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On the Hölder continuity for a class of vectorial problems

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Abstract: In this paper we prove local Hölder continuity of vectorial local minimizers of special classes of integral functionals with rank-one and polyconvex integrands. The energy densities satisfy suitable structure assumptions and may have neither radial nor quasi-diagonal structure. The regularity of minimizers is obtained by proving that each component stays in a suitable De Giorgi class and, from this, we conclude about the Hölder continuity. In the final section, we provide some non-trivial applications of our results.

Keywords: Holder, continuity, regularity, vectorial, minimizer, variational, integral

MSC: Primary: 49N60; Secondary: 35J50

1 Introduction

In this paper we establish Hölder regularity for vector-valued minimizers of a class of integral functionals of the Calculus of Variations. We shall apply such results to minimizers of quasiconvex integrands, therefore satisfying the natural condition to ensure existence in the vectorial setting.

For equations and scalar integrals, such a topic is strictly related to the celebrated De Giorgi result in [1]. Several generalizations in the scalar case have then been given, let us mention the contribution of Giaquinta-Giusti [2], establishing Hölder regularity for minima of non differentiable scalar functionals.

The question whether the previous theory and results extend to systems and vectorial integrals was solved in [3] by De Giorgi himself constructing an example of a second order linear elliptic system with solution $\frac{x}{|x|^{\gamma}}$, $\gamma > 1$ (see the nice survey [4]; we also refer to the paper [5] for the most recent result and an up-to-date bibliography on the subject). Motivated by the above mentioned counterexamples, in the mathematical literature there are two different research directions in the study of the regularity in the vector-valued setting: *partial regularity* as introduced by Morrey in [6], i.e., smoothness of solutions up to a set of zero Lebesgue measure, and *everywhere regularity* following Uhlenbeck [7]. For more exhaustive lists of references on such topics see for example [8–10].

Let us now introduce our working assumptions. Given $n,\ N \geq 2$, and a bounded open set $\Omega \subseteq \mathbb{R}^n$, let $f : \Omega\times\mathbb{R}^{N\times n} \to \mathbb{R}$ be a function such that there exist Carathéodory functions $F_\alpha : \Omega\times\mathbb{R}^n \to \mathbb{R}$, $\alpha \in \{1, \dots, N\}$, and $G : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$, such that for all $\xi \in \mathbb{R}^{N\times n}$ and for $\mathcal{L}^n$-a.e. $x \in \Omega$

$$f(x, \xi) := \sum_{\alpha=1}^{N} F_\alpha(x, \xi^\alpha) + G(x, \xi). \tag{1.1}$$

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Here, we have adopted the notation

$$\xi = \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^N \end{pmatrix} \tag{1.2}$$

where $\xi^\alpha \in \mathbb{R}^n$, $\alpha \in \{1, \dots, N\}$, is the α-th row of the matrix ξ.

Furthermore, we assume on each function F_α the following growth conditions: there exist an exponent $p \in (1, n)$, constants k_1, $k_2 > 0$ and a non-negative function $a \in L^\sigma_{\mathrm{loc}}(\Omega)$, $\sigma > 1$, such that for all $\alpha \in \{1, \dots, N\}$, for all $\lambda \in \mathbb{R}^n$ and for $\mathcal{L}^n$-a.e. $x \in \Omega$

$$k_1|\lambda|^p - a(x) \le F_\alpha(x, \lambda) \le k_2|\lambda|^p + a(x)\,. \tag{1.3}$$

In addition, we assume that G is rank-one convex and satisfies for all $\xi \in \mathbb{R}^{N\times n}$ and for $\mathcal{L}^n$-a.e. $x \in \Omega$

$$|G(x, \xi)| \le k_2|\xi|^q + b(x) \tag{1.4}$$

for some $q \in [1, p)$, and a non-negative function $b \in L^\sigma_{\mathrm{loc}}(\Omega)$ (for the precise definition of rank-one convexity and other generalized convexity conditions see Section 2).

Consider the energy functional $\mathscr{F}$ defined for every map $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ and for every measurable subset $E \subset\subset \Omega$ by

$$\mathscr{F}(u; E) := \int_E f\big(x, Du(x)\big)\, dx\,.$$

The main result of the paper concerns the regularity of local minimizers of the functional $\mathscr{F}$. We recall for convenience that a function $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is a local minimizer of $\mathscr{F}$ if for all $\varphi \in W^{1,p}(\Omega, \mathbb{R}^N)$ with $\operatorname{supp}\varphi \Subset \Omega$

$$\mathscr{F}(u; \operatorname{supp}\varphi) \le \mathscr{F}(u + \varphi; \operatorname{supp}\varphi).$$

Theorem 1.1. *Let f satisfy* (1.1) *and the growth conditions* (1.3), (1.4) *with $p \in (1, n)$. Assume further that*

$$1 \le q < \frac{p^2}{n} \quad \text{and} \quad \sigma > \frac{n}{p}. \tag{1.5}$$

Then the local minimizers $u \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ of $\mathscr{F}$ are locally Hölder continuous.

Existence of local minimizers for $\mathscr{F}$ is not assured under the conditions of Theorem 1.1 since f might fail to be quasiconvex under the given assumptions. In the statement we have chosen to underline the only conditions needed to establish the regularity result. For the existence issue see [11–13]. Despite this, we shall give some non-trivial applications of Theorem 1.1 in Section 5. In particular, by using the function introduced by Zhang in [14], we construct examples of genuinely quasiconvex integrands f, which are not convex, and satisfying (1.1)-(1.4). Furthermore, by considering the well-know Šverák's example [15], we exhibit an example of a convex energy density f satisfying the regularity assumptions with non-convex principal part F and with the perturbation G rank-one convex but not quasiconvex. For more examples see Section 5.

The special structure of the energy density f in (1.1) permits to prove Hölder regularity by applying the De Giorgi methods to each scalar component u^α of the minimizer u. More precisely, inspired by [16], we show that each component u^α satisfies a Caccioppoli type inequality, and then it is local Hölder continuous by applying the De Giorgi's arguments; see [8, 17]. As regards the application of the techniques of De Giorgi in the vector-valued case but in a different framework we quote [18]; for related Hölder continuity results for systems we quote [19–21]. We remark that in [22] local γ-Hölder continuity for every $\gamma \in (0, 1)$ has been proved for stationary points of similar variational integrals with rank-one convex lower order perturbations G differentiable at every point and with principal part $F(\xi) = |\xi|^p$.

In Section 4 we consider the case of polyconvex integrands. Precisely, the Hölder continuity of local minimizers is obtained under the same structural assumptions on F and suitable polyconvex lower order perturbations G depending only on the $(N-1) \times (N-1)$ minors of the gradient, see Theorem 4.1. The more rigid structure of the energy density f allows to obtain regularity results under weaker assumptions on the exponents when compared to Theorem 1.1, see Remark 4.2 and Example 1 in Section 5. We notice that in the recent papers [16, 23] the local boundedness of minimizers has been established for more general energy functionals $\mathscr{F}$ with polyconvex integrands and under less restrictive conditions on the growth exponents.

We remark that the assumption $p < n$ is not restrictive. Indeed, it is well-known that the regularity results still hold true if $p \geq n$, even without assuming the special structure of f in (1.1). This is a consequence of the p-growth satisfied by f, the Sobolev embedding, if $p > n$, together with the higher integrability of the gradient if $p = n$ (see [8, Theorem 6.7]).

We finally resume the contents of the paper. In Section 2 we introduce the various convexity notions in the vectorial setting of the Calculus of Variations and we recall De Giorgi's regularity result. Section 3 is devoted to the proof of Theorem 1.1. In Section 4 we deal with functionals with a polyconvex lower order term G. Finally, in Section 5 we provide several non trivial examples of application of our regularity results.

2 Preliminaries

2.1 Convexity conditions

Motivated by applications to nonlinear elasticity, J. Ball in 1977 pointed out in [11] that convexity of the energy density is an unrealistic assumption in the vectorial case. Indeed, it conflicts, for instance, with the natural requirement of frame-indifference for the elastic energy. Then, in the vector-valued setting $N > 1$, different convexity notions with respect to the gradient variable ξ play an important role. We recall all of them in what follows.

