $M_0$  in  $V \times H$ .

**Remark 2.7.** If  $(\phi, \psi) \in V$  and  $(\alpha, \beta) \in H$ , we shall often write  $(\phi, \psi, \alpha, \beta)$  instead of  $((\phi, \psi), (\alpha, \beta))$

eq2.8A

**Remark 2.8.** If  $(\phi, \psi, \alpha, \beta)$  belongs to  $V \times H$  and its components are real valued, then the components of  $e^{tM_0}(\phi, \psi, \alpha, \beta)$  are real valued. In case  $(\phi, \psi, \alpha, \beta) \in W \times V$ , This can be easily deduced from the uniqueness of the solution of the problem

$$\left\{ \begin{array}{l} D_t^2 u(t, x) = A_0(x, D_x)u(t, x), \quad (t, x) \in \mathbb{R} \times \Omega, \\ D_t^2 \gamma u(t, x') = \nabla_\tau \cdot (B(x') \nabla_\tau \gamma u)(t, x') - \frac{\partial u}{\partial \nu_A}(t, x') - \gamma u(t, x'), \\ (t, x') \in \mathbb{R} \times \partial\Omega, \\ u(0, x) = \phi(x), \quad x \in \Omega, \\ D_t u(0, x) = \alpha(x), \quad x \in \Omega, \end{array} \right.$$

which follows from Corollary 2.6. The general case follows by continuity.

re2.8

**Remark 2.9.** Suppose that the assumptions (A1)-(A4) are satisfied. Let  $u_0 \in H^2(\Omega)$ ,  $\gamma u_0 \in H^2(\partial\Omega)$ ,  $u_1 \in H^1(\Omega)$ ,  $\gamma u_1 \in H^1(\partial\Omega)$ , so that  $(u_0, \gamma u_0, u_1, \gamma u_1) \in W \times V$ . Let

$$(\phi(t), \psi(t), \alpha(t), \beta(t)) = e^{tM_0}(u_0, \gamma u_0, u_1, \gamma u_1). \quad (2.7)$$

eq0.14

Then  $\phi \in \cap_{j=0}^2 C^{2-j}(\mathbb{R}; H^j(\Omega))$ ,  $\psi \in \cap_{j=0}^2 C^{2-j}(\mathbb{R}; H^j(\partial\Omega))$ ,  $\psi = \gamma\phi$ ,  $\alpha = D_t\phi$ ,  $\beta = D_t\psi = \gamma\alpha$ . Moreover,

$$D_t^2 \phi(t, x) = A_0(x, D_x)\phi(t, x), \quad (t, x) \in \mathbb{R} \times \Omega.$$

So  $\phi$  is also the solution of the mixed Cauchy-Dirichlet problem

$$\left\{ \begin{array}{l} D_t^2 \phi(t, x) = A_0(x, D_x)\phi(t, x), \quad (t, x) \in (a, b) \times \Omega, \\ \phi(t, x') = \psi(t, x'), \quad (t, x') \in (a, b) \times \partial\Omega, \\ \phi(0, x) = u_0(x), \quad x \in \Omega, \\ D_t \phi(0, x) = u_1(x), \quad x \in \Omega. \end{array} \right. \quad (2.8)$$

eq2.8

Now we want to replace  $-\frac{\partial}{\partial \nu_A} - \gamma$  with an essentially arbitrary differential operator of order not exceeding one. To this aim, the key fact is the following result, following the idea of [8], Theorem 2.1. For a slightly different situation, see also [5].

th2.10

**Theorem 2.10.** Suppose that the conditions (A1)-(A4) are fulfilled. Let  $a, b \in \mathbb{R}$ , with  $a < b$ . Then there exists  $C(a, b)$  positive such that,  $\forall (u_0, \gamma u_0, u_1, \gamma u_1)$  belonging to  $W \times V$ ,

$$\left\| \frac{\partial \phi}{\partial \nu} \right\|_{L^2((a, b) \times \partial\Omega)} \leq C \|(u_0, \gamma u_0, u_1, \gamma u_1)\|_{V \times H},$$

with  $\phi(t)$  first component of  $e^{tM_0}(u_0, \gamma u_0, u_1, \gamma u_1)$  (see Remark 2.9).

