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HIGHER DIMENSIONAL ELLIPTIC FIBRATIONS AND ZARISKI DECOMPOSITIONS

ANTONELLA GRASSI AND DAVID WEN

ABSTRACT. We study the existence and properties of birationally equivalent models for elliptically fibered varieties. In particular these have either the structure of Mori fiber spaces or, assuming some standard conjectures, minimal models with a Zariski decomposition compatible with the elliptic fibration. We prove relations between the birational invariants of the elliptically fibered variety, the base of the fibration and of its Jacobian.

1. INTRODUCTION

The geometry of elliptic surfaces is well understood by the work of Kodaira. In particular, when the Kodaira dimension of an elliptic surface is non-negative, the minimal model has a birationally equivalent elliptic fibration. Kodaira's canonical bundle formula for relatively minimal elliptic surfaces relates the canonical bundle of the surface to the pullback of the canonical divisor of the base curve and a \mathbb{Q} -divisor Λ supported on the loci of the image of singular fibers, the discriminant locus of the fibration. The first author showed that the fibration structure on an elliptic threefold is compatible with the minimal model algorithm and in addition, that a generalization of Kodaira's formula for the canonical divisor holds on the (relative) minimal model [10, 11]. An ingredient in the proof of [10] is to show the existence of an appropriate combination of the Zariski Decomposition Theorem for surfaces and a relative version of the minimal model program. A challenge in dimension 4 (and higher, which we address here, is the existence of different definition(s) of Zariski decompositions and their relation with minimal models.

This paper addresses the case of elliptic fibrations of varieties of dimension ≥ 4 . In the following $\Pi : Y \rightarrow T$ is an elliptic fibration between normal complex projective varieties where $\dim Y = n$. Then there exists a birationally equivalent elliptic fibration $\pi : X \rightarrow B$, where X, B are smooth and the fibration has nice properties, in particular there exists an effective \mathbb{Q} -divisor Λ , supported on the discriminant of the fibration (Theorem 2 and Lemma 16).

Theorem (Proposition 18, Theorems 26, 33, 40, Corollary 34, 41). *Let $\pi : X \rightarrow B$ be an elliptic fibration between smooth varieties, Λ the discriminant \mathbb{Q} -divisor (Lemma 16). Then*

- (1) $\kappa(X) = \kappa(B, \Lambda)$.
- (2) *If $K_B + \Lambda$ is not pseudo-effective, there exists a birational equivalent fibration $\bar{\pi} : \bar{X} \rightarrow \bar{B}$, \bar{X} with \mathbb{Q} -factorial terminal singularities, $(\bar{B}, \bar{\Lambda})$ with klt singularities such that $K_{\bar{X}} \equiv \bar{\pi}^*(K_{\bar{B}} + \bar{\Lambda})$. X is birationally a Mori fiber space.*
- (3) *If $K_B + \Lambda$ is pseudo-effective, equivalently $\kappa(X) \geq 0$, and klt flips exist and terminate in dimension $n - 1$, there exists a birational equivalent fibration $\bar{X} \rightarrow \bar{B}$, \bar{X} minimal, with \mathbb{Q} -factorial terminal singularities, $(\bar{B}, \bar{\Lambda})$ with \mathbb{Q} -factorial klt singularities such that $K_{\bar{X}} \equiv \bar{\pi}^*(K_{\bar{B}} + \bar{\Lambda})$.*
- (4) *There is a birationally equivalent fibration $\bar{\pi} : \bar{X}_1 \rightarrow \bar{B}_1$, with the same properties of $\bar{X} \rightarrow \bar{B}$ in either (2) or (3) above, which is equidimensional over an open set $U \subset \bar{B}$ with $\text{codim}(\bar{B} \setminus U) \geq 3$.*

To prove part of the theorem above we show the compatibility of a Zariski type decomposition, the Fujita-Zariski decomposition, with the elliptic fibration. We prove that the compatibility plays a role in keeping track of the birational modifications of the steps in the MMP. More specifically, we prove:

Theorem (Theorem 30, 31). *Let $\pi : X \rightarrow B$ be an elliptic fibration as above and $\dim X = n$.*

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- (1) K_X birationally admits a Fujita-Zariski decomposition if and only if $K_B + \Lambda$ birationally admits a Fujita-Zariski decomposition.
- (2) If $K_B + \Lambda$ is pseudo-effective, equivalently if $\kappa(X) \geq 0$, and klt flips exist and terminate in dimension $n - 1$, K_X birationally admits a Fujita-Zariski decomposition compatible with the elliptic fibration structure.

Corollary (Corollary 36). *Let $\pi : Y \rightarrow T$ be an elliptic fibration, $\kappa(Y) \geq 0$ and $\dim Y \leq 5$. There exists a birational equivalent fibration $\bar{X} \rightarrow \bar{B}$, \bar{X} minimal, $(\bar{B}, \bar{\Lambda})$ with \mathbb{Q} -factorial klt singularities such that $K_{\bar{X}} \equiv \bar{\pi}^*(K_{\bar{B}} + \bar{\Lambda})$ and K_Y birationally admits a Fujita-Zariski decomposition compatible with the elliptic fibration structure.*

Corollary (Proposition 18, Theorem 26, Corollary 39). *Let $\pi : Y \rightarrow T$ be an elliptic fibration*

- (1) *If $\dim(Y) = 4$ there exists a birationally equivalent fibration $\bar{X} \rightarrow \bar{B}$, \bar{X} with \mathbb{Q} -factorial terminal singularities, $(\bar{B}, \bar{\Lambda})$ with klt singularities such that $K_{\bar{X}} \equiv \bar{\pi}^*(K_{\bar{B}} + \bar{\Lambda})$. Either Y is birationally a Mori fiber space or \bar{X} is a good minimal model.*
- (2) *If $\kappa(Y) = n - 1$, there exists a birationally equivalent fibration $\bar{\pi} : \bar{X} \rightarrow \bar{B}$ such that \bar{X} is a good minimal model with \mathbb{Q} -factorial terminal singularities, $K_{\bar{X}} \equiv_{\mathbb{Q}} \bar{\pi}^*(K_{\bar{B}} + \bar{\Lambda})$ and $(\bar{B}, \bar{\Lambda})$ has klt singularities.*

In Section 2, we review standard definitions and relevant results about elliptic fibrations, minimal model theory and generalized Zariski decompositions. We also highlight the different generalizations of the Zariski Decomposition, their properties and their relationship with minimal model theory and with the structure of elliptic fibrations. In Section 3, we prove results on the relations between the birational invariants of the elliptically fibered variety, the base of the fibration and of its Jacobian. In particular we have:

Corollary (Corollary 29). *If X is birationally a Calabi-Yau variety, so is $J(X)$.*

Some of these results are applied in the proofs of the theorems in Section 4. We also prove the relevant parts of the theorems stated above. In Section 4 we construct a compatible Zariski decomposition for elliptically fibered varieties of non-negative Kodaira dimension. Sections 5.1 and 5.2 contain applications, namely we prove existence results and related implications on abundance and other generalized Zariski decompositions for elliptic fibrations. Finally, in Section 6 we prove results on the dimension of the fibers on special birational models, which we construct. These statements replace the equidimensionality results for dimension 3. In fact, while it is easy to fabricate examples of minimal elliptic threefolds which are not equidimensional starting from ones which are, many smooth Calabi-Yau threefolds have a natural elliptic fibration which is not equidimensional. Theorem 40 and Corollary 41 are stronger than what one could get from a (relative) log-minimal model run, in at least two aspects. Namely, the singularities are terminal, while a minimal model run gives klt singularities, as in Example 3.1, but sometimes the model does not have desired properties, as in Example 3.2. In addition, not only are there no exceptional divisors in the fibers outside a codimension 3 set, but the fibration is actually equidimensional there.

We present applications all throughout the paper. The techniques of this paper set a foundation for generalization to the case of fibration of Calabi-Yau varieties and as well as log pairs [36]. Unless otherwise specified, the varieties in this paper are assumed to be complex, projective and normal.

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2. NOTATION-RESULTS

An elliptic fibration is a morphism, $\pi : X \rightarrow B$ between normal projective varieties, whose general fibers are smooth genus one curves with or without a marked point; the complement of the image of the smooth fibers in B is the discriminant of the fibration. If π has a section, namely if the general elliptic curve has a marked point, then X is a (smooth) resolution of W , the Weierstrass model of the fibration, [31].