Definition 2.1. A function $f = f(x, \xi) : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is said to be

(a) *rank-one convex*: if for all $\lambda \in [0, 1]$ and for all $\xi, \eta \in \mathbb{R}^{N\times n}$ with $\operatorname{rank}(\xi - \eta) \leq 1$

$$f(x, \lambda\xi + (1-\lambda)\eta) \leq \lambda f(x, \xi) + (1-\lambda) f(x, \eta) \tag{2.1}$$

for $\mathcal{L}^n$ a.e. $x \in \Omega$;

(b) *quasiconvex* (in Morrey' sense): if f is Carathéodory, $f(x, \cdot)$ is locally integrable, and

$$\mathcal{L}^n(\Omega) f(x, \xi) \leq \int_\Omega f\left(x, \xi + D\varphi(y)\right) dy, \tag{2.2}$$

for every $\xi \in \mathbb{R}^{N\times n}$, $\varphi \in C_c^\infty\left(\Omega, \mathbb{R}^N\right)$, and for $\mathcal{L}^n$ a.e. $x \in \Omega$;

(c) *polyconvex*: if there exists a function $g : \Omega \times \mathbb{R}^\tau \to \mathbb{R}$, with $g(x, \cdot)$ convex for $\mathcal{L}^n$ a.e. $x \in \Omega$, such that

$$f(x, \xi) = g\big(x, T(\xi)\big)\,. \tag{2.3}$$

In the last item we have adopted the standard notation

$$T(\xi) = \left(\xi, \operatorname{adj}_2\xi, \dots, \operatorname{adj}_i\xi, \dots, \operatorname{adj}_{N\wedge n}\xi\right).$$

for every matrix $\xi \in \mathbb{R}^{N\times n}$, where $\operatorname{adj}_i\xi$ is the adjugate matrix of order $i \in \{2, \dots, N\wedge n\}$ of ξ, that is the $\binom{N}{i}\times\binom{n}{i}$ matrix of all minors of order i of ξ. We will denote by $(\operatorname{adj}_i\xi)^\alpha$ the α-row of such a matrix. In particular, $\operatorname{adj}_1\xi := \xi$ if $i = 1$. Thus $T(\xi)$ is a vector in $\mathbb{R}^\tau$, with

$$\tau = \tau(N, n) := \sum_{i=1}^{N\wedge n} \binom{N}{i}$$

It is well-known that

$$f \text{ convex} \implies f \text{ polyconvex} \implies f \text{ quasiconvex} \implies f \text{ rank-one convex},$$

and that in the scalar case all these notions are equivalent (see for instance [13, Theorem 5.3]).

On the other hand, none of the previous implications can be reversed except for some particular cases. We refer to [13, Chapter 5] for several examples and counterexamples. In particular, in Section 5 we shall extensively deal with Šverák's celebrated counterexample to the reverse of the last implication above.

2.2 De Giorgi classes

In this section we recall the well-known regularity result in the scalar case due to De Giorgi [1].

Definition 2.2. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, open set and $v : \Omega \to \mathbb{R}$. We say that $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ belongs to the De Giorgi class $\mathrm{DG}^+(\Omega, p, y, y_\star, \delta)$, $p > 1$, y and $\delta > 0$, $y_\star \geq 0$ if

$$\int\limits_{B_{\sigma\rho}(x_0)} |D(v-k)_+|^p\,dx \leq \frac{y}{(1-\sigma)^p \rho^p} \int\limits_{B_\rho(x_0)} (v-k)_+^p\,dx + y_\star \big(\mathcal{L}^n(\{v>k\} \cap B_\rho(x_0))\big)^{1-\frac{p}{n}+p\delta} \tag{2.4}$$

for all $k \in \mathbb{R}$, $\sigma \in (0,1)$ and all pair of balls $B_{\sigma\rho}(x_0) \subset B_\rho(x_0) \subset\subset \Omega$.

The De Giorgi class $\mathrm{DG}^-(\Omega, p, y, y_\star, \delta)$ is defined similary with $(v-k)_+$ replaced by $(v-k)_-$.

Finally, we set $\mathrm{DG}(\Omega, p, y, y_\star, \delta) = \mathrm{DG}^+(\Omega, p, y, y_\star, \delta) \cap \mathrm{DG}^-(\Omega, p, y, y_\star, \delta)$.

(2.4) is a *Caccioppoli type* inequality on super-/sub-level sets and contains several informations on the smoothness of the function u. Indeed, functions in the De Giorgi classes have remarkable regularity properties. In particular, they are locally bounded and locally Hölder continuous in Ω (see [17, Theorems 2.1 and 3.1] and [8, Chapter 7]).

Theorem 2.3. *Let $v \in DG(\Omega, p, y, y_\star, \delta)$ and $\tau \in (0,1)$. There exists a constant $C > 1$ depending only upon the data and independent of v, such that for every pair of balls $B_{\tau\rho}(x_0) \subset B_\rho(x_0) \subset\subset \Omega$*

$$\|v\|_{L^\infty(B_{\tau\rho}(x_0))} \leq \max\Big\{y_\star \rho^{n\delta}; \frac{C}{(1-\tau)^{\frac{1}{\delta}}}\Big(\frac{1}{\mathcal{L}^n(B_\rho(x_0))}\int\limits_{B_\rho(x_0)} |v|^p\,dx\Big)^{\frac{1}{p}}\Big\},$$

moreover, there exists $\tilde{\alpha} \in (0,1)$ depending only upon the data and independent of v, such that

$$\operatorname{osc}(v, B_\rho(x_0)) \leq C \max\Big\{y_\star \rho^{n\delta}; \Big(\frac{\rho}{R}\Big)^{\tilde{\alpha}} \operatorname{osc}(v, B_R(x_0))\Big\}$$

where $\operatorname{osc}(v, B_\rho(x_0)) := \operatorname{ess\,sup}_{B_\rho(x_0)} v - \operatorname{ess\,inf}_{B_\rho(x_0)} v$. *Therefore, $v \in C^{0,\tilde{\alpha}_0}_{\mathrm{loc}}(\Omega)$ with $\tilde{\alpha}_0 := \tilde{\alpha} \wedge (n\delta)$.*

3 Proof of Theorem 1.1

The specific structure (1.1) of the energy density f yields a Caccioppoli inequality on every sub-/superlevel set for any component u^α of u. To provide the precise statement we introduce the following notation: given $x_0 \in \Omega$, $0 < t < \operatorname{dist}(x_0, \partial\Omega)$, and with fixed $k \in \mathbb{R}$ and $\alpha \in \{1, \dots, N\}$ set

$$A^\alpha_{k,t,x_0} := \{x \in B_t(x_0) : u^\alpha(x) > k\} \quad \text{and} \quad B^\alpha_{k,t,x_0} := \{x \in B_t(x_0) : u^\alpha(x) < k\}. \tag{3.1}$$

Proposition 3.1 (Caccioppoli inequality on sub-/superlevel sets)**.** *Let f be as in* (1.1), *satisfying the growth conditions* (1.3), (1.4). *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ be a local minimizer of $\mathscr{F}$.*

Then there exists $c = c(k_1, k_2, p, q, n) > 0$, such that for all $x_0 \in \Omega$ and for every $0 < \rho < R < R_0 \wedge \operatorname{dist}(x_0, \partial\Omega)$, with $\mathcal{L}^n(B_{R_0}) \leq 1$, and $\alpha \in \{1, \dots, N\}$ we have

$$\int\limits_{A^\alpha_{k,\rho,x_0}} |Du^\alpha|^p\,dx \leq c \int\limits_{A^\alpha_{k,R,x_0}} \Big(\frac{|u^\alpha - k|}{R-\rho}\Big)^p\,dx$$

$$+\,c\Big(1+\|a\|_{L^\sigma(B_R(x_0))}+\|b\|_{L^\sigma(B_R(x_0))}+\|Du\|^q_{L^p(B_R(x_0),\mathbb{R}^{N\times n})}\Big)\big(\mathcal{L}^n(A^\alpha_{k,R,x_0})\big)^\vartheta \tag{3.2}$$

where $\vartheta := \min\{1-\frac{q}{p}, 1-\frac{1}{\sigma}\}$. *The same inequality holds substituting* A^α_{k,R,x_0} *with* B^α_{k,R,x_0}.

