On the Semi-Global Stability of an EK-Like Filter

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***Abstract*— This paper proposes to apply the Kalman-like observer paradigm to general nonlinear systems by linearization along the estimated trajectory, similarly to an Extended Kalman Filter. The main difference is that the quadratic Riccati equation is replaced by a linear Lyapunov equation which can be solved and explicitly related to a determinability Gramian. This allows to show by Lyapunov analysis and without any ad-hoc assumption on the Riccati solution, that the resulting observer, called Extended Kalman-like Filter, can be made semi-globally convergent if the input is actively used to a) stabilize the (unknown) true trajectory, b) sufficiently excite the determinability of the linearized systems along the (known) estimated trajectory. A class of systems where this compromise can be reached is provided.**

***Index Terms*— nonlinear observers, Extended Kalman Filter, Kalman-like observers**

I. Introduction

In many practical applications, whether it be for control or surveillance purposes, it is often crucial to estimate in real-time the state of the plant. This *observer design* problem was solved for linear systems in the early 60s with the so-called Kalman-Bucy and Luenberger observers [16], [19] for time-varying and time-invariant systems respectively. Research has striven ever since to extend those methods to nonlinear systems but, unfortunately, no systematic practical design exists yet. Methods are usually limited to particular classes of systems that are transformable into a particular *normal* form, such as triangular forms [7], [9] or linear forms with output injection [1], [14] (among many others).

The appeal of the Kalman filter mainly lies in its robustness and simplicity, since it is made of a copy of the dynamics and a correction gain obtained from a dynamic Riccati equation, whose parameters can be linked to physical quantities like noise covariance. That is why a very popular method consists in applying the Kalman filter also to nonlinear dynamics by using the linearized model, leading to the so-called *Extended Kalman Filter* (EKF). Unfortunately, the linearization is carried out along the estimated trajectory, which introduces a loop in the analysis and only local convergence can be proved. More importantly, the stability analysis is performed under an ad-hoc lower/upper-boundedness assumption on the Riccati solution that depends on the estimate itself and thus cannot be verified [3], [5], [24], [25], [28]. Very few exceptions exist in the particular context of *uniformly observable* systems in triangular form [10], [17].

In parallel to the Kalman filters, the so-called *Kalman-like* observers, first for state-affine systems [4], [11], and later also for triangular systems using high-gain [2], were also developed. The main difference is that the quadratic Riccati equation is replaced by a linear Lyapunov equation which can be solved and explicitly related to the so-called *observability/determinability Gramian*. However, as far as we know, this Kalman-like design has not been used in an *extended* fashion for general nonlinear systems.

In this paper, we thus introduce the *Extended Kalman-like Filter*. We show that its basin of attraction is characterised by the bounds on the Lyapunov matrix which, unlike in the EKF, can be explicitly expressed in terms of a certain determinability Gramian. In particular, the basin of attraction can be enlarged provided the plant's trajectories remain bounded and the determinability of the linearized dynamics along the known estimate is sufficiently excited. If the input can be chosen to satisfy those two conditions, then the observer becomes semi-global. As far as we know, it is the first result proving the possibility of non-local convergence of an EK-likeF, or even an EKF, without any ad-hoc assumption on the Riccati solution.

The idea of optimising the input to maximise observability appears mostly in the literature of *sensor positioning* [30] or *active sensing* [8], [12], [13]. Similarly, [23] proposed a closed-loop optimal experiment design for online parameter identification in nonlinear systems where the input is chosen to maximize the information content while ensuring asymptotic stability. Then, [26] proposed an optimal input selection method to improve the performance of the discrete Kalman filter for LTV systems. More recently, [27] proposed a gradient optimization approach to maximize observability of nonlinear systems, with the hope of improving the performance of the EKF. Of course, optimizing observability means first quantifying observability. For that, some indicators have been developed, generally based on the observability Gramian or an empirical version of it [18], [21], [26], [29]. Actually, as noticed in [27], observer design requires to reconstruct the current state and not its initial condition and it is thus more natural to consider a *constructibility*, or *determinability* Gramian. It is indeed that Gramian that naturally appears in the bounds of the Kalman-like Lyapunov matrix and it is thus the one we propose to optimize.

Notation : We denote the set of natural and real numbers with $\mathbb{N}$ and $\mathbb{R}$ respectively. The interior of a set $\mathcal{X}$ is denoted $\mathrm{int}(\mathcal{X})$. I denotes the identity matrix of appropriate dimension, and given a matrix $S \in \mathbb{R}^{m\times n}$, this paper denotes with $\underline{\sigma}(S)$ and $\overline{\sigma}(S)$ the smallest and largest singular values of S.