*Proof.* We continue to employ the notation (2.7). Concerning the proof, it suffices to consider the case that the components of  $(u_0, \gamma u_0, u_1, \gamma u_1)$  are real valued, so that, by Remark 2.8,  $\phi$  is real valued.

We set

$$N := \|u_0\|_{H^1(\Omega)}^2 + \|\gamma u_0\|_{H^1(\partial\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2 + \|\gamma u_1\|_{L^2(\partial\Omega)}^2.$$

By simplicity of notation, we shall write, given a certain expression  $E$ , that  $E = O(N)$  if there exists  $C$  positive, possibly depending on  $(a, b)$ , but not  $(u_0, \gamma u_0, u_1, \gamma u_1)$ , such that

$$|E| \leq C \|(u_0, \gamma u_0, u_1, \gamma u_1)\|_{V \times H}^2.$$

For example,

$$\max_{t \in [a, b]} \{\phi(t)\|_{H^1(\Omega)} + \|\psi(t)\|_{H^1(\partial\Omega)} + \|\alpha(t)\|_{L^2(\Omega)} + \|\beta(t)\|_{L^2(\partial\Omega)}\} = O(N).$$

We introduce a function  $h \in C^1(\bar{\Omega}; \mathbb{R}^n)$  such that, for each  $j \in \{1, \dots, n\}$ , if  $x' \in \partial\Omega$ ,

$$h_j(x') = \sum_{i=1}^n a_{i,j}(x') \nu_i(x').$$

If  $b \in H^1(\Omega)$ , we have, by Green's formula,

$$\int_{\Omega} h(x) \cdot \nabla_x b(x) dx = \int_{\partial\Omega} A(x') b(x') d\sigma(x') - \int_{\Omega} \operatorname{div}_x(h)(x) b(x) dx, \quad (2.9) \quad \boxed{\text{eq3.15}}$$

with

$$A(x') = \sum_{i=1}^n \sum_{j=1}^n a_{i,j}(x') \nu_i(x') \nu_j(x'), \quad x' \in \partial\Omega.$$

We have also that in  $(a, b) \times \partial\Omega$ , for  $j = 1, \dots, n$ , employing the notation (2.7),

$$D_{x_j} \phi(t, x') = \nu_j(x') \frac{\partial \phi}{\partial \nu}(t, x') + T_j \psi(t, x'), \quad (2.10) \quad \boxed{\text{eq3.16A}}$$

with  $T_j$  differential operator of order one in  $\partial\Omega$ , with coefficients in  $C^1(\partial\Omega)$ . Then, by Remark 2.9,

$$\begin{aligned} & \int_{(a,b) \times \Omega} D_t^2 \phi(t, x) h(x) \cdot \nabla_x \phi(t, x) dt dx \\ &= \int_{(a,b) \times \Omega} A_0(x, D_x) \phi(t, x) h(x) \cdot \nabla_x \phi(t, x) dt dx. \end{aligned} \quad (2.11) \quad \boxed{\text{eq3.16}}$$

Now,

$$\begin{aligned} & \int_{(a,b) \times \Omega} D_t^2 \phi(t, x) h(x) \cdot \nabla_x \phi(t, x) dt dx \\ &= \int_{\Omega} \alpha(b, x) h(x) \cdot \nabla_x \phi(b, x) dx - \int_{\Omega} \alpha(a, x) h(x) \cdot \nabla_x \phi(a, x) dx \\ & \quad - \int_{(a,b) \times \Omega} \alpha(t, x) h(x) \cdot \nabla_x \alpha(t, x) dt dx. \end{aligned}$$