Proof. Without loss of generality we may assume $\alpha = 1$. For the sake of notational convenience we drop the x_0-dependence in the notation of the corresponding sub-/superlevel set. We start off with proving the inequality on the super-level sets. Given $0 < \rho < s < t < R < R_0 \wedge \operatorname{dist}(x_0, \partial\Omega)$, with $\mathcal{L}^n(B_{R_0}) \le 1$, consider a smooth cut-off function $\eta \in C^\infty_0(B_t)$ satisfying

$$0 \le \eta \le 1, \quad \eta \equiv 1 \text{ in } B_s(x_0), \quad |D\eta| \le \frac{2}{t-s}. \tag{3.3}$$

With fixed $k \in \mathbb{R}$, define $w \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ by

$$w^1 := \max(u^1 - k, 0), \quad w^\alpha := 0 \quad \alpha \in \{2,\dots,N\}$$

and

$$\varphi := -\eta^p w.$$

We have $\varphi = 0$ $\mathcal{L}^n$-a.e. in $\Omega \setminus (\{\eta > 0\} \cap \{u^1 > k\})$, thus

$$Du + D\varphi = Du \qquad \mathcal{L}^n\text{-a.e. in } \Omega \setminus (\{\eta > 0\} \cap \{u^1 > k\}). \tag{3.4}$$

Set

$$\mathbb{A} := \begin{pmatrix} p\eta^{-1}(k-u^1)D\eta \\ Du^2 \\ \vdots \\ Du^N \end{pmatrix}. \tag{3.5}$$

Then notice that $\mathcal{L}^n$-a.e. in $\{\eta > 0\} \cap \{u^1 > k\}$

$$Du + D\varphi = \begin{pmatrix} (1-\eta^p)Du^1 + p\eta^{p-1}(k-u^1)D\eta \\ Du^2 \\ \vdots \\ Du^N \end{pmatrix} = (1-\eta^p)Du + \eta^p\mathbb{A}. \tag{3.6}$$

Thus, since $Du - \mathbb{A}$ is a rank-one matrix, the rank-one-convexity of G yields

$$G(x, Du + D\varphi) \le (1-\eta^p)G(x, Du) + \eta^p G(x, \mathbb{A}) \qquad \mathcal{L}^n\text{-a.e. in } \{\eta > 0\} \cap \{u^1 > k\}. \tag{3.7}$$

By the local minimality of u, (3.4), (3.7) and taking into account that $\mathcal{L}^n$-a.e. in Ω

$$F_\alpha(x, (Du + D\varphi)^\alpha) = F_\alpha(x, Du^\alpha) \qquad \alpha \in \{2, \cdots, N\}$$

we have

$$\int\limits_{A^1_{k,t}\cap\{\eta>0\}} \Big(F_1(x, Du^1) + G(x, Du)\Big)\,dx$$
$$\le \int\limits_{A^1_{k,t}\cap\{\eta>0\}} \Big(F_1(x, (Du + D\varphi)^1) + (1-\eta^p)G(x, Du) + \eta^p G(x, \mathbb{A})\Big)\,dx\,.$$

The latter inequality and (3.4) imply that

$$\int\limits_{A^1_{k,t}} F_1(x, Du^1)\,dx$$

$$\leq \int_{A^1_{k,t}} F_1(x,(Du+D\varphi)^1)\,dx + \int_{A^1_{k,t}\cap\{\eta>0\}} \eta^p \left(G(x,\mathbb{A}) - G(x,Du)\right)\,dx\,. \tag{3.8}$$

By (1.3), (3.6), the convexity of $t \mapsto |t|^p$ and (3.3), we get

$$\begin{aligned}
\int_{A^1_{k,t}} F_1(x,(Du+D\varphi)^1)\,dx &\leq \int_{A^1_{k,t}} \left(k_2|(Du+D\varphi)^1|^p + a(x)\right)\,dx \\
&\leq \int_{A^1_{k,t}} \left(k_2(1-\eta^p)|Du^1|^p + k_2|p(k-u^1)D\eta|^p + a(x)\right)\,dx \\
&\leq c \int_{A^1_{k,t}\setminus A^1_{k,s}} |Du^1|^p\,dx + c\int_{A^1_{k,t}} \left(\left(\frac{u^1-k}{t-s}\right)^p + a(x)\right)dx
\end{aligned}$$

with $c = c(k_2, p)$. Therefore, (3.8) implies

$$\begin{aligned}
\int_{A^1_{k,t}} F_1(x,Du^1)\,dx \leq c &\int_{A^1_{k,t}\setminus A^1_{k,s}} |Du^1|^p\,dx + c\int_{A^1_{k,t}} \left(\left(\frac{u^1-k}{t-s}\right)^p + a(x)\right)dx \\
&+ \int_{A^1_{k,t}\cap\{\eta>0\}} \eta^p \left(G(x,\mathbb{A}) - G(x,Du)\right)\,dx\,.
\end{aligned} \tag{3.9}$$

We now estimate the last integral at the right hand side. The growth condition in (1.4) for G, Hölder's and Young's inequalities imply, for some $c = c(k_2, p, q) > 0$,

$$\begin{aligned}
\int_{A^1_{k,t}\cap\{\eta>0\}} \eta^p \Big(G(x,\mathbb{A}) - G(x,Du)\Big)\,dx &\leq c\int_{A^1_{k,t}} \left(\left(\frac{u^1-k}{t-s}\right)^q + |Du|^q + b(x)\right)dx \\
&\leq c\int_{A^1_{k,t}} \left(\frac{u^1-k}{t-s}\right)^p dx + c\,\mathcal{L}^n(A^1_{k,t}) \\
&\quad + c\|Du\|^q_{L^p(B_t,\mathbb{R}^{N\times n})}\left(\mathcal{L}^n(A^1_{k,t})\right)^{1-\frac{q}{p}} + c\,\|b\|_{L^\sigma(B_t)}\left(\mathcal{L}^n(A^1_{k,t})\right)^{1-\frac{1}{\sigma}}.
\end{aligned} \tag{3.10}$$

Hence, by taking into account estimates (1.3), (3.9) and (3.10), we obtain

$$\begin{aligned}
k_1\int_{A^1_{k,s}} |Du^1|^p\,dx \leq c&\int_{A^1_{k,t}\setminus A^1_{k,s}} |Du^1|^p\,dx + c\int_{A^1_{k,t}} \left(\frac{u^1-k}{t-s}\right)^p dx + c\,\mathcal{L}^n(A^1_{k,t}) \\
&+ c\|Du\|^q_{L^p(B_t,\mathbb{R}^{N\times n})}\left(\mathcal{L}^n(A^1_{k,t})\right)^{1-\frac{q}{p}} + c\,\left(\|a\|_{L^\sigma(B_t)} + \|b\|_{L^\sigma(B_t)}\right)\left(\mathcal{L}^n(A^1_{k,t})\right)^{1-\frac{1}{\sigma}}
\end{aligned}$$

for $c = c(k_2, p, q) > 0$. By hole-filling, i.e. adding to both sides

$$c\int_{A^1_{k,s}} |Du^1|^p\,dx,$$

we obtain

$$\begin{aligned}
\int_{A^1_{k,s}} |Du^1|^p\,dx \leq \frac{c}{k_1+c}\Bigg[&\int_{A^1_{k,t}} |Du^1|^p\,dx + \int_{A^1_{k,t}} \left(\frac{u^1-k}{t-s}\right)^p dx + \mathcal{L}^n(A^1_{k,t}) \\
&+\|Du\|^q_{L^p(B_t,\mathbb{R}^{N\times n})}\left(\mathcal{L}^n(A^1_{k,t})\right)^{1-\frac{q}{p}} + \left(\|a\|_{L^\sigma(B_t)} + \|b\|_{L^\sigma(B_t)}\right)\left(\mathcal{L}^n(A^1_{k,t})\right)^{1-\frac{1}{\sigma}}\Bigg]
\end{aligned}$$

for $c = c(k_2, p, q) > 0$. By Lemma 3.2 below we get

$$\int_{A^1_{k,\rho}} |Du^1|^p\,dx \le c \int_{A^1_{k,R}} \left(\frac{u^1-k}{R-\rho}\right)^p dx + c\,\mathcal{L}^n(A^1_{k,R}) + c\|Du\|^q_{L^p(B_R,\mathbb{R}^{N\times n})} \left(\mathcal{L}^n(A^1_{k,R})\right)^{1-\frac{q}{p}}$$
$$+ c\left(\|a\|_{L^\sigma(B_R)} + \|b\|_{L^\sigma(B_R)}\right)\left(\mathcal{L}^n(A^1_{k,R})\right)^{1-\frac{1}{\sigma}}, \tag{3.11}$$

for $c = c(k_1, k_2, p, q) > 0$. Estimate (3.2) follows at once from (3.11), by taking into account that $\mathcal{L}^n(A^1_{k,R}) \le \mathcal{L}^n(B_{R_0}) \le 1$.

In conclusion, the analogous estimate with $B^1_{k,R}$ in place of $A^1_{k,R}$ follows from (3.2) itself since $-u$ is a local minimizer of the integral functional with energy density $\tilde{f}(x,\xi) := f(x,-\xi)$. □

The following lemma finds an important application in the hole-filling method. The proof can be found for example in [8, Lemma 6.1] .