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II. Extended Kalman vs Kalman-like filtering

Consider a plant

$$\dot{x} = f(x,u) \quad , \quad y = h(x,u) \tag{1}$$

with state $x \in \mathbb{R}^{d_x}$, input $u \in \mathbb{R}^{d_u}$, output $y \in \mathbb{R}^{d_y}$, and maps $f,\, h \in \mathcal{C}^1$. As mentioned in the introduction, a popular observer for (1) is the Extended Kalman Filter (EKF)

$$\dot{\hat{x}} = f(\hat{x},u) + P\frac{\partial h}{\partial x}(\hat{x},u)^\top R^{-1}(y - h(\hat{x},u)) \tag{2a}$$

$$\begin{aligned}\dot{P} = \frac{\partial f}{\partial x}(\hat{x},u)P + P\frac{\partial f}{\partial x}(\hat{x},u)^\top + Q \\ - P\frac{\partial h}{\partial x}(\hat{x},u)^\top R^{-1}\frac{\partial h}{\partial x}(\hat{x},u)P\end{aligned} \tag{2b}$$

with $R = R^\top > 0$ and $Q = Q^\top > 0$. It is obtained by copying the dynamics (1) and using a gain $P(\partial h/\partial x)^\top R^{-1}$, that mimics the Kalman-Bucy filter gain for linear systems ([15], [16]), where the linear dynamics matrices are simply replaced by the linearized pair $(\partial f/\partial x, \partial h/\partial x)$ around the only available estimate $(\hat{x},u)$. Unfortunately, apart from specific triangular structures [10], [17], only local convergence of the estimation error $\hat{x} - x$ can be ensured, under the additional ad-hoc assumption that there exist $\underline{p}, \overline{p} > 0$ such that (see [3], [5], [24], [25], [28])

$$\underline{p}I \leq P(t) \leq \overline{p}I \qquad \forall t \geq 0\ . \tag{3}$$

But the trajectory of P depends on that of $\hat{x}$ itself, so that assuming (3) somehow introduces a loop in the stability analysis based on the Lyapunov function

$$V(x,\hat{x},t) = (x-\hat{x})^\top P(t)^{-1}(x-\hat{x})\ .$$

Note that in [3], a global result is derived in the case where the output is linear, but still under an ad-hoc assumption of the type (3).

A slight variation of (2b) was proposed in [25]

$$\begin{aligned}\dot{P} = \lambda P + \frac{\partial f}{\partial x}(\hat{x},u)P + P\frac{\partial f}{\partial x}(\hat{x},u)^\top + Q \\ - P\frac{\partial h}{\partial x}(\hat{x},u)^\top R^{-1}\frac{\partial h}{\partial x}(\hat{x},u)P\end{aligned} \tag{4}$$

where the additional linear term λP provides exponential stability. Indeed, $S(t) := P(t)^{-1}$ follows the dynamics

$$\begin{aligned}\dot{S} = -\lambda S - \frac{\partial f}{\partial x}(\hat{x},u)^\top S - S\frac{\partial f}{\partial x}(\hat{x},u) - SQS \\ + \frac{\partial h}{\partial x}(\hat{x},u)^\top R^{-1}\frac{\partial h}{\partial x}(\hat{x},u)\end{aligned} \tag{5}$$

and $\lambda > 0$ somehow dictates the exponential convergence rate. However, the assumption (3) is still made and the loop remains.

On the other hand, parallel to the Kalman school following [15], [16], Kalman-like observers were introduced, originally for linear time-varying systems as an optimal solution to a deterministic optimisation problem [4], [11] and later extended to some categories of nonlinear triangular systems [2]. This optimization problem does not take into account errors in the model so that the matrix Q is taken equal to 0, while the linear term λP is kept for stability purposes and models a "forgetting factor" on the past measurements (see [4]). This thus makes the dynamics (5) linear in S and explicitly solvable, unlike the Riccati-based equations (2b), (4), or (5) which are all quadratic. As will be detailed in Section III, the expression of S is then directly related to a *constructibility Gramian* and bounds of the form

$$\underline{s}I \leq S(t) \leq \overline{s}I \qquad \forall t \geq 0 \tag{6}$$

are explicitly obtained through persistence of excitation conditions on the input. Indeed, in [2], [4], [11], the matrices involved in the dynamics of S are independent from the estimate trajectory $\hat{x}$ and the invertibility of S is thus only related to the excitation capability of u.

In this paper, we propose to apply this design to general nonlinear systems in an EKF-like fashion, namely by considering the following "Extended Kalman-like observer"

$$\dot{\hat{x}} = f(\hat{x},u) + S^{-1}\frac{\partial h}{\partial x}(\hat{x},u)^\top R^{-1}(y - h(\hat{x},u)), \tag{7a}$$

$$\begin{aligned}\dot{S} = -\lambda S - \frac{\partial f}{\partial x}(\hat{x},u)^\top S - S\frac{\partial f}{\partial x}(\hat{x},u) \\ + \frac{\partial h}{\partial x}(\hat{x},u)^\top R^{-1}\frac{\partial h}{\partial x}(\hat{x},u)\end{aligned} \tag{7b}$$

with $R = R^\top$ a positive matrix and λ a positive scalar. We start by deriving in Section III explicit bounds for the solutions S to (7b). In particular, we show that S remains positive definite if a certain determinability Gramian of the linearized dynamics is sufficiently excited. Then, in Section IV we study the basin of attraction of the observer (7) and show semi-global convergence if the input can be chosen to stabilize the plant while sufficiently exciting the determinability Gramian along the estimated trajectory. Finally, we illustrate this result by an example in Section V where this dual goal can be achieved.