We have

$$\begin{aligned} & \left| \int_{\Omega} \alpha(b, x) h(x) \cdot \nabla_x \phi(b, x) dx - \int_{\Omega} \alpha(a, x) h(x) \cdot \nabla_x \phi(a, x) dx \right| \\ & \leq C(a, b) (\|\alpha(b, \cdot)\|_{L^2(\Omega)} \|\phi(b, \cdot)\|_{H^1(\Omega)} + \|\alpha(a, \cdot)\|_{L^2(\Omega)} \|\phi\|_{C([a,b]; H^1(\Omega))}) \\ & = O(N). \end{aligned}$$

Moreover, by (2.9),

$$\begin{aligned} & \left| \int_{(a,b) \times \Omega} \alpha(t, x) h(x) \cdot \nabla_x \alpha(t, x) dt dx \right| \\ &= \frac{1}{2} \left| \int_{(a,b) \times \Omega} h(x) \cdot \nabla_x \alpha^2(t, x) dt dx \right| \\ &= \frac{1}{2} \left| \int_a^b \left( \int_{\partial\Omega} A(x') \beta(t, x')^2 d\sigma \right) dt - \int_a^b \left( \int_{\Omega} \operatorname{div}_x h(x) \alpha(t, x)^2 dx \right) dt \right| \\ & \leq C (\|\beta\|_{C([a,b]; L^2(\partial\Omega))}^2 + \|\alpha\|_{C([a,b]; L^2(\Omega))}^2) \\ & = O(N). \end{aligned}$$



So, by (2.11),

$$\int_{(a,b) \times \Omega} A_0(x, D_x) \phi(t, x) h(x) \cdot \nabla_x \phi(t, x) dt dx = O(N).$$

We have

$$\begin{aligned} & \int_{(a,b) \times \Omega} A_0(x, D_x) \phi(t, x) h(x) \cdot \nabla_x \phi(t, x) dt dx \\ &= \int_{(a,b) \times \Omega} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n D_{x_i}(a_{ij}(x) D_{x_j} \phi(t, x) h_k(x) D_{x_k} \phi(t, x)) dt dx \\ & \quad - \int_{(a,b) \times \Omega} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n a_{ij}(x) D_{x_j} \phi(t, x) D_{x_i}(h_k(x) D_{x_k} \phi(t, x)) dt dx \\ &= \int_{(a,b) \times \Omega} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n D_{x_i}(a_{ij}(x) D_{x_j} \phi(t, x) h_k(x) D_{x_k} \phi(t, x)) dt dx \\ & \quad - \int_{(a,b) \times \Omega} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n a_{ij}(x) D_{x_j} \phi(t, x) h_k(x) D_{x_i x_k} \phi(t, x) h_k(x) dt dx \\ & \quad - \int_{(a,b) \times \Omega} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n a_{ij}(x) D_{x_j} \phi(t, x) D_{x_k} \phi(t, x) D_{x_i} h_k(x) dt dx \\ &:= I_1 - I_2 + O(N). \end{aligned}$$

Moreover,

$$\begin{aligned} -I_2 &= - \int_{(a,b) \times \Omega} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n D_{x_k}(a_{ij}(x) D_{x_j} \phi(t, x) D_{x_i} \phi(t, x) h_k(x)) dt dx \\ & \quad + \int_{(a,b) \times \Omega} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n D_{x_k}(a_{ij}(x) D_{x_j} \phi(t, x) h_k(x)) D_{x_i} \phi(t, x) dt dx \\ &= - \int_{(a,b) \times \Omega} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n D_{x_k}(a_{ij}(x) D_{x_j} \phi(t, x) D_{x_i} \phi(t, x) h_k(x)) dt dx \\ & \quad + \int_{(a,b) \times \Omega} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n a_{ij}(x) D_{x_k x_j} \phi(t, x) h_k(x) D_{x_i} \phi(t, x) dt dx \\ & \quad + \int_{(a,b) \times \Omega} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n D_{x_j} \phi(t, x) D_{x_i} \phi(t, x) D_{x_k}(a_{ij}(x) h_k(x)) dt dx \\ &:= -I_3 + I_2 + O(N). \end{aligned}$$