Lemma 3.2. *Let $h : [r, R_0] \to \mathbb{R}$ be a non-negative bounded function and $0 < \vartheta < 1$, $A, B \ge 0$ and $\beta > 0$. Assume that*

$$h(s) \le \vartheta h(t) + \frac{A}{(t-s)^\beta} + B,$$

for all $r \le s < t \le R_0$. Then

$$h(r) \le \frac{cA}{(R_0-r)^\beta} + cB,$$

where $c = c(\vartheta, \beta) > 0$.

We are now ready to prove the local Hölder continuity of local minimizers.

Proof of Theorem 1.1. We use Proposition 3.1 for u, with the exponents p, q satisfying (1.5) or, equivalently,

$$\vartheta > 1 - \frac{p}{n}, \tag{3.12}$$

recalling that $\vartheta = \min\{1 - \frac{q}{p}, 1 - \frac{1}{\sigma}\}$. Then inequality (2.4) holds for each u^α, $\alpha \in \{1, \dots, N\}$, i.e. u^α belongs to a suitable De Giorgi's class and Theorem 2.3 ensures that u^α is locally Hölder continuous. □

4 The polyconvex case

In this section we deal with the case of a suitable class of polyconvex functions G. We will exploit their specific structure to obtain Hölder continuity results not included in Theorem 1.1. We shall use extensively the notation introduced in Section 2.1.

For $u \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ and $E \subset\subset \Omega$ a measurable set, we shall consider functionals

$$\mathcal{F}(u;E) := \int_E f\left(x, Du(x)\right) dx\,,$$

with Carathéodory integrands $f : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$, $n \ge N \ge 3$, satisfying

$$f(x,\xi) := \sum_{\alpha=1}^N F_\alpha(x,\xi^\alpha) + G(x,\xi). \tag{4.1}$$

We assume that the functions F_α are as in the previous section. In particular, we assume that

there exist $p \in (1, n)$, $k_1,\ k_2 > 0$ and a non-negative function $a \in L^\sigma_{\mathrm{loc}}(\Omega)$, $\sigma > 1$, such that

$$k_1|\lambda|^p - a(x) \le F_\alpha(x,\lambda) \le k_2|\lambda|^p + a(x) \tag{4.2}$$

for all $\lambda \in \mathbb{R}^n$ and for $\mathcal{L}^n$-a.e. $x \in \Omega$

As far as $G : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is concerned, G depends only on $(N-1)\times(N-1)$ minors of ξ as follows:

$$G(x,\xi) := \sum_{\alpha=1}^{N} G_\alpha\left(x, (\mathrm{adj}_{N-1}\xi)^\alpha\right). \tag{4.3}$$

For every $\alpha \in \{1,\cdots,N\}$ we assume that $G_\alpha : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, $\lambda \mapsto G_\alpha(x,\lambda)$ convex, such that

there exist $r \in [1,p)$ and a non-negative function $b \in L^\sigma_{\mathrm{loc}}(\Omega)$, $\sigma > 1$, such that

$$0 \le G_\alpha(x,\lambda) \le k_2|\lambda|^r + b(x) \tag{4.4}$$

for all $\lambda \in \mathbb{R}^N$ and for $\mathcal{L}^n$-a.e. $x \in \Omega$.

Theorem 4.1. *Let f be as in* (4.1)*, and assume* (4.2)–(4.4)*. If*

$$1 \le r < \frac{p^2}{(N-2)n+p} \qquad \text{and} \qquad \sigma > \frac{n}{p}, \tag{4.5}$$

then the local minimizers $u \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ of $\mathcal{F}$ are locally Hölder continuous.

Remark 4.2. *A comparison between Theorem 4.1 and Theorem 1.1 is in order. We do it in the case $n = N = 3$. By* (4.3)*, the function*

$$G(x,\xi) := \sum_{\alpha=1}^{3} G_\alpha(x, (\mathrm{adj}_2\,\xi)^\alpha)$$

is a polyconvex function, satisfying

$$0 \le G(x,\xi) \le c\left(|\xi|^{2r} + b(x) + 1\right) \qquad \forall \xi \in \mathbb{R}^{3\times 3}$$

for a positive costant c depending on p and k_2.

By Theorem 1.1 we get that if $\sigma > \frac{3}{p}$ and

$$1 \le r < \frac{p^2}{6},$$

then the $W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^3)$-local minimizers of $\mathcal{F}$ are Hölder continuous.

The Hölder regularity of the local minimizers can be obtained by Theorem 4.1 under the following weaker condition on r

$$1 \le r < \frac{p^2}{p+3}.$$

The key result to establish Theorem 4.1 is, as in the case of Theorem 1.1, the following Caccioppoli's type inequality which improves Proposition 3.1. We state it only for the first component u^1 of u. We recall that the super-(sub-)level sets are defined as in (3.1). Moreover, we use here the following notation:

$$\hat{u} := (u^2, u^3, \cdots, u^N).$$

For the sake of simplicity, in the Lebesgue norms we will avoid to indicate the target space of the functions involved.

Proposition 4.3 (Caccioppoli inequality on sub-/superlevel sets)**.** *Let f be as in* (4.1)*, and assume that F_α and G satisfy* (4.2)–(4.4)*. Assume that*

$$1 \le r < \frac{p}{N-1}, \qquad \sigma > 1. \tag{4.6}$$

If $u \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ is a local minimizer of $\mathcal{F}$, then there exists $c = c(n,N,p,k_1,k_2,r) > 0$, such that for all $x_0 \in \Omega$ and for every $0 < \rho < R < R_0 \wedge \mathrm{dist}(x_0,\partial\Omega)$, with $\mathcal{L}^n(B_{R_0}) \le 1$, we have

$$\int_{A^1_{k,\rho,x_0}} |Du^1|^p\,dx \le c \int_{A^1_{k,R,x_0}} \left(\frac{|u^1-k|}{R-\rho}\right)^p dx$$

$$+ c\left(\|a+b\|_{L^\sigma(B_R(x_0))} + \|D\hat{u}\|_{L^p(B_R)}^{\frac{(N-2)rp}{p-r}}\right)\left(\mathcal{L}^n(A^1_{k,R,x_0})\right)^{\vartheta}, \tag{4.7}$$

where

$$\vartheta := \min\left\{1 - \frac{(N-2)r}{p-r}, 1 - \frac{1}{\sigma}\right\}. \tag{4.8}$$

The same inequality holds substituting A^1_{k,R,x_0} with B^1_{k,R,x_0}.

We limit ourselves to exhibit the proof of Proposition 4.3, given that Theorem 4.1 follows with the same lines of the proof of Theorem 1.1.

Proof of Proposition 4.3. The proof is the same of that of Proposition 3.1 up to inequality (3.9). By keeping the notation introduced there, (3.9) and the left inequality in (4.2) imply

$$\begin{aligned} k_1 \int\limits_{A^1_{k,t}\cap\{\eta>0\}} \eta^p |Du^1|^p\,dx &\le c \int\limits_{A^1_{k,t}\setminus A^1_{k,s}} |Du^1|^p\,dx + c\int\limits_{A^1_{k,t}} \left(\left(\frac{u^1-k}{t-s}\right)^p + a(x)\right)dx \\ &\quad + \int\limits_{A^1_{k,t}\cap\{\eta>0\}} \eta^p\left(G(x,\mathbb{A}) - G(x,Du)\right)dx, \end{aligned} \tag{4.9}$$

with c depending on p, k_2. We exploit next the specific structure of G. Taking into account the definition of $\mathbb{A}$, see (3.5), we have

$$G_1(x, (\mathrm{adj}_{N-1}\,\mathbb{A})^1) = G_1(x, (\mathrm{adj}_{N-1}\,Du)^1),$$

therefore

$$G(x,\mathbb{A}) - G(x,Du) = \sum_{\alpha=2}^N \left(G_\alpha\left(x, (\mathrm{adj}_{N-1}\mathbb{A})^\alpha\right) - G_\alpha\left(x, (\mathrm{adj}_{N-1}Du)^\alpha\right)\right).$$

Using the growth condition (4.4), that in particular implies that G_α is non-negative, we get

$$\begin{aligned} &\sum_{\alpha=2}^N \left(G_\alpha\left(x, (\mathrm{adj}_{N-1}\mathbb{A})^\alpha\right) - G_\alpha\left(x, (\mathrm{adj}_{N-1}Du)^\alpha\right)\right) \\ &\le \sum_{\alpha=2}^N G_\alpha\left(x, (\mathrm{adj}_{N-1}\mathbb{A})^\alpha\right) \le c\sum_{\alpha=2}^N \left(|(\mathrm{adj}_{N-1}\,\mathbb{A})^\alpha|^r + b(x)\right) \end{aligned}$$

with c depending on k_2.