Remark 1: The dynamics (7) are equivalent to

$$\dot{\hat{x}} = f(\hat{x},u) + P\frac{\partial h}{\partial x}(\hat{x},u)^\top R^{-1}(y - h(\hat{x},u)), \tag{8a}$$

$$\begin{aligned}\dot{P} = \lambda P + \frac{\partial f}{\partial x}(\hat{x},u)P + P\frac{\partial f}{\partial x}(\hat{x},u)^\top \\ - P\frac{\partial h}{\partial x}(\hat{x},u)^\top R^{-1}\frac{\partial h}{\partial x}(\hat{x},u)P\end{aligned} \tag{8b}$$

which avoid the computation of S^{-1} online, similarly to the standard EKF (2) and the modified EKF (4). □

III. Determinability Gramian

A. Determinability vs observability

Let us define the auxiliary linear dynamics

$$\dot{\chi} = \frac{\partial f}{\partial x}(\hat{x},u)\chi \qquad y_\chi = \frac{\partial h}{\partial x}(\hat{x},u)\chi \tag{9}$$

with input $(\hat{x},u)$. The transition matrix $\Psi_{\hat{x},u} : \mathbb{R}^2 \to \mathbb{R}^{d_x \times d_x}$ of system (9) is defined as the unique matrix verifying

$$\frac{\partial \Psi_{\hat{x},u}}{\partial \tau}(\tau,s) = \frac{\partial f}{\partial x}(\hat{x}(\tau),u(\tau))\Psi_{\hat{x},u}(\tau,s) \tag{10}$$

for all $(\tau,s) \in \mathbb{R}^2$ and with $\Psi_{\hat{x},u}(s,s) = I$. The associated determinability Gramian $\mathcal{G}_{\hat{x},u} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{d_x \times d_x}$ relative to

a positive matrix R [27], is then defined by

$$\mathcal{G}_{\hat{x},u}(t_1,t_2) = \int_{t_1}^{t_2} \Psi_{\hat{x},u}(\tau,t_2)^\top \frac{\partial h}{\partial x}(\hat{x}(\tau),u(\tau))^\top R^{-1} \frac{\partial h}{\partial x}(\hat{x}(\tau),u(\tau))\Psi_{\hat{x},u}(\tau,t_2)d\tau \tag{11}$$

for all $(t_1,t_2) \in \mathbb{R}^2$. The transition matrix is used to express the solutions to system (9) since for any $(\tau,s) \in \mathbb{R}^2$, we have

$$\chi(\tau) = \Psi_{\hat{x},u}(\tau,s)\chi(s). \tag{12}$$

It follows from (9) that $y_\chi(\tau) = \frac{\partial h}{\partial x}(\hat{x},u)\Psi_{\hat{x},u}(\tau,s)\chi(s)$ and therefore,

$$\chi(t)^\top \mathcal{G}_{\hat{x},u}(t_0,t)\chi(t) = \int_{t_0}^{t} y_\chi^\top(\tau)R^{-1}y_\chi(\tau)\,d\tau$$

for all $t_0, t \in \mathbb{R}^2$. It follows that the determinability or constructibility of the linear system (9), namely the property that for t sufficiently large

$$y_\chi(\tau) = 0 \quad \forall \tau \in [t_0,t] \quad \Longrightarrow \quad \chi(t) = 0$$

is characterized by the invertibility of $\mathcal{G}_{\hat{x},u}(t_0,t)$. It is thus related to the ability to reconstruct the value of the solution $\chi(t)$ at a given time t, based on the knowledge of the past output on $[t_0,t]$.

Instead, we could also have considered the more classical observability Gramian [6] whose invertibility rather conditions the ability to reconstruct the initial condition $\chi(t_0)$, based on the knowledge of the output on $[t_0,t]$, namely the observability of (9). Since the solutions to (9) verify

$$\exists \tau \,:\, \chi(\tau) = 0 \quad \Longleftrightarrow \quad \chi(t) = 0 \quad \forall t\ ,$$

both properties are qualitatively equivalent. But, as soon as a quantification of this observability is needed for observer design, as below, the natural quantity to handle is the determinability Gramian, as also explained in [27]. Indeed, we are interested in the ability to reconstruct the current state of the plant, based on its past output, not its initial condition.

B. Persistent determinability

Unlike the quadratic Riccati equations of the Kalman literature, the dynamics (7b) in the Kalman-like observers are linear and therefore explicitly solvable. This allows to explicitly relates its expression to a determinability Gramian and derive explicit bounds of the form (3).