Definition 1. The elliptic fibrations $\psi : Y \rightarrow T$ and π are birationally equivalent if there exist birational maps $f : X \dashrightarrow Y$ and $g : B \dashrightarrow T$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \dashrightarrow^f & Y \\ \pi \downarrow & & \downarrow \psi \\ B & \dashrightarrow^g & T \end{array}$$

Building on the work of Kodaira, Kawamata and Morikawi [18,23,30], Fujita and Nakayama proved results in [8, Thm 2.14 and Thm 2.15] and in [31, Thm. 0.2] respectively, which can be combined to the following:

Theorem 2. Let $\pi : X \rightarrow B$ an elliptic fibration between smooth varieties. Then the discriminant locus is a divisor; assume that it has simple normal crossing. In addition:

- (1) The \mathbf{J} -invariants of the fibers extends to a morphism $\mathbf{J} : B \rightarrow \mathbb{P}^1$.
- (2) $\pi_*(K_{X/B})$ is a line bundle.
- (3) $12\pi_*(K_{X/B}) = \mathcal{O}_S(\sum 12a_k D_k) \otimes \mathbf{J}^* \mathcal{O}_{\mathbb{P}^1}(1)$ where a_k are the rational numbers corresponding to the type of singularities over the general point of D_k , the irreducible components of the ramification locus. \mathbf{J} has a pole of order b_k along D_k , we write $\mathbf{J}^* \mathcal{O}_{\mathbb{P}^1}(1) = \sum b_k D_k$.
- (4) $K_X \equiv_{\mathbb{Q}} \pi^* \left(K_B + \pi_*(K_{X/B}) + \sum \frac{m_i-1}{m_i} Y_i \right) + E - G$, where:
- (5) The general fiber of $\pi^{-1}(Y_i)$ is a multiple fiber of multiplicity m_i
- (6) $E|_{\pi^{-1}(C)}$ is a union of a finite numbers of proper transforms of exceptional curves, for C a general curve on B .
- (7) G is an effective \mathbb{Q} -divisor and $\text{codim } \pi(G) \geq 2$,
- (8) $\pi_* \mathcal{O}_X([mE]) = \mathcal{O}_B$, $\forall m \in \mathbb{N}$
- (9) $\pi^* \left(\frac{m_i-1}{m_i} Y_i \right) + E - G$ is effective.

Definition 3. With the same hypothesis and notation of Theorem 2, we define the divisors:

$$\Delta_{X/B} \stackrel{\text{def}}{=} \sum a_k D_k + \frac{1}{12} J, \quad \Lambda_{X/B} \stackrel{\text{def}}{=} \Delta_{X/B} + \sum \frac{m_i-1}{m_i} Y_i$$

where J is an effective divisor whose associated line bundle is equivalent to $\mathbf{J}^* \mathcal{O}_{\mathbb{P}^1}(1)$. When the notation is clear from the context, we will write Δ and Λ . Δ and Λ are \mathbb{Q} -divisor.

The pairs (B, Δ) and (B, Λ) are *klt*, by [26, Prop. 2.41], since Δ and Λ are simple normal crossing divisors with rational coefficients in $(0, 1)$. More generally, they are log pairs which are of the form (B, D) where B is a normal (projective) variety, D_i prime divisors, $D = \sum a_i D_i$ and $K_B + D$ \mathbb{Q} -Cartier.

Definition 4. (i): A log resolution of (B, D) is a resolution $f : \tilde{B} \rightarrow B$, such that the union of $\sum a_i f_*^{-1}(D_i)$, the strict transform of D , and the exceptional locus of f are supported on divisors with simple normal crossings.

(ii) We then write $K_{\tilde{B}} + \sum a_i f_*^{-1}(D_i) = f^*(K_B + D) + \sum a(E_j, X, D) E_j$, where $a(E_j, X, D)$ are the discrepancies.

(iii) The pair (B, D) is

terminal: if for any (equivalently for every) log resolution f , $a(E_j, X, D) > 0, \forall j$.

klt: if for any (equivalently for every) log resolution f , $a(E_j, X, D) > -1, \forall j$.

lc: (log canonical) if for any (equivalently for every) log resolution f , $a(E_j, X, D) \geq -1, \forall j$.

dlt: (divisorially log terminal) if there is a log resolution f such that $a(E_j, X, D) > -1$ for every exceptional divisors E_j .

plt: (purely log terminal) if for any log resolution f , $a(E_j, X, D) > -1$, for every coefficient of an exceptional divisor E_j .

In the following (X, D) is always a lc pair.

Definition 5. Minimal models, log minimal models etc.

MM: \bar{X} is a minimal model if $(\bar{X}, 0)$ has terminal singularities, $K_{\bar{X}}$ is nef and \bar{X} is \mathbb{Q} -factorial.

Neg. Contr.: $\psi : B \dashrightarrow \bar{B}$ is a $(K_B + D)$ -negative contraction if ψ^{-1} does not contract any divisor and there exists a resolution \tilde{B}

$$\begin{array}{ccc} & \tilde{B} & \\ g \swarrow & & \searrow h \\ (B, D) & \xrightarrow{\psi} & (\bar{B}, \psi_*(D)) \end{array}$$

such that $g^*(K_B + D) - h^*(K_{\bar{B}} + \bar{D}) = \sum a_j E_j$, $a_j > 0$ and E_j exceptional for h .

LMM-A: $(\bar{B}, \psi_*(D))$ is a log minimal model for (B, D) if ψ is a $(K_B + D)$ -negative contraction and $(K_{\bar{B}} + \psi_*(D))$ is nef.

LBM: (\bar{B}, \bar{D}) is a log birational model of (B, D) if

$\psi : B \dashrightarrow \bar{B}$ is birational and $\bar{D} \stackrel{\text{def}}{=} \psi_*(D) + E$, where E is the reduced exceptional divisor of ψ^{-1} .

LMM-B: ([2]) A log birational model (\bar{B}, \bar{D}) is a log minimal model for (B, D) if (\bar{B}, \bar{D}) is \mathbb{Q} -factorial dlt, $(K_{\bar{B}} + \bar{D})$ is nef and $a(E_j, B, D) < a(E_j, \bar{B}, \bar{D})$, for E_j divisor in B , exceptional for ψ .

GOOD A log minimal model $(\bar{B}, \psi_*(D))$ is good if $K_{\bar{B}} + \psi_*(D)$ is semi-ample.

Remark 6. The definition LMM-B allows for ψ^{-1} exceptional divisors. Furthermore, the Negativity Lemma implies that a log minimal model according to A is a log minimal model in the sense of B; the two definitions are equivalent for plt pairs [2].

Definition 7 (Zariski Decompositions [2] - [8] - [6, 19, 29] - [33]). Let X be a normal, projective variety with a proper map $\pi : X \rightarrow Z$ and D a \mathbb{R} -divisor on X . We have that $D = P + N$ is called:

W: A Weak Zariski decomposition over Z , if P is π -nef and N is effective.

FZ-A: A Fujita-Zariski decomposition over Z , if it is a Weak Zariski decomposition and we have that for every projective birational morphism $f : W \rightarrow X$, where W is normal, and $f^*D = P' + N'$ with P' nef over Z , then we have $P' \leq f^*P$.

CKM: A CKM-Zariski decomposition over Z , if it is a Weak Zariski decomposition and we have that $\pi_*\mathcal{O}_X(mP) \rightarrow \pi_*\mathcal{O}_X(mD)$ is an isomorphism for all $m \in \mathbb{N}$.

If we have that $Z = \text{Spec}(\mathbb{C})$ then we will refer to $D = P + N$ as simply the (Weak, Fujita, CKM) Zariski decomposition. Additionally, for the case where $Z = \text{Spec}(\mathbb{C})$ and X smooth, we have the following original definition of the Fujita-Zariski decomposition.

Num. Fixed: Let E be an effective \mathbb{Q} -divisor and L be a \mathbb{Q} -divisor on X . We say E clutches L if, for any effective \mathbb{Q} -divisor F where $L - F$ is nef, we have that $E - F$ is effective. We say E is numerically fixed by L if for any birational morphism $\pi : W \rightarrow X$ we have that π^*E clutches π^*L .

FZ-B: $D = P + N$ is a Fujita-Zariski decomposition if N is numerically fixed by D .

Additionally, if we also assume that D is pseudo-effective we have the sectional decomposition (sometimes called the Nakayama-Zariski decomposition).

NZ: Let A be a fixed ample divisor on X . Given a prime divisor Γ on X , define

$$\sigma_\Gamma(D) = \min\{\text{mult}_\Gamma(D') \mid D' \geq 0, D' \sim_{\mathbb{Q}} D + \epsilon A \text{ for some } \epsilon > 0\}$$

This definition is independent of the choice of A . Furthermore, it was also shown in [33] that for only finitely many Γ that $\sigma_\Gamma(D) > 0$. This allows us to define the following decomposition.

Let $N_\sigma(D) = \sum_\Gamma \sigma_\Gamma(D) \Gamma$ and $P_\sigma(D) = D - N_\sigma(D)$, then we call $D = P_\sigma(D) + N_\sigma(D)$ the sectional decomposition. If we have also that $P_\sigma(D)$ is nef then we refer to this as the Nakayama-Zariski decomposition of D .