Denote $\hat{u} := (u^2, u^3, \cdots, u^N)$. For every $\alpha \in \{2,\cdots,N\}$ we have

$$|(\mathrm{adj}_{N-1}\,\mathbb{A})^\alpha| \le c|\mathbb{A}^1||\,\mathrm{adj}_{N-2}\,D\hat{u}|$$

with c depending on n and N. Since $r < p$ we can use the Young's inequality with exponents $\frac{p}{r}$ and $\frac{p}{p-r}$, so we have, $\mathcal{L}^n$-a.e. in $A^1_{k,t}\cap\{\eta>0\}$,

$$\left(|\mathbb{A}^1||\,\mathrm{adj}_{N-2}\,D\hat{u}|\right)^r \le c\left(|\mathbb{A}^1|^p + |\,\mathrm{adj}_{N-2}\,D\hat{u}|)^{\frac{rp}{p-r}}\right) \le c\left\{\left(\frac{u^1-k}{t-s}\right)^p + |D\hat{u}|^{\frac{(N-2)rp}{p-r}}\right\},$$

with $c = c(n, N, p, r)$.

Collecting the above inequalities, we get

$$\begin{aligned} \int\limits_{A^1_{k,t}\cap\{\eta>0\}} \eta^p\left(G(x,\mathbb{A}) - G(x,Du)\right)dx &\le c\int\limits_{A^1_{k,t}\cap\{\eta>0\}} \left(\frac{u^1-k}{t-s}\right)^p dx \\ &\quad + c\int\limits_{A^1_{k,t}} |D\hat{u}|^{\frac{(N-2)rp}{p-r}}\,dx + c\int\limits_{A^1_{k,t}} b(x)\,dx, \end{aligned} \tag{4.10}$$

with $c = c(k_2, n, N, p, r) > 0$. By (4.6) $\frac{(N-2)r}{p-r} < 1$ therefore by Hölder's inequality we get

$$\int_{A^1_{k,t}} |D\hat{u}|^{\frac{(N-2)rp}{p-r}}\,dx \le \left(\int_{B_t} |D\hat{u}|^p\,dx\right)^{\frac{(N-2)r}{p-r}} \left(\mathcal{L}^n(A^1_{k,t})\right)^{1-\frac{(N-2)r}{p-r}}.$$

Analogously,

$$\int_{A^1_{k,t}} (a(x)+b(x))\,dx \le \|a+b\|_{L^\sigma(B_t)} \left(\mathcal{L}^n(A^1_{k,t})\right)^{1-\frac{1}{\sigma}}.$$

Therefore by (4.9) and (4.10) we get

$$\begin{aligned} k_1 \int_{A^1_{k,s}} |Du^1|^p\,dx &\le c \int_{A^1_{k,t}\setminus A^1_{k,s}} |Du^1|^p\,dx + c\int_{A^1_{k,t}} \left(\frac{u^1-k}{t-s}\right)^p dx \\ &\quad + c\|D\hat{u}\|^{\frac{(N-2)rp}{p-r}}_{L^p(B_t)} \left(\mathcal{L}^n(A^1_{k,t})\right)^{1-\frac{(N-2)r}{p-r}} + c\|a+b\|_{L^\sigma(B_t)} \left(\mathcal{L}^n(A^1_{k,t})\right)^{1-\frac{1}{\sigma}}, \end{aligned} \tag{4.11}$$

with $c = c(n, N, p, k_2, r) > 0$.

We now proceed as in the proof of Proposition 3.2: adding to both sides of (4.11)

$$c\int_{A^1_{k,s}} |Du^1|^p\,dx$$

and using Lemma 3.2 we obtain that

$$\begin{aligned} \int_{A^1_{k,\rho}} |Du^1|^p\,dx &\le c\int_{A^1_{k,R}} \left(\frac{u^1-k}{R-\rho}\right)^p dx \\ &\quad + c\left\{\|D\hat{u}\|^{\frac{(N-2)rp}{p-r}}_{L^p(B_R)} + \|a+b\|_{L^\sigma(B_R)}\right\} \left(\mathcal{L}^n(A^1_{k,R})\right)^{\vartheta} \end{aligned}$$

with ϑ as in (4.8) and $c = c(n, N, p, k_1, k_2, r) > 0$.

In conclusion, the analogous estimate with $B^1_{k,t}$ in place of $A^1_{k,t}$ follows from (4.7) itself since $-u$ is a local minimizer of the integral functional with energy density $\tilde{f}(x,\xi) := f(x,-\xi)$. □

5 Examples

We provide some non trivial applications of Theorems 1.1 and 4.1. In particular, we infer Hölder continuity of local minimizers to vectorial variational problems with quasiconvex integrands. The energy densities that we consider satisfy (1.1)-(1.4) and have neither radial nor quasi-diagonal structure. More in details, the integrands in Examples 1 and 2 are not convex, respectively they are polyconvex and quasiconvex, being F convex but G only polyconvex in the first case, and quasiconvex in the second. In Example 3 we construct a convex density though with non-convex principal part. Instead, the energy density f in Examples 4 and 5 is convex. In particular, in the first one F is convex and G is the rank-one convex non-quasiconvex function introduced by Šverák in [15]; in the second we construct a non-convex integrand $\widetilde{F}$ by modifying F in Example 4, keeping the same G.

Being in all cases the resulting f quasiconvex, existence of local minimizers for the corresponding functional $\mathscr{F}$ easily follows from the Direct Method of the Calculus of Variations.

Example 1

Let $n = N = 3$ and consider $f : \mathbb{R}^{3\times 3} \to \mathbb{R}$,

$$f(\xi) := \sum_{\alpha=1}^{3} |\xi^\alpha|^p + \Big(1 + |\big(\operatorname{adj}_2 \xi\big)_1^1 - 1|\Big)^r,$$

with $p \geq 1$ and $r \geq 1$. We recall that, for all $\xi \in \mathbb{R}^{3\times 3}$, $\operatorname{adj}_2 \xi \in \mathbb{R}^{3\times 3}$ denotes the adjugate matrix of ξ of order 2, whose components are

$$(\operatorname{adj}_2 \xi)_i^y = (-1)^{y+i} \det \begin{pmatrix} \xi_k^\alpha & \xi_l^\alpha \\ \xi_k^\beta & \xi_l^\beta \end{pmatrix} \qquad y, i \in \{1, 2, 3\},$$

where $\alpha, \beta \in \{1, 2, 3\} \setminus \{y\}$, $\alpha < \beta$, and $k, l \in \{1, 2, 3\} \setminus \{i\}$, $k < l$.

We claim that f is a polyconvex, non-convex function satisfying the structure condition (4.1) with suitable F_α and G satisfying the growth conditions (4.2) and (4.4), respectively.

As far as the stucture is concerned, it is easy to see that (1.1) holds, if we define, for all $\alpha \in \{1, 2, 3\}$ and $\lambda \in \mathbb{R}^3$

$$F_\alpha(\lambda) = F(\lambda) := |\lambda|^p$$

and, for all $\xi \in \mathbb{R}^{3\times 3}$,

$$G(\xi) := h\big((\operatorname{adj}_2 \xi)_1^1\big),$$

with

$$h(t) := \big(1 + |t - 1|\big)^r, \qquad t \in \mathbb{R}.$$

The polyconvexity of f follows from the convexity of F and h (the latter holds true since $r \geq 1$), see e.g. [13]. Let us now prove that f is not convex. Consider the matrices $\xi_1 := \varepsilon\tilde{\xi}$ and $\xi_2 := -\xi_1$, $\varepsilon > 0$, where

$$\tilde{\xi} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We shall prove that for $\varepsilon > 0$ sufficiently small

$$f\Big(\frac{1}{2}(\xi_1 + \xi_2)\Big) > \frac{1}{2}\big(f(\xi_1) + f(\xi_2)\big), \tag{5.1}$$

thus establishing the claim. Indeed, on one hand the right hand side rewrites as

$$\frac{1}{2}\big(f(\xi_1) + f(\xi_2)\big) = f(\xi_1) = 2\,\varepsilon^p + (1 + |\varepsilon^2 - 1|)^r =: \varphi(\varepsilon),$$

while on the other hand the left hand side rewrites as

$$f\Big(\frac{1}{2}(\xi_1 + \xi_2)\Big) = f(\underline{0}) = \varphi(0).$$

Note that $\varphi \in C^2((-1, 1))$ since $p > 2$. Simple computations show that $\varphi'(0) = 0$ and $\varphi''(0) = -r\,2^r < 0$. Thus, for some $\delta \in (0, 1)$ and for all $\varepsilon \in (0, \delta)$ we have $\varphi'(\varepsilon) < \varphi'(0) = 0$. Thus $\varphi(\varepsilon) < \varphi(0)$, and inequality (5.1) follows at once.