Lemma 1: Consider $\widehat{\mathcal{X}} \subset \mathbb{R}^{d_x}$ and $\widehat{\mathcal{U}} \subset \mathbb{R}^{d_u}$ and assume

$$c_f := \sup_{(\hat{x},u)\in\widehat{\mathcal{X}}\times\mathcal{U}} \left|\frac{\partial f}{\partial x}(\hat{x},u)\right| < +\infty \tag{13a}$$

$$c_h := \sup_{(\hat{x},u)\in\widehat{\mathcal{X}}\times\mathcal{U}} \left|\frac{\partial h}{\partial x}(\hat{x},u)\right| < +\infty\ . \tag{13b}$$

Fix $R = R^\top > 0$. For any input $(\hat{x},u) \in \widehat{\mathcal{X}}\times\mathcal{U}$, any solution to (7b) initialized at $S(0) > 0$ is positive definite for all $t \geq 0$ and

- if $\lambda > 2c_f$, then

$$S(t) \leq \overline{s}_\lambda I \quad \forall t \geq 0$$
$$\text{with } \overline{s}_\lambda := \max\left\{\overline{\sigma}(S(0)), \frac{c_h^2}{\lambda - 2c_f}\frac{1}{\underline{\sigma}(R)}\right\}. \tag{14a}$$

- if there exist positive scalars $t_0, \bar{t}, \alpha$ such that

$$\mathcal{G}_{\hat{x},u}(t-\bar{t},t) \geq \alpha I \qquad \forall t \geq t_0 \geq \bar{t}\ , \tag{14b}$$

then

$$\underline{s}_{u,\lambda} I \leq S(t) \quad \forall\, t \geq t_0 \ \text{ with } \underline{s}_{u,\lambda} := \alpha e^{-\lambda\bar{t}}\ . \tag{14c}$$

Proof: The dynamics (7b) are linear in S so its solutions are explicitly given by

$$S(t) = e^{-\lambda t}\Psi_{\hat{x},u}(0,t)^\top S(0)\Psi_{\hat{x},u}(0,t) + \int_0^t e^{-\lambda(t-\tau)} \Psi_{\hat{x},u}(\tau,t)^\top \frac{\partial h}{\partial x}(\hat{x},u)^\top R^{-1}\frac{\partial h}{\partial x}(\hat{x},u)\Psi_{\hat{x},u}(\tau,t)d\tau.$$

Therefore, since $S(0) > 0$, S is symmetric and positive. Besides, $\Psi_{\hat{x},u}$ verifies

$$\Psi_{\hat{x},u}(\tau,t) = I + \int_t^\tau \frac{\partial f}{\partial x}(\hat{x}(s),u(s))\Psi_{\hat{x},u}(s,t)ds$$

so that for all $\tau \leq t$, $|\Psi_{\hat{x},u}(\tau,t)| \leq 1 + \int_\tau^t c_f\,|\Psi_{\hat{x},u}(s,t)|\,ds$ and by Gronwall's lemma, $|\Psi_{\hat{x},u}(\tau,t)| \leq e^{c_f(t-\tau)}$. It follows from the expression of S that for $\lambda > 2c_f$,

$$S(t) \leq e^{-(\lambda-2c_f)t}\overline{\sigma}(S(0))I + \int_0^t e^{-(\lambda-2c_f)(t-\tau)}\left|\frac{\partial h}{\partial x}(\hat{x},u)^\top R^{-1}\frac{\partial h}{\partial x}(\hat{x},u)\right| d\tau \leq \overline{s}_\lambda I\ .$$

On the other hand, if (14b) holds, then for all $t \geq t_0$, $S(t) \geq e^{-\lambda\bar{t}}\mathcal{G}_{\hat{x},u}(t-\bar{t},t) \geq \alpha e^{-\lambda\bar{t}}$, which concludes the proof. ■

From (14a), we deduce that S is bounded as long as $(\hat{x},u)$ remains in $\widehat{\mathcal{X}}\times\mathcal{U}$ where (13) holds. Besides, we conclude from (14c) that S is (uniformly) positive definite if $(\hat{x},u)$ ensures persistent determinability of the linearized system (9) as described by (14b). We show in the next section how those properties enable to prove semi-global convergence of the observer (7) under sufficient excitation.