We say D birationally admits a (Weak, Fujita, CKM, Nakayama) Zariski decomposition over Z if there exists some resolution $f : Y \rightarrow X$ such that $f^*(D)$ has a (Weak, Fujita, CKM, Nakayama) Zariski decomposition over Z .

Remark 8. There is a nesting of the above generalized Zariski decompositions as listed:

- (1) A Nakayama-Zariski decomposition (a sectional decomposition with nef positive part) is a Fujita-Zariski decomposition.
- (2) A Fujita-Zariski decomposition is a CKM-Zariski decomposition.
- (3) These are all Weak Zariski decompositions.
- (4) There are CKM-Zariski decompositions that are not Fujita-Zariski decompositions.
- (5) It is not known if there are Fujita-Zariski decompositions that are not Nakayama-Zariski decompositions.

Below we list some technical properties, relations and similarities of the different versions of the generalized Zariski decompositions.

Proposition 9 ([8, Cor. 1.9; Lemma 1.22]). *Let X be a smooth projective variety with E an effective \mathbb{Q} -divisor that is numerically fixed by a Cartier divisor L .*

- (1) *Let F be the smallest Cartier divisor such that $F - E$ is effective, then we have the following isomorphism of graded rings:*

$$\bigoplus_{t \geq 0} H^0(X, tL) \cong \bigoplus_{t \geq 0} H^0(X, tL - tF)$$

- (2) *$L - E$ admits a Fujita-Zariski decomposition if and only if L admits a Fujita-Zariski decomposition. Additionally, the nef parts of the decompositions are the same.*

Proposition 10 ([8, Prop. 1.10] - [9, Lemma 2.16]). *Let $f : M \rightarrow S$ be a surjective morphism of manifolds with connected fibers. Let X be a divisor on M such that $\dim f(X) < \dim S$. Suppose that for every irreducible component Z of $f(X)$ with $\dim Z = \dim S - 1$, there is a prime divisor D on M such that $f(D) = Z$ and $D \not\subset \text{Supp}(X)$.*

- (1) *X is numerically fixed by $X + f^*L$ for any \mathbb{Q} -Cartier divisor on S .*
- (2) *For any pseudoeffective \mathbb{R} -divisor L on S , $D \leq N_\sigma(f^*L + D)$ and $P_\sigma(f^*L + D) = P_\sigma(f^*L)$.*

Proposition 11 ([8, Prop. 1.24]). *Let $f : M \rightarrow S$ be a surjective morphism of manifolds with L , a \mathbb{Q} -Cartier divisor, on S and R an effective \mathbb{Q} -divisor on M such that $\dim f(R) \leq \dim S - 2$. Then $f^*L + R$ birationally admits a Fujita-Zariski decomposition if and only if L birationally admits a Fujita-Zariski decomposition.*

Proposition 12 ([33, Prop. V.1.14]). *Let D be a pseudo effective \mathbb{R} -divisor, then*

- (1) *$N_\sigma(D) = 0$ if and only if D is movable.*
- (2) *If $D - E$ is movable for an effective divisor E , then $N_\sigma(D) \leq E$.*

Proposition 13. *When $Z = \text{Spec}(\mathbb{C})$ and X is smooth, the two definitions of the Fujita-Zariski decomposition are equivalent.*

Proof. Let $D = P + N$ be a Fujita-Zariski decomposition in the sense of FZ-B. We will show that this implies the properties of FZ-A. Let $f : X' \rightarrow X$ be a birational morphism with $f^*(D) = P' + N'$ where P' is nef and N' is an effective \mathbb{Q} -Cartier divisor. We have that N is numerically fixed by D and so $f^*(N)$ clutches $f^*(D)$. As $f^*(D) - N' = P'$ is nef, we have that $N' - f^*(N)$ is effective. But we know that $N' = f^*(D) - P'$ and $f^*(N) = f^*(D) - f^*(P)$. So by replacing and simplifying we have that $f^*(P) - P'$ is effective.

Let $D = P + N$ be a Fujita-Zariski decomposition in the sense of FZ-A and we will show that N is numerically fixed by D , so given a birational morphism $f : X' \rightarrow X$ we will show that $f^*(N)$ clutches $f^*(D)$. Thus given an effective \mathbb{Q} -divisor N' such that $P' := f^*(D) - N'$ is nef, we want to show that $N' - f^*(N)$ is effective. We can assume that X' is normal, otherwise we can resolve singularities to get $\pi : W \rightarrow X'$, where we have that showing $\pi^*(N' - f^*(N))$ is effective is sufficient to show that $N' - f^*(N)$ is effective on X' . Thus without loss of generalities, we can assume X' is normal. Since $P + N$ is a Fujita-Zariski decomposition in the sense of FZ - A, we have that $f^*(P) - P'$ is effective. Replacing with $f^*(P) = f^*(D) - f^*(N)$ and $P' = f^*(D) - N'$, we get that $N' - f^*(N)$ is effective. This completes the argument that shows the two definitions are equivalent. ■

Remark 14. (1) The original definition of the Fujita-Zariski decomposition in [8] is equivalent to our definition of a divisor birationally admitting a Fujita-Zariski decomposition.

- (2) If $K_B + \Delta$ has a log minimal model, then it birationally has a Fujita (also CKM and Weak) Zariski decomposition. This is shown explicitly as part of the argument of [2, Thm 1.5].

- (3) A Fujita-Zariski decomposition and a Nakayama-Zariski decomposition of a divisor is unique. A CKM-Zariski and Weak Zariski decomposition of a divisor need not be unique.
- (4) Each of the above generalized Zariski decomposition for the canonical divisor has a different role in birational geometry and their relations to minimal models. The Nakayama-Zariski decomposition is more attune to work with abundance and good minimal models as seen in [9]. The Fujita-Zariski decomposition aligns with minimal models as seen below, and the CKM-Zariski decomposition corresponds to working with the canonical ring and, as a result, the canonical model.
- (5) Recent work in [13, 15] and [2] show that there is a strong correlation between the existence of log-minimal models and Zariski decompositions.

Theorem 15 ([25, 34, 35] - [20, 25] - [3] - [1]). *We have the following results in the theory of minimal models.*

- (1) *Flips for klt pairs exists in all dimension.*
- (2) *Any sequence of klt flips terminate in dimension 3.*
- (3) *A klt pair in dimension up to 4 either admits a minimal model or is birational to a Mori Fiber space.*
- (4) *A klt pair, (X, Δ) , such that $\dim(X) = 5$ and $\kappa(X, \Delta) \geq 0$ admits a minimal model.*
- (5) *The abundance conjecture holds for klt pairs of dimension ≤ 3 . Thus klt pairs of dimension up to 3 admit a good minimal model or are birational to a Mori Fiber space.*
- (6) *General type klt pairs admit a good minimal model.*

3. $X \& B$, RELATIVE MINIMAL MODELS, THE CANONICAL BUNDLE FORMULA, JACOBIANS

3.1. $X \& B$ and the discriminant. We recall the following application of Hironaka's flattening theorem:

Lemma 16 ([16]). *Let $\Pi : Y \rightarrow T$ be an elliptic fibration between normal varieties. Then there exist birational equivalent fibrations*

$$\begin{array}{ccccccc}
 Y & \longleftarrow & X_0 & \longleftarrow & X_1 & \longleftarrow & X \\
 \downarrow \Pi & & \downarrow \pi_0 & & \downarrow \pi_1 & & \downarrow \pi \\
 T & \longleftarrow & B_0 & \longleftarrow & B_1 & \longleftarrow & B \\
 & & & & \swarrow \psi_0 & & \\
 & & & & \psi & &
 \end{array}$$

where X_0, X, B_0, B and B_1 are smooth, π_1 is flat and $\pi : X \rightarrow B$ satisfy the hypothesis of Theorem 2.

Without loss of generality we can also assume $B_1 = B$ in the above Lemma.

Definition 17. *With the notation of Lemma 16 we set:*

$\Lambda_{B_0} \stackrel{\text{def}}{=} \psi_{0*}(\Lambda_{X/B})$ and $\Lambda_T \stackrel{\text{def}}{=} \psi_*(\Lambda_{X/B})$, where $\Lambda_{X/B}$ is as in Definition 3.

In particular the type of the singular fiber over a general point in the codimension 1 locus of the support of Λ_r is determined by the coefficient of its irreducible component in Λ_T .