By using Theorem 4.1 we have that, if $p \in \big(\frac{1}{2}(1 + \sqrt{13}), 3\big)$ and $r \in \big[1, \frac{p^2}{3+p}\big)$, then the $W^{1,p}_{\mathrm{loc}}$-local minimizers of the corresponding functional $\mathscr{F}$ are locally Hölder continuous.

We note that the arguments in [22, Theorem 1] do not apply since the function G is not differentiable.

Example 2

Let $n, N \geq 2$. Given two matrices ξ_1, ξ_2 in $\mathbb{R}^{N\times n}$ such that $\operatorname{rank}(\xi_1 - \xi_2) > 1$, define

$$K := \{\xi_1, \xi_2\}.$$

Denoting $Q\mathrm{dist}(\cdot, K)$ the quasiconvex envelope of the distance function from K, we define, for $q \geq 1$, the quasiconvex function $G : \mathbb{R}^{N\times n} \to [0, +\infty)$ by

$$G(\xi) := Q\mathrm{dist}(\xi, K) \vee \left(\mathrm{dist}(\xi, \mathrm{co}K)\right)^q,$$

where $\mathrm{co}K$ is the convex envelope of the set K. For all $\varrho \in [0, 1]$ define the energy density $f_\varrho : \mathbb{R}^{N\times n} \to \mathbb{R}$,

$$f_\varrho(\xi) := \varrho \sum_{\alpha=1}^{n} |\xi^\alpha|^p + (1 - \varrho)G(\xi),$$

and note that f_ϱ satisfies (1.1)-(1.4) and it is quasiconvex.

We claim that, fixed $p \geq 1$, there exists $\varrho_0 > 0$ such that, for every $\varrho \in (0, \varrho_0)$, f_ϱ is quasiconvex, but not convex. Given this for granted, by Theorem 1.1 we have that the $W^{1,p}_{\mathrm{loc}}$-local minimizers of the corresponding functional $\mathscr{F}$ are locally Hölder continuous provided that $1 \leq q < \frac{p^2}{n}$.

To prove the claim, we first observe that the function G is not convex, since $G^{-1}((-\infty, 0])$ turns out to be the set K, which is non-empty and non-convex. Indeed, by [14, Theorem 1.1, Example 4.3], the zero set of the quasiconvex function with linear growth $\xi \mapsto Q\mathrm{dist}(\xi, K)$ is K. This implies $G^{-1}((-\infty, 0]) = K$.

Next we consider the set $J := \{\varrho \in [0, 1] : f_\varrho \text{ is convex}\}$ and note that J is non-empty, as $1 \in J$, and closed, since convexity is stable under pointwise convergence. Since $0 \notin J$ we can find $\varrho_0 > 0$ such that $[0, \varrho_0) \cap J = \emptyset$. Hence, we conclude that f_ϱ is non-convex for $\varrho \in [0, \varrho_0)$.

Example 3

We give an example of an overall convex function f having non-convex principal part and convex lower order term.

Let $2 \leq q < p < n$, $\mu > 0$, and $B_1 := \{z \in \mathbb{R}^n : |z| < 1\}$. Given $\varphi \in C^\infty_c(B_1, [0, 1])$ with $\varphi(\underline{0}) = 1$ and $D^2\varphi(\underline{0})$ negative definite, let

$$F(\xi) := \sum_{\alpha=1}^{N} F_\alpha(\xi^\alpha)$$

where $F_\alpha(\lambda) = h(\lambda) := (\mu + |\lambda|^2)^{\frac{p}{2}}$ for $\alpha \in \{2, \dots, N\}$, $\lambda \in \mathbb{R}^n$, $F_1(\lambda) := h(\lambda) + M\,\varphi(\lambda)$, $M > 0$ to be chosen in what follows.

We claim that it is possible to find $M_\mu > 0$ such that for every $M \geq M_\mu$ and for all $\eta^1 \in \mathbb{R}^n \setminus \{\underline{0}\}$

$$\langle D^2 F_1(\underline{0})\eta^1, \eta^1\rangle < 0. \tag{5.2}$$

With this aim we first compute the Hessian matrices of F_α and F. Simple computations yield for all $\lambda,\ \zeta \in \mathbb{R}^n$.

$$\langle D^2 h(\lambda)\zeta, \zeta\rangle = p(\mu + |\lambda|^2)^{\frac{p}{2}-2}\left((\mu + |\lambda|^2)|\zeta|^2 + (p-2)\langle\lambda, \zeta\rangle^2\right). \tag{5.3}$$

Hence, if we set $F^\star_\alpha(\lambda) := h(\lambda)$ and $F^\star(\xi) := \sum_{\alpha=1}^N F^\star_\alpha(\xi^\alpha) = \sum_{\alpha=1}^N h(\xi^\alpha)$, being $p > 2$, we get that

$$\langle D^2 F^\star(\xi)\eta, \eta\rangle = \sum_{\alpha=1}^{N}\langle D^2 F^\star_\alpha(\xi^\alpha)\eta^\alpha, \eta^\alpha\rangle \geq p\sum_{\alpha=1}^{N}(\mu + |\xi^\alpha|^2)^{\frac{p}{2}-1}|\eta^\alpha|^2 \geq p\,\mu^{\frac{p}{2}-1}|\eta|^2. \tag{5.4}$$

We are now ready to show that F_1 is not convex. Indeed, we have

$$\langle D^2 F_1(\underline{0})\eta^1, \eta^1\rangle \overset{(5.3)}{=} p\mu^{\frac{p}{2}-1}|\eta^1|^2 + M\langle D^2\varphi(\underline{0})\eta^1, \eta^1\rangle \leq p\mu^{\frac{p}{2}-1}|\eta^1|^2 + M\Lambda|\eta^1|^2, \tag{5.5}$$

where $\Lambda < 0$ is the maximum eigenvalue of $D^2\varphi(\underline{0})$. Hence, (5.2) follows at once provided that $M > M_\mu := p\mu^{\frac{p}{2}-1}|\Lambda|^{-1}$.

In particular, the function F is not convex on $\mathbb{R}^{N\times n}$, since it is not convex with respect to the variable ξ^1. Indeed, if $\bar\eta \in \mathbb{R}^{N\times n}$ is such that $\bar\eta^\alpha = 0$ for $\alpha \in \{2,\dots,N\}$ and $|\bar\eta^1| > 0$ we conclude that

$$\langle D^2F(\underline{0})\bar\eta,\bar\eta\rangle = \langle D^2F_1(\underline{0})\bar\eta^1,\bar\eta^1\rangle \overset{(5.2)}{<} 0\,.$$

Let $\ell > 0$ and set

$$G(\xi) := \ell(\ell + |\xi|^2)^{\frac{q}{2}},$$

and recall that for all $\xi,\ \eta \in \mathbb{R}^{N\times n}$ (cf. (5.4)) being $q \geq 2$

$$\langle D^2G(\xi)\eta,\eta\rangle = q\ell(\ell+|\xi|^2)^{\frac{q}{2}-2}\Big((\ell+|\xi|^2)|\eta|^2 + (q-2)\langle\xi,\eta\rangle^2\Big) \geq q\ell^{\frac{q}{2}}|\eta|^2\,.$$

To show that $f := F + G$ is convex we compute its Hessian, being clearly $f \in C^2(\mathbb{R}^{N\times n})$. We have

$$\begin{aligned}\langle D^2f(\xi)\eta,\eta\rangle &\overset{(5.4)}{\geq} p\mu^{\frac{p}{2}-1}|\eta|^2 + M\langle D^2\varphi(\xi^1)\eta^1,\eta^1\rangle + q\ell^{\frac{q}{2}}|\eta|^2\\ &\geq \big(p\mu^{\frac{p}{2}-1} + q\ell^{\frac{q}{2}}\big)|\eta|^2 - M\sup_{|\xi^1|\leq 1,|z|\leq 1}|\langle D^2\varphi(\xi^1)z,z\rangle||\eta^1|^2\\ &\geq \Big(p\mu^{\frac{p}{2}-1} + q\ell^{\frac{q}{2}} - M\sup_{|\xi^1|\leq 1,|z|\leq 1}|\langle D^2\varphi(\xi^1)z,z\rangle|\Big)|\eta^1|^2 \geq 0\,,\end{aligned}$$

if, for instance, $\ell > \Big(q^{-1}M\sup_{|\xi^1|\leq 1,|z|\leq 1}|\langle D^2\varphi(\xi^1)z,z\rangle|\Big)^{\frac{2}{q}}$.

In conclusion, since f satisfies (1.1)-(1.4), its convexity assures the existence of $W^{1,p}_{\mathrm{loc}}$-local minimizers of the corresponding functional $\mathscr{F}$, which, in view of Theorem 1.1, are locally Hölder continuous.