So

$$I_2 = \frac{I_3}{2} + O(N).$$

We deduce that

$$I_1 - \frac{I_3}{2} = O(N). \quad (2.12) \quad \boxed{\text{eq3.18}}$$

We have

$$\begin{aligned} I_1 &= \int_{(a,b) \times \Omega} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n D_{x_i}(a_{ij}(x) D_{x_j} \phi(t, x) h_k(x) D_{x_k} \phi(t, x)) dt dx \\ &= \int_{(a,b) \times \partial \Omega} \sum_{i=1}^n \sum_{j=1}^n a_{ij}(x') D_{x_j} \phi(t, x') \nu_i(x') \sum_{k=1}^n h_k(x') D_{x_k} \phi(t, x') dt d\sigma \\ &= \int_{(a,b) \times \partial \Omega} (D_{\nu_A} \phi(t, x'))^2 dt d\sigma. \end{aligned}$$

By (2.10), we have

$$D_{\nu_A} \phi(t, x') = A(x') \frac{\partial \phi}{\partial \nu}(t, x') + \sum_{i=1}^n \sum_{j=1}^n a_{ij}(x') \nu_i(x') T_j \psi(t, x')$$

so that

$$\begin{aligned} I_1 &= \int_{(a,b) \times \partial \Omega} A(x')^2 \frac{\partial \phi}{\partial \nu}(t, x')^2 dt d\sigma \\ & \quad + \int_{(a,b) \times \partial \Omega} \frac{\partial \phi}{\partial \nu}(t, x') S_1 \psi(t, x') dt d\sigma + O(N), \end{aligned}$$

with  $S_1$  differential operator of order one in  $\partial\Omega$ , while

$$\begin{aligned}
I_3 &= \int_{(a,b) \times \partial\Omega} A(x') \sum_{i=1}^n \sum_{j=1}^n a_{ij}(x') D_{x_j} \phi(t, x') D_{x_i} \phi(t, x') dt d\sigma \\
&= \int_{(a,b) \times \partial\Omega} A(x') \sum_{i=1}^n \sum_{j=1}^n a_{ij}(x') (\nu_j(x') \frac{\partial \phi}{\partial \nu}(t, x') + T_j \psi(t, x')) \\
&\quad \times (\nu_i(x') \frac{\partial \phi}{\partial \nu}(t, x') + T_i \psi(t, x')) dt d\sigma \\
&= \int_{(a,b) \times \partial\Omega} A(x')^2 \frac{\partial \phi}{\partial \nu}(t, x')^2 dt d\sigma \\
&\quad + \int_{(a,b) \times \partial\Omega} \frac{\partial \phi}{\partial \nu}(t, x') S_2 \psi(t, x') dt d\sigma + O(N),
\end{aligned}$$

with  $S_2$  differential operator of order one in  $\partial\Omega$ . From (2.12) we deduce

$$\frac{1}{2} \int_{(a,b) \times \partial\Omega} A(x')^2 \frac{\partial \phi}{\partial \nu}(t, x')^2 dt d\sigma + \int_{(a,b) \times \partial\Omega} \frac{\partial \phi}{\partial \nu}(t, x') (S_1 - \frac{1}{2} S_2) \psi(t, x') dt d\sigma = O(N),$$

and, as  $A(x')^2$  is lower bounded by a positive constant, for some  $C_0$  positive independent of  $(u_0, \xi, v_0, \eta_0)$ ,  $s$ ,

$$\int_{(a,b) \times \partial\Omega} \frac{\partial u}{\partial \nu}(t, x')^2 dt d\sigma \leq C_0 [N + N^{1/2} (\int_{(s,T) \times \partial\Omega} \frac{\partial u}{\partial \nu}(t, x')^2 dt d\sigma)^{1/2}],$$

implying

$$\int_{(a,b) \times \partial\Omega} \frac{\partial u}{\partial \nu}(t, x')^2 dt d\sigma \leq \frac{(C_0 + \sqrt{C_0^2 + 4C_0})^2}{4} N.$$