IV. Basin of attraction of the EK-likeF

Consider a compact set $\mathcal{X} \subset \mathbb{R}^{d_x}$, a positive definite matrix S_0, a scalar $\mu > 1$ and define

$$\begin{aligned} v_0 &:= \max_{(x,\hat{x})\in\mathcal{X}\times\mathcal{X}} (x-\hat{x})^\top S_0(x-\hat{x}) \\ \widehat{\mathcal{X}} &:= \{\hat{x} \in \mathbb{R}^{d_x} : \exists\, x \in \mathcal{X} : (x-\hat{x})^\top S_0(x-\hat{x}) \leq \mu^2 v_0\}, \end{aligned} \tag{15}$$

along with the maps

$$\begin{aligned} \varphi_f(x,\hat{x},u) &:= f(x,u) - f(\hat{x},u) - \frac{\partial f}{\partial x}(\hat{x},u)(x-\hat{x}) \\ \varphi_h(x,\hat{x},u) &:= h(x,u) - h(\hat{x},u) - \frac{\partial h}{\partial x}(\hat{x},u)(x-\hat{x}). \end{aligned} \tag{16}$$

Theorem 1: Assume (13) holds for $\widehat{\mathcal{X}}$ defined above and assume there exist two scalars κ_f, $\kappa_h > 0$ such that

$$|\varphi_f(x,\hat{x},u)| \le \kappa_f |x-\hat{x}|^2 \qquad \forall (x,\hat{x},u) \in \mathcal{X}\times\widehat{\mathcal{X}}\times\mathcal{U}$$
$$|\varphi_h(x,\hat{x},u)| \le \kappa_h |x-\hat{x}|^2 \qquad \forall (x,\hat{x},u) \in \mathcal{X}\times\widehat{\mathcal{X}}\times\mathcal{U}$$

Choose $R = R^\top > 0$. There exist $\lambda > 2c_f$ and scalars $\alpha, \bar{t}, t_0, c, \lambda_m > 0$ such that for any $x : [0,+\infty) \to \mathbb{R}^{d_x}$ maximal solution to (1) initialised at $x_0 \in \mathcal{X}_0$ and any $(\hat{x}, S) : [0, t_\infty) \to \mathbb{R}^{d_x} \times \mathbb{R}^{d_x \times d_x}$ maximal solution to (7) initialised at $(\hat{x}_0, S_0)$ with $\hat{x}_0 \in \mathrm{int}(\mathcal{X})$ and input $u \in \mathcal{U}$ such that

- $x(t) \in \mathcal{X}$ for all $t \in [0,+\infty)$,
- the persistence of excitation condition (14b) holds with $\alpha, \bar{t}, t_0$ and with inputs $(\hat{x}, u)$ on $[0, t_\infty)$,

then, $t_\infty = +\infty$, and

$$\hat{x}(t) \in \widehat{\mathcal{X}} \quad \forall t \in [0,+\infty) \ ,$$

$$|\hat{x}(t) - x(t)| \le c\, e^{-\frac{\lambda_m}{2}(t-t_0)} \quad \forall t \in [t_0,+\infty) \, .$$

Proof: Since $\hat{x}_0 \in \mathrm{int}(\mathcal{X})$ and $S(0) = S_0$, by exploiting the continuity of the functions, there exists $t_0 > 0$ such that $\forall\, t \le t_0$,

$$\hat{x}(t) \in \mathcal{X} \subset \mathrm{int}(\widehat{\mathcal{X}}) \ \text{ and } \ S(t) \le \mu S_0.$$

Therefore, defining

$$V(x,\hat{x},t) = (x-\hat{x})^\top S(t)(x-\hat{x}) \ ,$$

we have $V(x(t_0), \hat{x}(t_0), t_0) \le \mu v_0$. Now define $t_1 > t_0$ the largest positive scalar such that $\hat{x}(t)$ is defined in $\mathrm{int}(\widehat{\mathcal{X}})$ for all $t \in [0, t_1)$. According to Lemma 1, we have

$$\underline{s}_{u,\lambda}|x-\hat{x}|^2 \le V(x,\hat{x},t) \le \overline{s}_\lambda |x-\hat{x}|^2 \quad \forall t \in [t_0, t_1) \quad (17)$$

with $\underline{s}_{u,\lambda}$ and $\overline{s}_\lambda$ defined in (14). Therefore, exploiting (16),

$$\dot{x} - \dot{\hat{x}} = \left[\frac{\partial f}{\partial x}(\hat{x},u) - S^{-1}\frac{\partial h}{\partial x}(\hat{x},u)^\top R^{-1}\frac{\partial h}{\partial x}(\hat{x},u)\right](x-\hat{x}) + \varphi_f(x,\hat{x},u) - S^{-1}\frac{\partial h}{\partial x}(\hat{x},u)^\top R^{-1}\varphi_h(x,\hat{x},u),$$

we have for all $t \in [t_0, t_1)$

$$\begin{aligned}\dot{V}(\hat{x},x,t) = {} & \varphi_f^\top S(x-\hat{x}) + (x-\hat{x})^\top S\,\varphi_f \\ & - \varphi_h^\top R^{-1}\frac{\partial h}{\partial x}(x-\hat{x}) - (x-\hat{x})^\top \frac{\partial h}{\partial x}^\top R^{-1}\,\varphi_h \\ & - \lambda V - (x-\hat{x})^\top \frac{\partial h}{\partial x}^\top R^{-1}\frac{\partial h}{\partial x}(x-\hat{x})\end{aligned}$$