Example 4

In what follows we construct an example of a convex energy density f satisfying (1.1)-(1.4) with G rank-one convex but not quasiconvex. With this aim we recall next the construction of Šverák's celebrated example in [15] in some details, following the presentation given in the book [13]. With this aim consider

$$L := \left\{\zeta \in \mathbb{R}^{3\times 2} : \zeta = \begin{pmatrix} x & 0\\ 0 & y\\ z & z\end{pmatrix} \text{ where } x,y,z \in \mathbb{R}\right\}, \tag{5.6}$$

and let $h : L \to \mathbb{R}$ be given by

$$h\begin{pmatrix} x & 0\\ 0 & y\\ z & z\end{pmatrix} = -xyz\,.$$

Let $P : \mathbb{R}^{3\times 2} \to L$ be defined as

$$P(\zeta) := \begin{pmatrix} \zeta_1^1 & 0\\ 0 & \zeta_2^2\\ \frac{1}{2}(\zeta_1^3+\zeta_2^3) & \frac{1}{2}(\zeta_1^3+\zeta_2^3)\end{pmatrix},$$

and set

$$g_{\varepsilon,\gamma}(\zeta) := h(P(\zeta)) + \varepsilon|\zeta|^2 + \varepsilon|\zeta|^4 + \gamma|\zeta - P(\zeta)|^2\,.$$

One can prove that there exists $\varepsilon_0 > 0$ such that $g_{\varepsilon,\gamma}$ is not quasiconvex if $\varepsilon \in (0,\varepsilon_0)$ for every $\gamma \geq 0$ (cf. [13, Theorem 5.50, Step 3]). In addition, for every $\varepsilon > 0$ one can find $\gamma(\varepsilon) > 0$ such that $g_{\varepsilon,\gamma(\varepsilon)}$ is rank-one convex (cf. [13, Theorem 5.50, Steps 4, 4' and 4"]).

It is convenient to recall more details of the proof of the rank-one convexity of $g_{\varepsilon,y(\varepsilon)}$. To begin with, since h is a homogeneous polynomial of degree three we have

$$\langle D^2 h(P(\zeta))z, z\rangle \geq -\vartheta|\zeta||z|^2 \tag{5.7}$$

for some $\vartheta > 0$ and for all $\zeta,\ z \in \mathbb{R}^{3\times 2}$. It turns then out that

$$\begin{aligned}\langle D^2 g_{\varepsilon,y}(\zeta)z, z\rangle &= \langle D^2 h(P(\zeta))z, z\rangle \\ &\quad + 2\varepsilon|z|^2 + 4\varepsilon|\zeta|^2|z|^2 + 8\varepsilon\langle\zeta, z\rangle^2 + 2y|z - P(z)|^2.\end{aligned}$$

for all $\zeta,\ z \in \mathbb{R}^{3\times 2}$. In particular, we conclude that for all $\varepsilon > 0$ and $y \geq 0$

$$\langle D^2 g_{\varepsilon,y}(\zeta)z, z\rangle \geq (4\varepsilon|\zeta| - \vartheta)|\zeta||z|^2 \geq \frac{\vartheta + 1}{4\varepsilon}|z|^2 \tag{5.8}$$

for all $z \in \mathbb{R}^{3\times 2}$ provided that $\zeta \in \mathbb{R}^{3\times 2}$ is such that $|\zeta| \geq \frac{\vartheta+1}{4\varepsilon}$. Note that the last inequality holds true independently of the fact that $\operatorname{rank}(z) = 1$. Therefore, the uniform convexity of $g_{\varepsilon,y}$ on $\mathbb{R}^{3\times 2} \setminus B_{\frac{\vartheta+1}{4\varepsilon}}$ follows (cf. [13, Theorem 5.50, Step 4']). The appropriate choice of $y(\varepsilon)$ establishes the rank-one convexity of $g_{\varepsilon,y(\varepsilon)}$ on the bounded set $B_{\frac{\vartheta+1}{4\varepsilon}}$.

We set $g_\varepsilon := g_{\varepsilon,y(\varepsilon)}$, for $\varepsilon \in (0, \varepsilon_0)$, in a way that g_ε is rank-one convex but not quasiconvex.

Let $n \geq 2$ and $N \geq 3$, let $\pi : \mathbb{R}^{N\times n} \to \mathbb{R}^{3\times 2}$ be the projection

$$\pi(\xi) = \begin{pmatrix} \xi_1^1 & \xi_2^1 \\ \xi_1^2 & \xi_2^2 \\ \xi_1^3 & \xi_2^3 \end{pmatrix},$$

and set

$$G_\varepsilon(\xi) := g_\varepsilon(\pi(\xi))$$

where $g_\varepsilon : \mathbb{R}^{3\times 2} \to \mathbb{R}$ is defined above. Then G_ε is rank-one convex and not quasiconvex (cf. [13, Theorem 5.50, Step 1]).

Let $\mu > 0$ and $F_\alpha(\lambda) := (\mu + |\lambda|^2)^{\frac{p}{2}}$ for all $\lambda \in \mathbb{R}^n$ and $\alpha \in \{1, \ldots, N\}$. We claim that the function

$$f(\xi) := \sum_{\alpha=1}^{N} F_\alpha(\xi^\alpha) + G_\varepsilon(\xi)$$

is convex for $\mu \geq \mu_\varepsilon > 0$ large enough. Given this for granted, f satisfies (1.1)-(1.4) with $q = 4$ and $p \in (2\sqrt{n}, n)$ if $n \geq 5$. Therefore, we conclude in view of Theorem 1.1 that the $W^{1,p}_{\mathrm{loc}}$-local minimizers of the corresponding functional $\mathscr{F}$ are locally Hölder continuous.

To prove the claim, since $f \in C^2(\mathbb{R}^{N\times n})$ we shall compute its Hessian. First note that $F(\xi) := \sum_{\alpha=1}^N F_\alpha(\xi^\alpha)$ is uniformly convex on $\mathbb{R}^{N\times n}$ in view of (5.4), which, together with (5.8), yields

$$\langle D^2 f(\xi)\eta, \eta\rangle \geq 0 \tag{5.9}$$

for all $\xi \in \mathbb{R}^{N\times n}$ such that $|\pi(\xi)| \geq \frac{\vartheta+1}{4\varepsilon}$ and for all $\eta \in \mathbb{R}^{N\times n}$. In addition, using again that $p > 2$, by (5.7) we have

$$\langle D^2 f(\xi)\eta, \eta\rangle \geq \langle D^2 F(\xi)\eta, \eta\rangle - \vartheta|\pi(\xi)||\pi(\eta)|^2 \geq \big(p\,\mu^{\frac{p}{2}-1} - \vartheta|\pi(\xi)|\big)|\pi(\eta)|^2. \tag{5.10}$$

Hence, the Hessian of f at ξ with $|\pi(\xi)| < \frac{\vartheta+1}{4\varepsilon}$ is non-negative provided that

$$\mu \geq \mu_\varepsilon := \left(\frac{\vartheta(\vartheta + 1)}{4p\varepsilon}\right)^{\frac{2}{p-2}}. \tag{5.11}$$

Example 5

Finally, we give an example that exploits the full strength of the assumptions of Theorem 1.1 on the leading term. By keeping the notation introduced in Example 4, we shall modify F_1 there to get a non-convex function so that the resulting principal term $\widetilde{F}$ is non-convex. On the other hand, the sum $\widetilde{F}+G_\varepsilon$ turns out to be convex exploiting the uniform convexity of G_ε on the subspace L for large values of the variable $\pi(\xi)$ (cf. (5.6) and (5.8)).