Proposition 18. *Let $X_0 \rightarrow B_0$ be an elliptic fibration between smooth varieties and X and B be as in Lemma 16. Then $\kappa(X) = \kappa(K_B + \Lambda)$.*

Proof. The statement holds for $\dim X_0 = 2$. When $\dim X_0 = 3$, from the proof of [8, Theorem 3.2] we can deduce that $\kappa(X) = \kappa(X, K_X + G) = \kappa(X, \pi^*(K_B + \Lambda) + E) = \kappa(X, \pi^*(K_B + \Lambda)) = \kappa(B, K_B + \Lambda)$, as in [10, Proposition 1.3]. In particular $h^0(mK_X) = h^0(m(K_B + \Lambda))$, for all $m \gg 0$. The arguments about the pluricanonical rings in the proof of [8, Theorem 3.2] and [10, Proposition 1.3] are independent of dimension and they can be extended to $\dim X_0 \geq 4$. ■

We also have:

Proposition 19. *Let $\pi : X \rightarrow B$ be an elliptic fibration between manifolds, with $\dim X = n$ and let Δ the component of the discriminant divisor associated to the invertible sheaf $\pi_*(K_{X/B})$.*

The following exact sequences and isomorphisms hold:

$$\begin{aligned}
 0 \rightarrow H^\ell(B, \mathcal{O}_B) \rightarrow H^\ell(X, \mathcal{O}_X) \rightarrow H^{\ell-1}(B, -\Delta) \rightarrow 0, \quad 1 < \ell < n-1 \\
 H^0(X, \mathcal{O}_X) \simeq H^0(B, \mathcal{O}_B), \quad H^0(X, K_X) \simeq H^0(B, K_B + \Delta).
 \end{aligned}$$

Proof. Recall that $R^1\pi_*(\mathcal{O}_X) \simeq \pi_*(K_{X/S}) \simeq \mathcal{O}_B(\Delta)$ is the effective divisor of the discriminant corresponding to the non-multiple fibers (Theorem 2). The proof is in [10, Prop.2.2]. ■

3.2. Relative minimal models.

Theorem 20. *Let $\pi : X \rightarrow B$ be an elliptic fibration between \mathbb{Q} -factorial varieties. Assume that $\text{codim Sing}(X) \geq 3$ and that $K_X \equiv_{\mathbb{Q}} \pi^*(L) + F$, where F is a \mathbb{Q} -effective \mathbb{Q} -divisor such that no irreducible component F_j of F is $F_j = \pi^*(\Gamma_j)$, for some Γ_j . Then*

- (i) K_X is not π -nef.
- (ii) *If in addition X has terminal singularities there is a relative good minimal model for X over B , that is, there exists a birational equivalent elliptic fibration $\pi_r : X_r \rightarrow B$, such that X_r has \mathbb{Q} -factorial terminal singularities, K_{X_r} is π_r -nef and π_r -semiample. In addition $K_{X_r} \equiv_{\mathbb{Q}} \pi_r^*(L)$.*
- (iii) X_r be as in (ii). There exists a \mathbb{Q} -divisor Λ_r such that $L = K_B + \Lambda_r$, Λ_r defined as in Definition 17 and (B, Λ_r) has klt singularities.

Proof. Note that the locus of terminal singularities has codimension ≥ 3 .

- (i) Let F_j be an irreducible component of F . If $\text{codim} \pi(F_j) \geq 2$, then there is an effective curve γ such that $F_j \cdot \gamma < 0$. If $\text{codim} \pi(F_j) = 1$, let C be a general curve in B , p a point in C and in the support of $\pi(F_j)$. Consider the elliptic surface $S \stackrel{\text{def}}{=} \pi^*C$, $S \rightarrow C$ and conclude that there is an effective (exceptional) curve γ in the fiber over p such that $K_X \cdot \gamma = F_j \cdot \gamma < 0$.
- (ii) Since the π -relative log canonical ring is finitely generated ([3], [22, Theorem 6.6]), the hypothesis of Theorem 2.12 in [14] (which generalizes [27]) are satisfied. Then there exists a relative good minimal model for X over B , that is, there exists a birational equivalent elliptic fibration $\pi_r : X_r \rightarrow B$, such that X_r K_{X_r} is π_r -nef and π_r -semiample. In particular the proof of [14][Theorem 2.12] shows that if X has \mathbb{Q} -factorial terminal singularities, so does X_r .

We now assume without loss of generality that there is a birational morphism $\mu : X \rightarrow X_r$, with $K_X \equiv \mu^*(K_{X_r}) + \sum a_i E_i$, E_i μ -exceptional and $a_i > 0$. We then have $\mu^*(K_{X_r} - \pi_r^*(L)) = F - \sum a_i E_i$. If F is not μ -exceptional, we can take γ as in part (i) and conclude by contradiction, since

$$0 \leq \mu^*(K_{X_r} - \pi_r^*(K_B + L)) \cdot \gamma = (F - \sum a_i E_i) \cdot \gamma < 0.$$

- (iii) It follows from Lemma 16 and [31, Theorem 0.4]. ■

Proposition 21. *Let $\pi_i : X_i \rightarrow B$ $i = 1, 2$ be birationally equivalent elliptic fibrations, X_i with terminal singularities, (B, Λ) with klt singularities and*

$$K_{X_i} \equiv \pi_i^*(K_B + \Lambda) + F_i, \quad F_i \text{ } \mathbb{Q}\text{-Cartier } \mathbb{Q}\text{-divisor.}$$

Then and F_1 is \mathbb{Q} -effective if and only if F_2 is.

Proof. We take a common resolution of X_1 and X_2 and conclude by the same argument as in the proof of [11, Lemma 1.5]. ■

Proposition 22. *Let $\pi : X \rightarrow B$ be an elliptic fibrations, X with terminal singularities, (B, Λ) with klt singularities and $K_X \equiv \pi^*(K_B + \Lambda) + F$, for some \mathbb{Q} -Cartier \mathbb{Q} -divisor F .*

Then F is \mathbb{Q} -effective if and only if there exists a birational equivalent elliptic fibration $\pi_r : X_r \rightarrow B$, X_r with terminal singularities, such that $K_{X_r} \equiv_{\mathbb{Q}} \pi_r^(K_B + \Lambda)$.*

Proof. The same argument as in the proof of [11, Corollary 1.4] applies. ■

Note that part (iii) of Theorem 20 assure that if F is \mathbb{Q} -effective, (B, Λ) is klt. In particular no birationally equivalent elliptic fibration $\pi_r : X_r \rightarrow B$, X_r with terminal singularities, such that $K_{X_r} \equiv_{\mathbb{Q}} \pi_r^*(K_B + \Lambda)$ can exist in the following example:

Example 3.1 (Terminal versus klt). Example 1.1 of [11] provides an elliptic fibration $\pi : X \rightarrow \mathbb{P}^2$, X smooth, such that $K_X \equiv \pi^*(K_B + \Lambda) - \frac{1}{3}D$, where D is an effective divisor. The discriminant of the elliptic fibrations consists of 2 different lines corresponding to multiple fibers of type $3I_0$. By Proposition 21 there is no birationally equivalent elliptic fibration $\phi : \bar{X} \rightarrow \mathbb{P}^2$, \bar{X} with terminal singularities, such that $K_{\bar{X}} = \phi^*(L)$,

for some L . However, in this particular example $K_X + \alpha D$ is not relatively nef over \mathbb{P}^2 if $\frac{1}{3} < \alpha < 1$ and we can explicitly apply the relative log minimal model Theorems of [14] and [27] for the klt pair $(X, \alpha D)$ to obtain a birational equivalent elliptic fibration $\bar{\pi} : X_r \rightarrow \mathbb{P}^2$ with $X_r \equiv \bar{\pi}^*(K_B + \Lambda)$. X_r has *klt singularities*.

We stress that it is not guaranteed that running the relative minimal model program on the klt pair $(X, \alpha D)$ would contract D , as in the example below, and give a birationally equivalent model $\bar{\phi} : X_r \rightarrow B$, X_r with klt singularities such that $K_{X_r} = \bar{\phi}^*(L)$, contrary to the claim in [24, Section 8].

Example 3.2 (MM run to klt singularities does not guarantee a pullback formula for the canonical divisor). Consider the elliptic threefold, $f : X \rightarrow \mathbb{C}^2$, $y^2 = x^3 + (s^2 - t^2)^4 x + s^6 t^6$ over $(s, t) \in \mathbb{C}^2$. X is a local Weierstrass model over a smooth surface. X has a log canonical singularity; in fact, there is a resolution $g : Y \rightarrow X$ such that $K_Y = g^*(K_X) - D = g^*f^*(L) - D$, for some divisor L on \mathbb{C}^2 with $D \subset Y$ being a surface over the origin in \mathbb{C}^2 . As D maps to a point on \mathbb{C}^2 we have that $-D$ is nef over \mathbb{C}^2 . Thus for $(Y, \alpha D)$, there is no value of $\alpha \in [0, 1]$ which when running the minimal model program over \mathbb{C}^2 would contract D , and obtain a birationally equivalent model $\bar{\phi} : X_r \rightarrow \mathbb{C}^2$, X_r with klt singularities such that $K_{X_r} = \bar{\phi}^*(L)$.