so that $\dot{V}$ is bounded from above by

$$\dot{V}(\hat{x},x,t) \le -\lambda_m\, V(\hat{x},x,t)$$

with

$$\lambda_m := \lambda - 2\left(\frac{\overline{s}_\lambda\,\kappa_f}{\underline{s}_{u,\lambda}} + \frac{c_h\,\kappa_h}{\underline{\sigma}(R)\underline{s}_{u,\lambda}}\right) d_m \qquad (18)$$

where $d_m := \max_{(x,\hat{x})\in\mathcal{X}\times\widehat{\mathcal{X}}} |x-\hat{x}|$ is well-defined because $\mathcal{X}\times\widehat{\mathcal{X}}$ is compact. If we ensure sufficient excitation $\frac{1}{\mu}\overline{s}_\lambda \le \underline{s}_{u,\lambda} (\le \overline{s}_\lambda)$, then, using the fact that $\overline{s}_\lambda \ge \overline{\sigma}(S_0)$,

$$\lambda_m \ge \lambda - 2\mu\left(\kappa_f + \frac{c_h\kappa_h}{\underline{\sigma}(R)\overline{\sigma}(S_0)}\right) d_m \ .$$

So, pick λ such that

$$\lambda > 2\mu\left(\kappa_f + \frac{c_h\kappa_h}{\underline{\sigma}(R)\overline{\sigma}(S_0)}\right) d_m \ , \qquad (19a)$$

and $(\alpha, \bar{t})$ such that

$$\overline{s}_\lambda \ge \alpha e^{-\lambda\bar{t}} \ge \frac{1}{\mu}\overline{s}_\lambda \qquad (19b)$$

which is feasible since $\mu > 1$. Then, $\lambda_m > 0$ and V decreases for all $t \in [t_0, t_1)$, namely

$$V(\hat{x}(t), x(t), t) \le \mu v_0 \qquad \forall t \in [t_0, t_1) \ .$$

This implies from (17) and from $\underline{s}_{u,\lambda} > \frac{\overline{s}_\lambda}{\mu} > \frac{\overline{\sigma}(S_0)}{\mu}$ that $\forall t \in [t_0, t_1)$,

$$(x(t)-\hat{x}(t))^\top S_0 (x(t)-\hat{x}(t)) \le \overline{\sigma}(S_0)|x(t)-\hat{x}(t)|^2 \le \frac{\overline{\sigma}(S_0)}{\underline{s}_{\lambda,u}}\mu v_0 < \mu^2 v_0.$$

We conclude that $\hat{x}(t)$ does not approach the boundary of $\widehat{\mathcal{X}}$ and therefore necessarily $t_1 = +\infty$. It finally follows that for all $t \ge t_0$,

$$|\hat{x}(t) - x(t)| \le \sqrt{\frac{V(\hat{x}(t),x(t),t)}{\underline{s}_{\lambda,u}}} \le \sqrt{\frac{\mu v_0}{\underline{s}_{\lambda,u}}}\, e^{-\frac{\lambda_m}{2}(t-t_0)}. \ \square$$

The bounds (13) and the existence of κ_f, $\kappa_h > 0$ are guaranteed if $\mathcal{U}$ is compact and f, h are $\mathcal{C}^2$. Then, the result says that, given a bounded set of initial conditions, if the input u can be chosen in $\mathcal{U}$ so that both

- the determinability of the linearized dynamics (9) around the (known) estimate $\hat{x}$ is sufficiently excited after t_0,
- the true trajectory x remains in the compact set $\mathcal{X}$,

then the observer stays in $\widehat{\mathcal{X}}$ and exponentially converges after t_0 with a decay rate given by (18). This is indeed a semi-global result. Of course, designing u to ensure both goals simultaneously may seem difficult, especially since x is unknown, but a class of systems where such a compromise appears feasible is presented in Section V.

Concerning the choice of the design parameters, condition (19a) says that λ should be chosen sufficiently large to compensate for the maximal estimation error d_m in $\mathcal{X}\times\widehat{\mathcal{X}}$ and the nonlinearities of f and h. The parameter $\mu > 1$ is only an analysis parameter describing how close the excitation $\underline{s}_{u,\lambda} := \alpha e^{-\lambda\bar{t}}$ is to its upper-bound $\overline{s}_\lambda$. In terms of design, (19b) says that this excitation should be maximized (see also (18)). Note that $\alpha e^{-\lambda\bar{t}}$ decreases with λ, so that the excitation time scale $\bar{t}$ must be adapted to λ. Also, increasing R or S_0 may seem to lighten the constraint (19a) on λ, but it may simultaneously decrease α from the Gramian definition (11) or increase $c_f, c_h, \kappa_f, \kappa_h$ through $\widehat{\mathcal{X}}$. So R and S_0 are fixed beforehand. Then, $\lambda > 2c_f$ is fixed, if possible satisfying (19a), but the latter constraint may be overestimated due to the conservativeness of the proof. The input u is finally used to maximise $\underline{s}_{u,\lambda}$ by increasing α and decreasing $\bar{t}$ (and t_0), i.e. by sufficiently exciting the determinability of (9) relative to R and sufficiently fast in the time scale imposed by λ. Observe also that the larger $\underline{s}_{u,\lambda}$, the closer the convergence rate λ_m to the maximal rate λ.