□

**co0.9** **Corollary 2.11.** *Suppose (A1)-(A4) hold. Let  $T \in \mathbb{R}^+$ ,  $u_0 \in H^2(\Omega)$  with  $\gamma u_0 \in H^2(\partial\Omega)$ ,  $u_1 \in H^1(\Omega)$  with  $\gamma u_1 \in H^1(\partial\Omega)$ , so that  $(u_0, \gamma u_0, u_1, \gamma u_1) \in W \times V$ . Let*

$$(\phi(t), \psi(t), \alpha(t), \beta(t)) = e^{tM_0}(u_0, \gamma u_0, u_1, \gamma u_1) \quad (2.13)$$

**eq2.9**

*Moreover, if  $x' \in \partial\Omega$ , let  $G(x', D_x)u(x') = \sum_{j=1}^n g_j(x') D_{x_j} u(x') + g_0(x') u(x')$ , with  $g_j \in L^\infty(\partial\Omega)$ . Then there exists  $C(T)$  positive, independent of  $u_0$  and  $u_1$ , such that*

$$\|G(\cdot, D_x)\phi\|_{L^2((-T,T) \times \partial\Omega)} \leq C(T) \|(u_0, \gamma u_0, u_1, \gamma u_1)\|_{V \times H}.$$

*Proof.* Let

$$g(x') = (g_1(x'), \dots, g_n(x')).$$

If we set

$$k(x') := g(x') \cdot \nu(x'),$$

then  $t(x') := g(x') - k(x')\nu(x')$  is tangential to  $\partial\Omega$  in  $x'$ . So

$$G(x', D_x)\phi(t, x') = k(x') \frac{\partial \phi}{\partial \nu}(t, x') + t(x') \cdot \nabla_\tau \psi(t, x') + g_0(x') \psi(t, x').$$

and

$$\|G(\cdot, D_x)\phi(t, \cdot)\|_{L^2(\partial\Omega)} \leq C_0 (\|\frac{\partial \phi}{\partial \nu}(t, \cdot)\|_{L^2(\partial\Omega)} + \|\psi(t, \cdot)\|_{H^1(\partial\Omega)}) \quad (2.14)$$

**eq0.16A**

So the conclusion follows from Theorem 2.10. □

Now we recall the following perturbation result of Miyadera type (see [4], Corollary 3.16):

**th0.10**

**Theorem 2.12.** *Let  $A$  be the infinitesimal generator of a strongly continuous semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $X$  and let  $B \in \mathcal{L}(D(A), X)$  satisfy, for some  $t_0 > 0$ ,  $q \in [0, 1)$ ,*

$$\int_0^{t_0} \|BT(t)x\| dt \leq q\|x\|, \quad \forall x \in D(A).$$

*Then  $A + B$ , with domain  $D(A)$ , is the infinitesimal generator of a strongly continuous semigroup in  $X$ .*

co2.13

**Corollary 2.13.** *Let  $A$  be the infinitesimal generator of a strongly continuous group  $(T(t))_{t \in \mathbb{R}}$  on a Banach space  $X$  and let  $B \in \mathcal{L}(D(A), X)$  satisfy, for some  $t_0 > 0$ ,  $q \in [0, 1]$ ,*

$$\int_{-t_0}^{t_0} \|BT(t)x\| dt \leq q\|x\|, \quad \forall x \in D(A).$$

*Then  $A + B$ , with domain  $D(A)$ , is the infinitesimal generator of a strongly continuous group in  $X$ .*

*Proof.* As  $((T(t))_{t \in \mathbb{R}}$  is a group, if we set  $T_-(t) := T(-t)$ , with  $t \geq 0$ ,  $(T_-(t))_{t \geq 0}$  is a strongly continuous semigroup with infinitesimal generator  $-A$ . By Theorem 2.12,  $-A - B$ , with domain  $D(A)$ , is the infinitesimal generator of a strongly continuous semigroup. As both  $\pm(A + B)$  are infinitesimal generators of strongly continuous semigroups,  $A + B$  is the infinitesimal generator of a strongly continuous group.  $\square$

Now we are able to prove Theorem 2.1.