In Theorem 40 we prove a more general and precise statement for the context of the above Examples 3.1 and 3.2.

Corollary 23. *Let $\pi : X_0 \rightarrow B_0$ be an elliptic fibrations between manifolds such that the ramification locus has simple normal crossing as in Theorem 2. Assume that either π is equidimensional or there are no multiple fibers. Then there exists a good minimal model X_r of X over B , that is a birational map $\mu : X \dashrightarrow X_r$ and a morphism $\pi_r : X_r \rightarrow B$ such that the diagram commutes, K_{X_r} is π_r -nef, $K_X \equiv_{\mathbb{Q}} \pi_r^*(K_B + \Lambda)$ and X_r has terminal singularities.*

Proof. In fact if there are no multiple fiber $F = E - G$ is effective; if π is equidimensional then $F = E$ is effective and we conclude by Proposition 22 ■

3.3. Birational equivalent elliptic fibrations, minimality, Mori fiber spaces and the canonical bundle formula.

Proposition 24. *Let $\pi : X \rightarrow B$ an elliptic fibration between manifolds. Assume that the ramification divisor of the fibration has simple normal crossing as in Theorem 2. Then, there is a birationally equivalent elliptic fibration $\pi_r : X_r \rightarrow B_r$ such that X_r has terminal singularities, (B_r, Λ_r) has klt singularities and $K_{X_r} \equiv_{\mathbb{Q}} \pi_r^*(K_{B_r} + \Lambda_r)$.*

Proof. There is a relative good minimal model $\pi_{r'} : X_r \rightarrow B$, by the proof of Theorem 2.12 in [14] and [3]. Note that [14] generalizes [27]). In particular there exist a birational morphism $\phi : B_r \rightarrow B$, a birationally equivalent fibration $\pi_r : X_r \rightarrow B_r$ and L_r ϕ -semiample such that $K_{X_r} \equiv \pi_r^*(L)$ and $\pi_{r'} = \phi \cdot \pi_r$. Note that $\dim B_r \geq \dim B$, that the morphism ϕ has to be birational; π_r is a birationally equivalent elliptic fibration. X_r has terminal singularities and then $K_{X_r} = \pi_r^*(K_{B_r} + \Lambda_r)$ and (B_r, Λ_r) is klt [31, Corollary 0.4]. ■

Examples 3.1 and 3.2 show that it can be necessary to birationally modify the base: $B_r \rightarrow B$.

Corollary 25. *Let $\pi : X \rightarrow B$ be an elliptic fibration between manifolds such that the ramification locus has simple normal crossing as in Theorem 2.*

- (1) *If $K_B + \Lambda$ is not pseudo effective, there exists a birational equivalent fibration $\bar{X} \rightarrow \bar{B}$, \bar{X} with terminal singularities, $(\bar{B}, \bar{\Lambda})$ with klt singularities such that $K_{\bar{X}} \equiv \bar{\pi}^*(K_{\bar{B}} + \bar{\Lambda})$. In addition X is birationally a Mori fiber space.*
- (2) *If $K_B + \Lambda$ is pseudo effective and klt flips exist and terminate in dimension $n - 1$, then there exists a birational equivalent fibration $\bar{X} \rightarrow \bar{B}$, \bar{X} minimal, with terminal singularities, $(\bar{B}, \bar{\Lambda})$ with klt singularities such that $K_{\bar{X}} \equiv \bar{\pi}^*(K_{\bar{B}} + \bar{\Lambda})$.*

Proof. Let $\pi_r : X_r \rightarrow B_r$ a birationally equivalent elliptic fibration as in Proposition 24. If $K_B + \Lambda$ is not pseudo effective, then by [3, Corollary 1.3.2] (B, Λ) is birationally a Mori fiber space, that is there exist a $(K_B + \Lambda)$ -negative birational contraction $\psi : B \dashrightarrow \bar{B}$ and a morphism $f : \bar{B} \rightarrow Z$ with connected fibers such that $\dim(Z) < \dim(B)$ and $(K_{\bar{B}} + \psi_*\Lambda)|_F$ is anti-ample for a general fiber F of f . We then conclude by applying Corollary 2.13 in [14] to every birational contraction and flip in ψ . This shows (1). Part (2) follows similarly. ■

Theorem 26. *Let $\pi : Y \rightarrow T$ be an elliptic fibration*

- (1) *If $\kappa(Y) \geq 0$ and klt flips exist and terminate in dimension $n - 1$, then there exists a birational equivalent fibration $\bar{X} \rightarrow \bar{B}$, \bar{X} minimal, with terminal singularities, $(\bar{B}, \bar{\Lambda})$ with klt singularities such that $K_{\bar{X}} \equiv \bar{\pi}^*(K_{\bar{B}} + \bar{\Lambda})$.*
- (2) *Let be Λ_T the \mathbb{Q} -divisor defined in Definition 17. If $K_T + \Lambda_T$ is not pseudo-effective there exists a birational equivalent fibration $\bar{X} \rightarrow \bar{B}$, \bar{X} with terminal singularities, $(\bar{B}, \bar{\Lambda})$ with klt singularities such that $K_{\bar{X}} \equiv \bar{\pi}^*(K_{\bar{B}} + \bar{\Lambda})$. In addition X is birationally a Mori fiber space.*

Proof. It follows from Corollary 25 and Proposition 18. ■

3.4. Jacobians.

Let $\pi : X \rightarrow B$ be an elliptic fibration between manifolds, with $\dim X = n$ and the ramification divisor a divisor with simple normal crossings. The corresponding Jacobian elliptic fibrations $\pi_J : J(X) \rightarrow B$ is defined birationally [7] from the relative minimal model of $X \rightarrow B$, which exists by Theorem 20.

Proposition 27. *Let $\pi : X \rightarrow B$ be an elliptic fibration between manifolds, with $\dim X = n$ and the ramification divisor a divisor with simple normal crossings. Let $\pi_J : J(X) \rightarrow B$ be the Jacobian fibration as above. Then:*

$$h^i(X, \mathcal{O}_X) = h^i(J(X), \mathcal{O}_{J(X)}), \quad 1 < i < n - 1.$$

Proof. In fact it follows from [7, Proposition 2.17] that the ramification divisor of the Jacobian fibration is a simple normal crossing divisor. In addition $\Delta = \Delta_J$. Proposition 19 implies the statement. ■

Proposition 28. *With the same hypothesis of Proposition 27, assume also that $h^0(X, K_X) = 1$ and $\kappa(X) = 0$, then $h^0(J(X), K_{J(X)}) = 1$ and $\kappa(J(X)) = 0$,*

Proof. Since $\Delta = \Delta_J$, Proposition 19 implies that $h^0(J(X), K_{J(X)}) = h^0(X, K_X) = 1$. Furthermore, as in the proof of Proposition 18, for $m \gg 0$,

$$h^0(X, mK_X) = h^0(B, m(K_B + \Lambda)) \geq h^0(B, m(K_B + \Delta)) = h^0(B, m(K_B + \Delta_J)) = h^0(J(X), mK_{J(X)}).$$
■

Corollary 29. *If X has birationally a trivial canonical divisor and $h^i(X, \mathcal{O}_X) = 0$, $0 < i < n$, that is if X is birationally a Calabi-Yau variety, so is $J(X)$.*

Proof. The statement follows from Proposition 19 and 28. ■

4. NON NEGATIVE KODAIRA DIMENSION, MINIMAL MODELS, ZARISKI DECOMPOSITION AND THE CANONICAL BUNDLE FORMULA

We prove, assuming standard conjectures in the theory of minimal models, a birational Fujita-Zariski decomposition for the canonical divisor for elliptic fibrations with non-negative Kodaira dimension. We use properties of the two definitions of the Fujita-Zariski decomposition. From [2], we use the relationship between Fujita-Zariski decomposition and minimal model theory; from [8], we use the relationship between Fujita-Zariski decomposition and the properties of numerically fixed divisors. This, in conjunction with the canonical bundle formula in Theorem 2, is enough to show a relationship between the total space and base space of an elliptic fibration through a birational Fujita-Zariski decomposition.

4.1. Generalized Zariski Decompositions for Elliptic Fibrations. In the following, we establish the compatibility of the Fujita-Zariski decomposition with elliptic fiber spaces which extends the arguments of [8] for elliptic threefold to higher dimensions. Furthermore, we will show the explicit decomposition in the case where we have existence of log minimal models for the base of the fiber space.