It is also worth highlighting that theoretically speaking, the computation of the Gramian involved in the excitation condition (14b) is possible online since it depends only on the known signals $(\hat{x}(\tau), u(\tau))$ over the past interval $\tau \in [t-\bar{t},\, t]$. Optimizing this Gramian along the estimated trajectory is not by default, because the true trajectory is unavailable, but indeed because this Gramian is the right one to consider to ensure bounds on S: only the determinability along the estimate matters, and not the true trajectory. Of course, both join asymptotically.

V. Example

A class of systems for which the compromise between stabilisation of x and excitation of $\hat{x}$ is conceivable is

$$\dot{x} = f(x, u_s) \quad , \quad y = h(x, u_o) \tag{20}$$

where the state x (stabilized by the input u_s) is known to remain in a compact set $\mathcal{X}$ and u_o is left to be chosen for observation purposes. An example is when u_o represents the position of a fully actuated drone/vehicle chasing an independent stable system x which transmits some information $h(x, u_o)$ about its position with respect to u_o. For instance, the distance $\|x - u_o\|$ or some measured magnetic field as in the transmitter/receiver devices used in avalanche search and rescues scenarios [20]. Let us consider a target whose position x should be estimated, and which moves according to a 2D Van der Pol oscillator

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & 4(1-x_1^2)x_2 - x_1 \end{array} \quad , \quad y = \|x - u_o\|^2$$

where u_o is the seeking drone position to be chosen.

Assume that the target is known to evolve in $\mathcal{X} = \{x \in \mathbb{R}^2 \,:\, \|x\| \leq 10\}$. The selection of $S_0 = I$ and a margin $\mu > 1$ leads us to $\widehat{\mathcal{X}} = \{x \in \mathbb{R}^2 \,:\, \|x\| \leq 31\}$ according to (15). With these data at hand, $c_f \approx 4 \cdot 10^3$ and λ is designed as $\lambda = 2.1 c_f$. Theorem 1 claims that the estimate $\hat{x}$ given by (7) can be made to stay in $\widehat{\mathcal{X}}$ and converge to x from any initial condition in $\mathrm{int}(\mathcal{X})$, if u_o is sufficiently exciting for the linear system

$$\dot{\chi} = \begin{pmatrix} 0 & 1 \\ -8\hat{x}_1\hat{x}_2 - 1 & 4(1-\hat{x}_1^2) \end{pmatrix} \chi,\; y_\chi = 2(\hat{x} - u_o)^\top \chi \tag{21}$$

We propose to choose $u_o = \hat{x} - \delta$ with $\delta(t) \in \mathbb{R}^2$ to be designed to guarantee persistent determinability in (21). This choice leads to the observability matrix

$$\mathcal{O} = 2 \begin{pmatrix} \delta_1 & \delta_2 \\ \dot{\delta}_1 + \delta_2(-8\hat{x}_1\hat{x}_2 - 1) & \dot{\delta}_2 + \delta_1 + 4\delta_2(1-\hat{x}_1^2) \end{pmatrix}$$

By choosing $\delta(t) = r(\cos(wt), \sin(wt))$, we notice that $\mathcal{O}^\top \mathcal{O} = r^2 M(w, \hat{x})$ with M invertible for $\hat{x} \in \widehat{\mathcal{X}}$ and w sufficiently large, thus ensuring at least persistent observability/determinability [6, Theorems 6.O11, 6.O12]. Now, in order to make this excitation *sufficient*, we may intuitively say that increasing $\mathcal{O}^\top \mathcal{O}$, namely r, improves observability, but w also needs to increase to maintain sufficiently fast excitation "far from $\hat{x}$". Figure 1 shows the estimation trajectories from 200 initial conditions in $\mathrm{int}(\mathcal{X})$ for $(r, w) = (5, 10)$ (left) and $(r, w) = (15, 30)$ (right) respectively. The converging trajectories are plotted in blue while the unstable ones appear in red. We see that, despite the selection of the control law guaranteeing the invertibility of the matrix M, the level of excitation $\underline{s}_\lambda$ associated to $(r, w) = (5, 10)$ is not sufficient to ensure stability from all the initial conditions belonging to $\mathcal{X}$. However, taking larger (r, w) enables to increase the level of excitation and enlarge the basin of attraction to the whole compact set $\mathcal{X}$. The corresponding estimation errors are plotted in Figure 2. Note that a failure of sufficient excitation could also have meant that λ was not large enough.