**Proof of Theorem 2.1** We set  $X = V \times H$ ,  $A = M_0$  and we introduce the following operator  $B$ :

$$\begin{cases} B : W \times V \rightarrow V \times H, \\ B(u_0, \gamma u_0, u_1, \gamma u_1) = (0, 0, 0, F(\cdot, D_x)u_0 + \frac{\partial u_0}{\partial \nu_A} + \gamma u_0). \end{cases}$$

Setting  $G(\cdot, D_x) = F(\cdot, D_x) + \frac{\partial}{\partial \nu_A} + \gamma$ , we have, taking  $t_0 \in (0, 1]$ , with the position (2.13),

$$\begin{aligned} & \int_{-t_0}^{t_0} \|Be^{tM_0}(u_0, \gamma u_0, u_1, \gamma u_1)\|_{V \times H} dt \\ & \leq (2t_0)^{1/2} (\int_{-t_0}^{t_0} \|G(\cdot, D_x)\phi(t)\|_{L^2(\partial\Omega)}^2 dt)^{1/2} \leq C(1)(2t_0)^{1/2} \|(u_0, \gamma u_0, u_1, \gamma u_1)\|_{V \times H}. \end{aligned}$$

So the assumptions of Corollary 2.13 are satisfied and the conclusion follows from the fact that  $M = M_0 + B$ .

### 3 Developments of Theorem 2.1

se3

We shall employ the following well known fact, concerning strongly continuous semigroups:

pr3.1A

**Proposition 3.1.** *Let  $A$  be the infinitesimal generator of a strongly continuous semigroup  $(T(t))_{t \geq 0}$  in the Banach space  $X$ . Let  $x \in D(A)$  and  $f \in W^{1,1}(0, T; X) + C([0, T]; X) \cap L^1(0, T; D(A))$ . Then the Cauchy problem*

$$\begin{cases} u'(t) = Au(t) + f(t), & t \in [0, T], \\ u(0) = x \end{cases}$$

*has a unique solution  $u$  in  $C^1([0, T]; X) \cap C([0, T]; D(A))$  given by the variation of parameter formula*

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds.$$

We consider the following problem:

$$\begin{cases} D_t^2 u(t, x) + a(x)D_t u(t, x) = A(x, D_x)u(t, x) + f(t, x), & (t, x) \in (0, T) \times \Omega, \\ D_t^2 \gamma u(t, x') + b(x')D_t \gamma u(t, x') = \nabla_\tau \cdot (B(x')\nabla_\tau \gamma u)(t, x') + F(x', D_x)u(t, x') + h(t, x'), \\ (t, x') \in (0, T) \times \partial\Omega, \\ u(0, x) = u_0(x), \quad x \in \Omega, \\ D_t u(0, x) = u_1(x), \quad x \in \Omega. \end{cases} \quad (3.1) \quad \text{eq3.1A}$$

We introduce the following assumptions:

(B1) (A1), (A3), (A4), (A5) hold;

$$A(x, D_x) = A_0(x, D_x) + \sum_{j=1}^n a_j(x) D_{x_j} + a_0(x),$$

with  $A_0$  as in (A2),  $a_j \in L^\infty(\Omega)$  ( $0 \leq j \leq n$ );

(B2)  $a \in L^\infty(\Omega)$ ,  $b \in L^\infty(\partial\Omega)$ .

Then we have:

**pr3.1** **Proposition 3.2.** Suppose that (B1)-(B2) hold. We introduce the following operator  $M_1$ :

$$\begin{cases} M_1 : V \times H \rightarrow V \times H, \\ M_1(v_0, v_1, w_0, w_1) = (0, 0, \sum_{j=1}^n a_j(\cdot) D_{x_j} v_0 + a_0(\cdot) v_0 - a(\cdot) w_0, -b(\cdot) w_1). \end{cases}$$

Then

(I)  $M + M_1$ , with domain  $W \times V$ , is the infinitesimal generator of a strongly continuous group in  $V \times H$ ;

(II) consider the problem (3.1), with  $T \in \mathbb{R}^+$ . Suppose, moreover, that:

(a)  $u_0 \in H^2(\Omega)$ ,  $\gamma u_0 \in H^2(\partial\Omega)$ ,  $u_1 \in H^1(\Omega)$ ,  $\gamma u_1 \in H^1(\partial\Omega)$ ;

(b)  $f(t, x) = f_1(t, x) + f_2(t, x)$ , with  $f_1 \in C([0, T]; L^2(\Omega)) \cap L^1(0, T; H^1(\Omega))$ ,  $\gamma f_1 \in C([0, T]; L^2(\partial\Omega)) \cap L^1(0, T; H^1(\partial\Omega))$ ,  $f_2 \in W^{1,1}(0, T; L^2(\Omega))$ ;

(c)  $h(t, x') = \gamma f_1(t, x') + h_1(t, x')$ , with  $h_1 \in W^{1,1}(0, T; L^2(\partial\Omega))$ .

Then (3.1) has a unique solution  $u$  belonging to  $\cap_{j=0}^2 C^{2-j}([0, T]; H^j(\Omega))$ , with  $\gamma u$  belonging to  $\cap_{j=0}^2 C^{2-j}([0, T]; H^j(\partial\Omega))$ .

*Proof.* (I) follows from Theorem 2.1 and the fact that  $M_1$  belongs to  $\mathcal{L}(V \times H)$ .

(II) We set  $\phi := u$ ,  $\psi := \gamma u$ ,  $\alpha := D_t u$ ,  $\beta := \gamma \alpha = D_t \psi$ . Then (3.1) can be written in the equivalent form

$$\begin{cases} (\phi'(t), \psi'(t), \alpha'(t), \beta'(t)) = (M + M_1)(\phi(t), \psi(t), \alpha(t), \beta(t)) + (0, 0, f(t), h(t)), & t \in [0, T], \\ (\phi(0), \psi(0), \alpha(0), \beta(0)) = (u_0, \gamma u_0, u_1, \gamma u_1). \end{cases} \quad (3.2) \quad \text{eq3.2}$$

Then  $(u_0, \gamma u_0, u_1, \gamma u_1)$  belongs to  $W \times V$ , while

$$(0, 0, f(t), h(t)) = (0, 0, f_1(t), \gamma f_1(t)) + (0, 0, f_2(t), h_1(t)),$$

with the first summand in

$$C([0, T]; V \times H) \cap L^1(0, T; W \times V) = C([0, T]; V \times H) \cap L^1(0, T; D(M + M_1)),$$

the second summand in  $W^{1,1}(0, T; V \times H)$ . By Proposition 3.1, (3.2) has a unique solution in  $C^1([0, T]; V \times H) \cap C([0, T]; W \times V)$ .  $\square$

We conclude with an application to (1.2).

**Proposition 3.3.** Consider the problem (1.2), with the assumption (B1) and  $T \in \mathbb{R}^+$ . Suppose, moreover, that:

(a)  $u_0 \in H^2(\Omega)$ ,  $\gamma u_0 \in H^2(\partial\Omega)$ ,  $u_1 \in H^1(\Omega)$ ,  $\gamma u_1 \in H^1(\partial\Omega)$ ;

(b)  $f \in C([0, T]; L^2(\Omega)) \cap L^1(0, T; H^1(\Omega))$ ,  $\gamma f \in C([0, T]; L^2(\partial\Omega)) \cap L^1(0, T; H^1(\partial\Omega))$ ;

(c)  $h \in W^{1,1}(0, T; L^2(\partial\Omega))$ .

Then (1.2) has a unique solution  $u$  belonging to  $\cap_{j=0}^2 C^{2-j}([0, T]; H^j(\Omega))$ , with  $\gamma u$  belonging to  $\cap_{j=0}^2 C^{2-j}([0, T]; H^j(\partial\Omega))$ . Here  $A(x', D_x)u(t, x')$  is intended as  $D_t^2 \gamma u - \gamma f$ .

*Proof.* The problem is equivalent to (3.1) with  $a \equiv 0$ ,  $b \equiv 0$  and  $h$  replaced by  $\gamma f + h$ . So the conclusion follows from Proposition 3.2.  $\square$

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