Theorem 30. *Given an elliptic fibration $X_0 \rightarrow B_0$, there exist a birationally equivalent fibration $X \rightarrow B$ and a \mathbb{Q} -divisor Λ on B such that K_X birationally admits a Fujita-Zariski decomposition if and only if $K_B + \Lambda$ birationally admits a Fujita-Zariski decomposition where X and (B, Λ) are as in Lemma 16.*

Proof. Without loss of generality we can assume that X_0 and B_0 are smooth, with ramification divisor Λ_0 having simple normal crossing. As in Lemma 16 we have:

$$\begin{array}{ccccc}
 & & \nu & & \\
 & \swarrow & & \searrow & \\
 X_0 & \longleftarrow & X_1 & \longleftarrow & X \\
 \pi_0 \downarrow & & \downarrow & & \downarrow \pi \\
 B_0 & \longleftarrow & B_1 & \longleftarrow & B
 \end{array}$$

where all the horizontal maps are birational morphisms, X_1 is the resolution of the flattening of π_0 and $\pi : X \rightarrow B$ is as in Theorem 2. We have $K_X = \pi^*(K_B + \Lambda) + E - G$ where (B, Λ) is a klt pair of dimension $n - 1$.

Assume that K_X birationally admits a Fujita-Zariski decomposition. Without loss of generality, we assume that K_X admits a Fujita-Zariski decomposition, in the sense of FZ-A, equivalently, FZ-B, as in Definition 7 and Remark 14. Then we have

$$P + N = K_X = \pi^*(K_B + \Lambda) + E - G$$

with P, N as in Definition 7. We will show that $K_B + \Lambda$ birationally admits a Fujita-Zariski decomposition.

We have G is a ν -exceptional effective divisor, since $X_1 \rightarrow B_1$ is equidimensional, as it is a flat morphism over a smooth base, and $\text{codim}(\pi(G)) \geq 2$. Furthermore $K_X = \nu^*(K_{X_0}) + F$, with F an effective ν -exceptional divisor, since X and X_0 are smooth. Then F is numerically fixed by K_X and $F + G$ is numerically fixed by $\nu^*(K_{X_0}) + F + G = K_X + G$, [8, Prop. 1.10]. Since F is numerically fixed by K_X and $K_X = P + N$ is a Fujita-Zariski decomposition then $K_X - F = \nu^*(K_{X_0})$ has a Fujita-Zariski decomposition by Lemma 9. Similarly, since $F + G$ is numerically fixed by $\nu^*(K_{X_0}) + F + G = K_X + G$, thus $K_X + G$ admits a Fujita-Zariski decomposition. In both cases P is the nef part of the decomposition. It follows that

$$\pi^*(K_B + \Lambda) + E = P + N + G$$

is a Fujita-Zariski decomposition. Since E is also numerically fixed by $\pi^*(K_B + \Lambda) + E$ (Theorem 2 and [8, Prop 1.10]); then $\pi^*(K_B + \Lambda)$ also admits a Fujita-Zariski decomposition (Lemma 9). Then $K_B + \Lambda$ birationally also admits a Fujita-Zariski decomposition by Proposition 11.

Assume now that $K_B + \Lambda$ birationally admits a Fujita-Zariski decomposition. Without loss of generality we assume that $K_B + \Lambda = P_\Lambda + N_\Lambda$ is a Fujita-Zariski decomposition. We have $\pi^*(K_B + \Lambda) = \pi^*(P_\Lambda) + h^*(N_\Lambda)$ is then a Fujita-Zariski decomposition (Proposition 11), with $\pi^*(P_\Lambda)$ the nef portion of the decomposition. The canonical bundle formula $K_X = \pi^*(K_B + \Lambda) + E - G$ and [8, Prop 1.10] imply that E is numerically fixed by $\pi^*(K_B + \Lambda) + E$. We have then a Fujita-Zariski decomposition for $K_X + G = \pi^*(K_B + \Lambda) + E = \pi^*(P_\Lambda) + \pi^*(N_\Lambda) + E$ with nef part $\pi^*(P_\Lambda)$. Similarly, with Lemma 9 applied to G , we deduce that K_X admits a Fujita-Zariski decomposition of the form

$$K_X = \pi^*(P_\Lambda) + \pi^*(N_\Lambda) + E - G.$$

Here $\pi^*(N) + E - G$ is effective and $\pi^*(P_\Lambda)$ is nef. ■

Theorem 31. *Let $\pi_0 : X_0 \rightarrow B_0$ be an elliptic fibration, $\dim X_0 = n$ and $\kappa(X_0) \geq 0$. Assume the existence of minimal models for klt pairs of non negative Kodaira dimension in dimension $n - 1$. There exist birationally equivalent fibrations and birational morphisms $\phi_{\tilde{B}}$ and ϕ_B*

$$\begin{array}{ccccc}
 X_0 & \longleftarrow & X & & \\
 \pi_0 \downarrow & & \downarrow \tilde{\pi} & \searrow \epsilon & \\
 & & \tilde{B} & & \\
 & \swarrow \pi & \downarrow \phi_B & \searrow \phi_{\tilde{B}} & \\
 B_0 & \longleftarrow & (B, \Lambda) & \dashrightarrow & (\tilde{B}, \bar{\Lambda})
 \end{array}$$

such that $K_X = \epsilon^*(K_{\tilde{B}} + \bar{\Lambda}) + \tilde{\pi}^*\Gamma + E - G$ is a Fujita-Zariski decomposition of $K_X = \pi^*(K_B + \Lambda) + E - G$

where

- $(\bar{B}, \bar{\Lambda})$ is a log minimal model of the klt pair (B, Λ)
- Γ is an $\phi_{\bar{B}}$ -exceptional effective \mathbb{Q} -divisor.
- $P = \epsilon^*(K_{\bar{B}} + \bar{\Lambda})$ is the nef part and $N = \tilde{\pi}^*\Gamma + E - G$ the effective part of the Fujita-Zariski decomposition.

Proof. As in Theorem 30 we have the birationally equivalent fibrations:

$$\begin{array}{ccccc} X_0 & \longleftarrow & X_1 & \longleftarrow & X \\ \pi_0 \downarrow & & \downarrow & & \downarrow \pi \\ B_0 & \longleftarrow & B_1 & \longleftarrow & B \end{array}$$

and $K_X = \pi^*(K_B + \Lambda) + E - G$, where (B, Λ) is a klt pair of dimension $n - 1$. By the hypotheses $0 \leq \kappa(X) = \kappa(B, K_B + \Lambda)$ (Proposition 18) and existence of minimal models for klt pairs of dimension $n - 1$, (B, Λ) has log minimal model $(\bar{B}, \bar{\Lambda})$. Let \tilde{B} be a common log resolution of (B, Λ) and $(\bar{B}, \bar{\Lambda})$ and \tilde{X} be a resolution of $X \times_B \tilde{B}$. As in Theorem 30 we can assume without loss of generalities $\tilde{X} = X$. We have the following commutative diagram:

$$\begin{array}{ccccc} X_0 & \longleftarrow & & & X \\ \pi_0 \downarrow & & \swarrow \pi & & \downarrow \tilde{\pi} \\ & & \tilde{B} & & \\ & \searrow \phi_B & & \searrow \phi_{\bar{B}} & \\ B_0 & \longleftarrow & (B, \Lambda) & \dashrightarrow & (\bar{B}, \bar{\Lambda}) \end{array}$$

By the Negativity Lemma, [26, Lemma 3.39], we have $\phi_B^*(K_B + \Lambda) = \phi_{\bar{B}}^*(K_{\bar{B}} + \bar{\Lambda}) + \Gamma$ with Γ effective and $\phi_{\bar{B}}$ -exceptional. From the arguments of [2, Thm. 1.5], $\phi_B^*(K_B + \Lambda) = \phi_{\bar{B}}^*(K_{\bar{B}} + \bar{\Lambda}) + \Gamma$ is a Fujita-Zariski decomposition of $\phi_B^*(K_B + \Lambda)$ with $\phi_{\bar{B}}^*(K_{\bar{B}} + \bar{\Lambda}) = P_{\Lambda}$ the nef part and $\Gamma = N_{\Lambda}$. Then $K_B + \Lambda$ birationally admits a Fujita-Zariski decomposition and so by the arguments of Theorem 30 we have that:

$$\begin{aligned} K_X &= \tilde{\pi}^*h^*(K_{\bar{B}} + \bar{\Lambda}) + \tilde{\pi}^*(\Gamma) + E - G \\ &= \epsilon^*(K_{\bar{B}} + \bar{\Lambda}) + \tilde{\pi}^*(\Gamma) + E - G. \end{aligned}$$

■

Corollary 32. *Under the assumption of the hypothesis and notation of Theorem 31, the canonical model of X is isomorphic to the log canonical model of (B, Λ) .*

Proof. A Fujita-Zariski decomposition is a CKM-Zariski decomposition (8). In Theorems 30 and 31, we showed that $P = \pi^*(P_{\Lambda})$. Then, up to a change in grading, the canonical rings of X and (B, Λ) are isomorphic and the canonical models are isomorphic. ■

4.2. The Zariski Decompositions and Minimal Models for Elliptic Fibrations.

We now use our results of Zariski decomposition and elliptic fibrations (Theorems 31 and 20) to give a different proof of part (2) in Theorem 26. Note that the statement is stronger. In particular, \bar{B} is \mathbb{Q} -factorial.