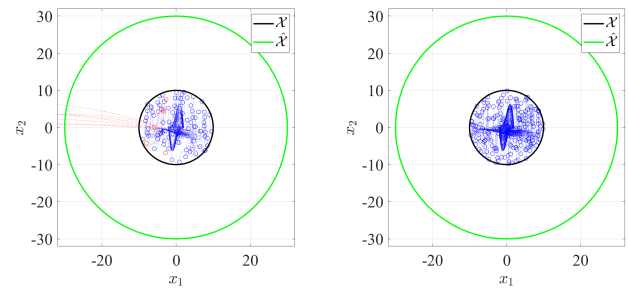

Fig. 1. Converging (blue) and non converging (red) estimate trajectories for initial conditions randomly distributed in $\boldsymbol{\mathcal{X}}$ with slowly exciting control law (left) and faster exciting control law (right).

Figure 3 depicts what can happen when the control u is not sufficiently exciting. By reducing w and slowing down the seeker to a point where u stops being exciting for the steady state Van der Pol trajectories, the estimation error starts by converging and then diverges. In fact, during the transient the eigenvalues of S remain sufficiently away from 0. But once $\hat{x}$ has converged to the Van der Pol manifold, the excitation no longer is sufficient, one eigenvalue of S goes to zero and the estimator blows up. It is indeed the excitation along $\hat{x}$ that matters, but, in order to stabilize the zero-estimation error, determinability along x becomes necessary in steady state.

Now, when we consider a larger set $\mathcal{X}$, the parameters c_f and λ increase, so does $\widehat{\mathcal{X}}$. Therefore, w has to be increased accordingly, to speed up the excitation and guarantee observability, as instructed by the theory. Note that this semi-globality is rendered possible by the fact that u_o actively adapts to $\hat{x}$ instead of being based on x, and guarantees the sufficient excitation of the determinability of the linearized system (21) even for large initial conditions $\hat{x}(0)$. But more work is necessary to provide effective optimization of the determinability Gramian. In the general context, although the Gramian depends only on known signals $(\hat{x}, u_o)$ and can be computed online, choosing u_o via an MPC-like optimization method would require to know the future of $\hat{x}$. Unfortunately, the latter is unknown since it depends on the future of the output y. Note that a gradient-based optimization algorithm avoiding this difficulty was recently presented in [27] and could be used in this context.

VI. Conclusion

We have shown that an EK-likeF can be made semi-globally convergent if the input is used to both stabilize the system and sufficiently excite the determinability of the linearized system

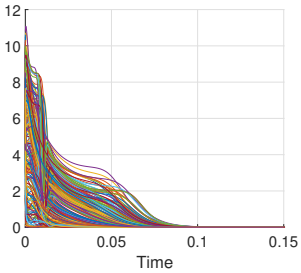

Fig. 2. Norm of the estimation error with sufficiently exciting control law from random initial conditions in $\mathcal{X}$, as in Figure 1 (right).

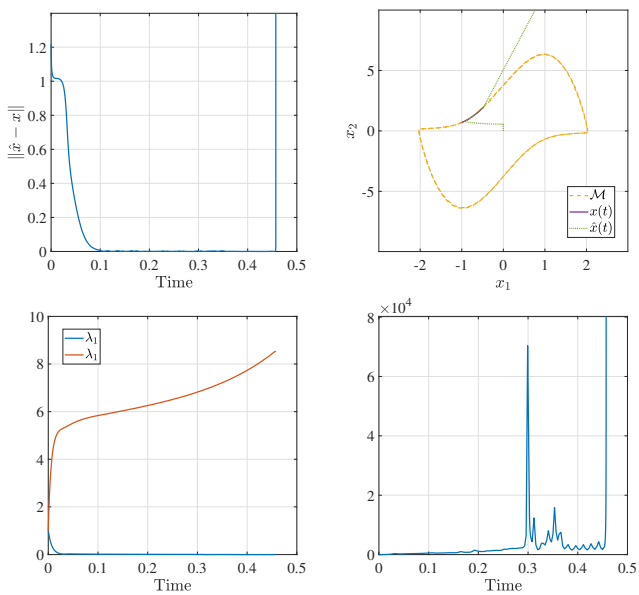

Fig. 3. Case of input not sufficiently exciting. When the frequency ω is too small the estimation diverges. (top-left) estimation error, (top-right) geometrical distance from the estimation to the Van Der Pol manifold $\mathcal{M}$, (bottom-left) eigenvalues of S and (bottom-right) condition number of S

along the estimated trajectory. This confirms the intuition of [27] that the determinability Gramian should be optimized to improve the observer performances. This statement is made in [27] for the standard EKF without proof of convergence, whereas considering the EK-likeF instead allows here to explicitly study the basin of attraction and prove stability. It is nonetheless possible that a similar result applies for the EKF, since the lower-bound of the Riccati solution is related to the same determinability Gramian as exhibited in [22].

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