Theorem 33. *Let $\pi_0 : X_0 \rightarrow B_0$ be elliptic fibration, with $\dim(X) = n$ and $\kappa(X) \geq 0$.*

Assume one of the following:

- (1) *Log minimal models for klt pairs of non negative Kodaira dimension in dimension $n - 1$ exist.*
- (2) *Any sequence of flips for generalized klt pairs of dimension at most $n - 2$ terminates and $K_B + \Lambda$ admits a weak Zariski decomposition.*

There exists a birationally equivalent fibration $\tilde{\pi} : \tilde{X} \rightarrow \bar{B}$ such that

- \bar{B} is normal and \mathbb{Q} -factorial.

- There exists a effective divisor $\bar{\Lambda}$ on \bar{B} such that $(\bar{B}, \bar{\Lambda})$ is a klt pair.
- \bar{X} has at worst terminal singularities.
- $K_{\bar{X}} \equiv_{\mathbb{Q}} \tilde{\pi}^*(K_{\bar{B}} + \bar{\Lambda})$
- $K_{\bar{X}}$ is nef

Proof. Assumption (2) ensures the existence of a minimal model for (B, Λ) [13, Thm. 1]. Let $(\bar{B}, \bar{\Lambda})$ be a minimal model as in Theorem 31. We have the following diagram:

$$\begin{array}{ccccc}
 X_0 & \xleftarrow{\quad} & X & & \\
 \downarrow \pi_0 & & \downarrow \pi & \searrow \epsilon & \\
 & & \tilde{B} & & \\
 & \swarrow \phi_B & & \searrow \phi_B & \\
 B_0 & \xleftarrow{\quad} & (B, \Lambda) & \dashrightarrow & (\bar{B}, \bar{\Lambda})
 \end{array}$$

and the following Fujita-Zariski decomposition of the canonical divisor of K_X :

$$K_X = \epsilon^*(K_{\bar{B}} + \bar{\Lambda}) + \tilde{\pi}^*\Gamma + E - G, \epsilon^*(K_{\bar{B}} + \bar{\Lambda}) \text{ nef and } \tilde{\pi}^*\Gamma + E - G \text{ effective.}$$

We apply Theorem 20 to the relative MMP with respect to $\epsilon : X \rightarrow \bar{B}$:

$$\begin{array}{ccccc}
 X & \dashrightarrow & \bar{X} & & \\
 \downarrow \pi & & \downarrow \tilde{\pi} & \searrow \epsilon & \downarrow \tilde{\pi} \\
 & & \tilde{B} & & \\
 \swarrow \phi_B & & & \searrow \phi_B & \\
 (B, \Lambda) & \dashrightarrow & (\bar{B}, \bar{\Lambda}) & &
 \end{array}$$

To apply Theorem 20 we want to show that no component of the effective divisor $\tilde{\pi}^*\Gamma + E - G$ is a pullback of some \mathbb{Q} -divisor on \bar{B} .

It is sufficient to show that no component of $\tilde{\pi}^*\Gamma$ and E contains the pullback of a divisor on \bar{B} , since they contain all the components of $\tilde{\pi}^*\Gamma + E - G$. We have that Γ is contracted by ϕ_B thus $\tilde{\pi}^*\Gamma$ cannot contain the pullback of a divisor on \bar{B} . The components of E can map down to a space of codimension 1 or to a space of codimension ≥ 2 on \bar{B} .

We then need to show that when $\epsilon_*(D)$ has codimension one in \bar{B} , then D does not contain the fiber over the points in its image on \bar{B} .

Assuming that to be the case, $\tilde{\pi}_*(D)$ will be an effective divisor. Furthermore, $\tilde{\pi}_*(D)$ cannot be contracted by $\phi_B : \tilde{B} \rightarrow B$, because then it would mean D would map to a space of codimension ≥ 2 on B and since $(\bar{B}, \bar{\Lambda})$ is a log minimal model of (B, Λ) , $\tilde{\pi}_*(D)$ would also be contracted by $\phi_{\bar{B}}$. Since $\tilde{\pi}_*(D)$ is not contracted by ϕ , then D is exceptional, in the sense of Theorem 2; in particular D does not contain preimage of general points on its image in \tilde{B} and D is not a pullback of a divisor on \tilde{B} and a fortiori of \bar{B} also.

By Theorem 20, we will have $K_{\bar{X}} \equiv_{\mathbb{Q}} \tilde{\pi}^*(K_{\bar{B}} + \bar{\Lambda})$ and $K_{\bar{X}}$ is nef since it is numerically the pullback of a log canonical divisor of a log minimal model. \bar{X} has at worst terminal singularities since it is obtained from running a relative MMP on a smooth variety. \blacksquare

5. APPLICATIONS

5.1. Existence of Zariski decompositions and minimal models.

Corollary 34. *Assume the existence of minimal models for klt pairs in dimension $n - 1$ with non-negative Kodaira dimension. Given an elliptic n -fold, $\pi : X \rightarrow B$, then we have that K_X birationally admits a Fujita-Zariski decomposition.*

Corollary 35. *Let $\pi : Y \rightarrow T$ be an elliptic fibration with $\dim(Y) = n$ and $\kappa(Y) \geq 0$. If generalized klt flips terminate in dimension up to $n - 1$, then any minimal model program for Y terminates.*

Proof. Theorem 31 establishes a weak Zariski decomposition for K_X and the results follow from [13, Thm. 1]. \blacksquare

Since minimal model exist for *klt* pairs of non-negative Kodaira dimension of dimension up to 4 we have the following:

Corollary 36. *An elliptically fibered variety of dimension $n \leq 5$ with non-negative Kodaira dimension has a birationally equivalent fibration $\bar{\pi} : \bar{X} \rightarrow \bar{B}$ where \bar{X} is a minimal model and $K_{\bar{X}} \equiv_{\mathbb{Q}} \bar{\pi}^*(K_{\bar{B}} + \bar{\Lambda})$.*

Theorem 37. *Assume termination of flips for dlt pairs in dimension $n - 2$. Let $X \rightarrow B$ and (B, Λ) as in Lemma 16. X has a minimal model if and only if (B, Λ) has a log minimal model.*

Proof. K_X birationally admits a Fujita-Zariski decomposition if and only if $K_B + \Lambda$ birationally admits a Fujita-Zariski decomposition (Theorem 30). If (B, Λ) has a log minimal model then following the argument in the proof of Theorem 33 we can construct a minimal model of X .

If X has a minimal model, the arguments of [2, Thm. 1.5] show that K_X birationally admits a Fujita-Zariski decomposition. Then $K_B + \Lambda$ birationally admits a Fujita-Zariski decomposition (Theorem 30). Now since $\dim(B) = n - 1$, (B, Λ) has a log minimal model [2, Thm. 1.5]. \blacksquare

5.2. Abundance and Elliptic Fibrations.

In the previous section we proved the compatibility of the Fujita-Zariski decomposition with elliptic fibrations and minimal models. Now we turn our attention to good minimal models. Associated to a good minimal models we have the Nakayama-Zariski decomposition (Definition 7). We prove that the Fujita-Zariski decomposition when there is a good minimal model is also a Nakayama-Zariski decomposition.

Corollary 38. *Let $\pi_0 : X_0 \rightarrow B_0$ be an elliptic fibration, $\dim X_0 = n$ and $\kappa(X_0) \geq 0$. Assume the existence of good minimal models for *klt* pairs of non negative Kodaira dimension in dimension $n - 1$. Then the Fujita-Zariski decomposition in Theorem 31 is also a Nakayama-Zariski decomposition.*

Proof. Using the notation and set up as in Theorem 31, the Fujita-Zariski decomposition of K_X is given by:

$$K_X = \epsilon^*(K_{\bar{B}} + \bar{\Lambda}) + \tilde{\pi}^*(\Gamma) + E - G$$

By assumption K_X is pseudoeffective and so it also has a Nakayama-Zariski decomposition:

$$K_X = P_{\sigma}(K_X) + N_{\sigma}(K_X);$$

we will show that $\epsilon^*(K_{\bar{B}} + \bar{\Lambda}) = P_{\sigma}(K_X)$ and $\tilde{\pi}^*(\Gamma) + E - G = N_{\sigma}(K_X)$.

As $(\bar{B}, \bar{\Lambda})$ is a good minimal model, $K_{\bar{B}} + \bar{\Lambda}$ is semiample. From the arguments in Theorem 31, we have that $\tilde{\pi}^*(\Gamma) + E - G$ is ϵ -degenerate thus by [9, Lemma 2.16] we have:

$$\begin{aligned} \tilde{\pi}^*(\Gamma) + E - G &\leq N_{\sigma}(K_X) \\ P_{\sigma}(\epsilon^*(K_{\bar{B}} + \bar{\Lambda})) &= P_{\sigma}(K_X) \end{aligned}$$

From [9, Lemma 2.9], we have that for any pseudoeffective divisor D , we have $N_{\sigma}(D)$ is contained in $B_-(D)$ where:

$$B_-(D) = \bigcup_{\epsilon > 0} Bs(D + \epsilon A)$$

and $Bs(F)$ denotes the base locus of F and A is any ample divisor. The definition is independent of the choice of A . Now as we have that $\epsilon^*(K_{\bar{B}} + \bar{\Lambda})$ is semiample, we must have that $B_-(\epsilon^*(K_{\bar{B}} + \bar{\Lambda})) = \emptyset$, so that $N_{\sigma}(\epsilon^*(K_{\bar{B}} + \bar{\Lambda})) = 0$. This implies that:

$$P_{\sigma}(K_X) = \epsilon^*(K_{\bar{B}} + \bar{\Lambda})$$

and

$$N_{\sigma}(K_X) = \tilde{\pi}^*(\Gamma) + E - G. \quad \blacksquare$$

Corollary 39. *[Proposition 18, Theorem 26] Let $\pi : Y \rightarrow T$ be an elliptic fibration*

- (1) If $\dim(Y) = 4$ there exists a birational equivalent fibration $\bar{X} \rightarrow \bar{B}$, \bar{X} with \mathbb{Q} -factorial terminal singularities, $(\bar{B}, \bar{\Lambda})$ with klt singularities such that $K_{\bar{X}} \equiv \bar{\pi}^*(K_{\bar{B}} + \bar{\Lambda})$ and either Y is birationally a Mori fiber space or \bar{X} is a good minimal model.
- (2) If $\kappa(Y) = n - 1$, there exists a birationally equivalent fibration $\bar{\pi} : \bar{X} \rightarrow \bar{B}$ such that \bar{X} is a good minimal model, $K_{\bar{X}} \equiv_{\mathbb{Q}} \bar{\pi}^*(K_{\bar{B}} + \bar{\Lambda})$ and $(\bar{B}, \bar{\Lambda})$ has klt singularities.

Proof. It follows from Proposition 18, Theorem 15 and Theorem 26. See also [27, Thm. 4.4], [9, Cor. 4.5]. \blacksquare

6. THE DIMENSION OF THE FIBERS AND EQUIDIMENSIONALITY, UP TO BIRATIONAL EQUIVALENCE

While it is easy to fabricate examples of minimal elliptic threefolds which are not equidimensional starting from ones which are, many smooth Calabi-Yau threefolds have a natural elliptic fibration which is not equidimensional. These examples were mostly found during searches to provide evidence that very large classes of Calabi-Yau threefolds are birationally elliptically fibered [4, 17].

If $\dim(X) = 3$, there always exists a birational equivalent elliptic fibration, minimal or a Mori fiber space, which is equidimensional [11, Cor. 8.2]. By contrast there is an example of a non-equidimensional elliptic fourfold for which it is not known if an equidimensional model exists [5]. Examples of local Calabi-Yau fourfolds in generalized Weierstrass form with possibly non-equidimensional elliptic fibrations are also described in [28]. In the example in [5] a particular fiber contains a smooth surface and the fibration is otherwise equidimensional. Corollary 41 proves that this is what it can be generally expected.

Theorem 40 and Corollary 41 are stronger than what one could obtain from a log-minimal model run, in at least two aspects. First, the singularities in our models are terminal, while a minimal model run gives log terminal singularities, as in Example 3.1. See [12] for an analysis of terminal versus log-terminal singularity in this context. In addition we prove that not only there are no exceptional divisors in the fibers outside a codimension 3 set, but the fibration is equidimensional there.

Theorem 40. *Let $f : X \rightarrow B$ be an elliptic fibration such that X has at worst \mathbb{Q} -factorial terminal singularities and $K_X = f^*(L)$ where L is a \mathbb{Q} -Cartier divisor on B . Then there exists a birationally equivalent elliptic fibration $h : Y \rightarrow T$ and a \mathbb{Q} -divisor Λ_T such that:*

- (1) Y has at worst \mathbb{Q} -factorial terminal singularities.
- (2) $K_Y = h^*(K_T + \Lambda_T)$ where (T, Λ_T) is klt.
- (3) There is no effective divisor E in Y such that $\text{codim } h(E) \geq 2$.

Proof. Let $\rho(X/B)$ be the rank of the relative Neron-Severi group of $f : X \rightarrow B$; it has finite rank and we proceed by induction on this invariant. If E is an effective divisor on X such that $f(E)$ has codimension ≥ 2 , then we can take an effective Cartier divisor without fixed component, C , on B that contains $f(E)$ and we have that:

$$f^*(C) = D + F$$

where F is the maximal component of $f^*(C)$ such $\text{codim}(\pi(F)) \geq 2$ and $D = f^*(C) - F$. Then $\text{Supp}(E) \leq \text{Supp}(F)$ and $\text{codim } h(D) = 1$. $K_X + D$ is f -nef if and only if D is f -nef.

If D is not f -nef, the f -minimal program on the log pair $(X, \epsilon D)$ for $0 < \epsilon \ll 1$ produces a relatively minimal pair $(Y, \epsilon D')$ over B [14, Thm. 2.12]. Furthermore D is f -movable [21, Def. 1.1] so that running the relative log minimal model program on $(X, \epsilon D)$ over B results in a sequence of D -flips. As K_X is numerically trivial over B , we have that this sequence of D -flips is a sequence of flops. We have the diagram:

$$\begin{array}{ccc} (X, \epsilon D) & \xrightarrow{\phi} & (Y, \epsilon D') \\ f \downarrow & \swarrow g & \\ B & & \end{array}$$

where $K_Y = g^*(L)$, $D' = \phi_* D$ and $K_Y + \epsilon D'$ is g -nef. So if D was not f -nef we can obtain a birational model Y by a sequence of flops and we can reduce to the case where we have $g : Y \rightarrow B$ with a g -nef divisor D' .

If D' is g -nef, then D' is g -semiample [32, Thm. A.4], and there exists a morphism $h : Y \rightarrow T$ that factors as follows:

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ f \downarrow & \swarrow g & \downarrow h \\ B & \xleftarrow{\psi} & T \end{array}$$

Since $K_Y = g^*(L) = h^*(\psi^*(L))$, and by letting $L_Y = \psi^*(L)$ we have that $K_Y = h^*(L_Y)$, L_Y \mathbb{Q} -divisor on T . Furthermore, ψ is not an isomorphism since D' is numerically trivial over T but not over B . $\dim(T) = \dim(B)$ since $D' \leq g^*(C)$ and so is not f -ample. This implies that $\rho(Y/T) < \rho(X/B)$ and that $h : Y \rightarrow T$ also satisfies the hypothesis of the theorem. As the rank of the Neron-Severi group is finite, this process must eventually terminate. \blacksquare

Corollary 41. *Let $\pi_0 : X_0 \rightarrow B_0$ be an elliptic fibrations as in Theorem 26 or Theorem 33, then there exists a birationally equivalent elliptic fibration $h : Y \rightarrow T$, where Y is a relatively minimal model over T and h is equidimensional over an open set $U \subset T$ whose complement has codimension ≥ 3 . If furthermore we have that $\kappa(K_{X_0}) \geq 0$, we can take Y to be a minimal model.*

Proof. From Theorem 26 or 33, we obtain a birationally equivalent elliptic fibration $\bar{\pi} : \bar{X} \rightarrow \bar{B}$ that satisfies the hypothesis of Theorem 40, namely a birationally equivalent fibration $h : Y \rightarrow T$ such that there is no effective divisor E in Y such that $\text{codim } h(E) \geq 2$. If furthermore $\kappa(K_{X_0}) \geq 0$, we can take \bar{X} to be a minimal model and, since Y is obtained via a sequence of flops from \bar{X} , we have that Y is also a minimal model.

We will prove that the general fibers of h over subvarieties of codimension ≤ 2 are 1-dimensional. Let $S \subset T$ be an irreducible closed subvariety of T . If S has codimension 1 then we have that $h^{-1}(S)$ has codimension 1, and a general fiber over S is 1 dimensional. Let now S be of codimension 2. Then $h^{-1}(S)$ has codimension ≤ 2 . Since no divisors of Y maps down to space of codimension 2, we must have that $h^{-1}(S)$ has codimension 2. By counting the dimensions, we have that the general fibers over S is 1 dimensional. Thus general fibers of h over subvarieties of codimension ≤ 2 are 1-dimensional. Thus, we have that h is equidimensional over some open set $U \subset T$ whose complement has codimension ≥ 3 . \blacksquare

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