Consider a function $\varphi:\mathbb{R}^n\to[0,2]$, $\varphi\in C_c^\infty(B_3)$, with $\varphi(\underline{0})=2$,

$$D^2\varphi(\underline{0})=-2\,\mathrm{Id}_{n\times n} \tag{5.12}$$

and

$$\sup_{x\in\mathbb{R}^n}\sup_{z\in\mathbb{R}^n,\,|z|\le1}|\langle D^2\varphi(x)z,z\rangle|=\sup_{z\in\mathbb{R}^n,\,|z|\le1}|\langle D^2\varphi(\underline{0})z,z\rangle|=2=|\Lambda| \tag{5.13}$$

where $\Lambda:=-2<0$ is the (unique) eigenvalue of $D^2\varphi(\underline{0})$ (see Lemma 5.1 below for the existence of such a function φ). Let

$$\widetilde{F}(\xi):=\sum_{\alpha=1}^N\widetilde{F}_\alpha(\xi^\alpha)$$

where

$$\widetilde{F}_\alpha=F_\alpha \text{ for } \alpha\in\{2,\dots,N\}, \text{ and } \widetilde{F}_1:=F_1+M\,\varphi\circ\sigma,$$

$M>0$ to be chosen in what follows, and $\sigma:\mathbb{R}^n\to\mathbb{R}^n$ defined by

$$\sigma(\xi^1):=(\xi^1_1,0,\dots,0)\,.$$

Note that $\widetilde{F}\in C^2(\mathbb{R}^{N\times n})$. In particular, $\widetilde{F}_1=F_1$ for all $\xi\in\mathbb{R}^{N\times n}\setminus\sigma^{-1}(\mathrm{supp}\varphi)$, and for such points $D^2\widetilde{F}_1(\xi^1)=D^2F_1(\xi^1)$. Moreover, it is possible to find $M_\mu>0$ such that for every $M>M_\mu$ and for some $\bar{\eta}^1\in\mathbb{R}^n\setminus\{\underline{0}\}$ (independent of M)

$$\langle D^2\widetilde{F}_1(\underline{0})\bar{\eta}^1,\bar{\eta}^1\rangle<0. \tag{5.14}$$

Indeed, arguing as to obtain (5.5), and using (5.12), for all $\bar{\eta}^1\in\mathbb{R}^n$ such that $|\bar{\eta}^1|=|\sigma(\bar{\eta}^1)|>0$ we get

$$\langle D^2\widetilde{F}_1(\underline{0})\bar{\eta}^1,\bar{\eta}^1\rangle\le\big(p\mu^{\frac{p}{2}-1}+\Lambda M\big)|\sigma(\bar{\eta}^1)|^2<0, \tag{5.15}$$

provided that

$$M>M_\mu:=p\mu^{\frac{p}{2}-1}|\Lambda|^{-1}. \tag{5.16}$$

In particular, the function $\widetilde{F}$ is not convex on $\mathbb{R}^{N\times n}$, since it is not convex with respect to the variable ξ^1. Indeed, if $\bar{\eta}\in\mathbb{R}^{N\times n}$ is such that $\bar{\eta}^\alpha=0$ for $\alpha\in\{2,\dots,N\}$ and $|\bar{\eta}^1|=|\sigma(\bar{\eta}^1)|>0$ we conclude that

$$\langle D^2\widetilde{F}(\underline{0})\bar{\eta},\bar{\eta}\rangle=\langle D^2\widetilde{F}_1(\underline{0})\bar{\eta}^1,\bar{\eta}^1\rangle\overset{(5.15)}{<}0\,.$$

For fix $\varepsilon\in(0,\varepsilon_0)$ consider $z_\varepsilon\in\mathbb{R}^n$ given by $z_\varepsilon:=\big(\frac{\vartheta+1}{4\varepsilon}+3,0,\dots,0\big)$.

Let G_ε be the rank-one convex, non-quasiconvex function introduced in Example 4, we claim that the integrand

$$\widetilde{f}(\xi):=\widetilde{F}(\xi)+G_\varepsilon(\xi^1-z_\varepsilon,\xi^2,\dots,\xi^N)=\sum_{\alpha=1}^N(\mu+|\xi^\alpha|^2)^{\frac{p}{2}}+M\varphi(\sigma(\xi^1))+G_\varepsilon(\xi^1-z_\varepsilon,\xi^2,\dots,\xi^N)$$

is convex for all $\mu\ge\mu_\varepsilon=\big(\frac{\vartheta(\vartheta+1)}{4p\varepsilon}\big)^{\frac{2}{p-2}}$ and for all $M\le M_\mu+|\Lambda|^{-1}\frac{\vartheta+1}{4\varepsilon}$ (the value of μ_ε has been introduced in (5.11)).

With this aim, it suffices to check the Hessian of $\widetilde{f}$ being $\widetilde{f}\in C^2(\mathbb{R}^{N\times n})$. First note that $\widetilde{f}$ coincides with a variant of the function f in Example 3 on the open set $\Sigma:=\{\xi\in\mathbb{R}^{N\times n}:\ \sigma(\xi^1)\notin\mathrm{supp}\varphi\}$. More precisely, if $\xi\in\Sigma$

$$\widetilde{f}(\xi)=F(\xi)+G_\varepsilon(\xi^1-z_\varepsilon,\xi^2,\dots,\xi^N)\,.$$

Then its convexity for all $\xi \in \mathbb{R}^{N\times n}$ such that $|(\xi^1 - z_\varepsilon, \xi^2, \dots, \xi^N)| \geq \frac{\vartheta+1}{4\varepsilon}$ follows at once from (5.8) and (5.4). Instead, if $|(\xi^1 - z_\varepsilon, \xi^2, \dots, \xi^N)| < \frac{\vartheta+1}{4\varepsilon}$, arguing as in (5.10), we get

$$\langle D^2\widetilde{f}(\xi)\eta, \eta\rangle \geq \langle D^2 F(\xi)\eta, \eta\rangle - \vartheta|\pi(\xi^1 - z_\varepsilon, \xi^2, \dots, \xi^N)||\pi(\eta)|^2 \overset{(5.4)}{\geq} \Big(p\,\mu^{\frac{p}{2}-1} - \vartheta\frac{\vartheta+1}{4\varepsilon}\Big)|\pi(\eta)|^2 \geq 0$$

thanks to the choice $\mu \geq \mu_\varepsilon$.

On the other hand, by the convexity of each F_α with respect to ξ^α (cf. (5.3)) and by taking into account that $\sigma(\xi^1) \in \operatorname{supp}\varphi \subset B_3$ yields $|\pi(\xi^1 - z_\varepsilon)| \geq |\xi^1_1 - \frac{\vartheta+1}{4\varepsilon} - 3| \geq \frac{\vartheta+1}{4\varepsilon}$, we have

$$\begin{aligned}
\langle D^2\widetilde{f}(\xi)\eta, \eta\rangle &\overset{(5.4)}{\geq} p\mu^{\frac{p}{2}-1}|\eta|^2 + M\langle D^2(\varphi\circ\sigma)(\xi^1)\eta^1, \eta^1\rangle + \langle D^2 G_\varepsilon(\xi^1 - z_\varepsilon, \xi^2, \dots, \xi^N)\eta, \eta\rangle \\
&= p\mu^{\frac{p}{2}-1}|\eta|^2 + M\langle D^2\varphi(\sigma(\xi^1))\sigma(\eta^1), \sigma(\eta^1)\rangle + \langle D^2 G_\varepsilon(\xi^1 - z_\varepsilon, \xi^2, \dots, \xi^N)\eta, \eta\rangle \\
&\overset{(5.8)}{\geq} p\mu^{\frac{p}{2}-1}|\eta|^2 - M\sup_{|\sigma(\xi^1)|\leq 3,\ |z|\leq 1}|\langle D^2\varphi(\sigma(\xi^1))z, z\rangle||\sigma(\eta^1)|^2 + \frac{\vartheta+1}{4\varepsilon}|\pi(\eta)|^2 \\
&\geq \Big(p\mu^{\frac{p}{2}-1} - M\sup_{|\sigma(\xi^1)|\leq 3,\ |z|\leq 1}|\langle D^2\varphi(\sigma(\xi^1))z, z\rangle| + \frac{\vartheta+1}{4\varepsilon}\Big)|\sigma(\eta^1)|^2 .
\end{aligned}$$

Thus, the Hessian of $\widetilde{f}$ at such ξ's is nonnegative provided that

$$p\mu^{\frac{p}{2}-1} + \frac{\vartheta+1}{4\varepsilon} \geq M\sup_{|\sigma(\xi^1)|\leq 3,\ |z|\leq 1}|\langle D^2\varphi(\sigma(\xi^1))z, z\rangle| .$$

In conclusion, we have to ensure the following two inequalities

$$M_\mu = p\mu^{\frac{p}{2}-1}|\Lambda|^{-1} < M \leq \Big(\sup_{|\sigma(\xi^1)|\leq 3,\ |z|\leq 1}|\langle D^2\varphi(\sigma(\xi^1))z, z\rangle|\Big)^{-1}\Big(p\mu^{\frac{p}{2}-1} + \frac{\vartheta+1}{4\varepsilon}\Big). \tag{5.17}$$

Since by (5.13)

$$\sup_{|\sigma(\xi^1)|\leq 3,\ |z|\leq 1}|\langle D^2\varphi(\sigma(\xi^1))z, z\rangle| = |\Lambda| ,$$

then (5.17) holds for every M such that $M_\mu < M \leq M_\mu + |\Lambda|^{-1}\frac{\vartheta+1}{4\varepsilon}$.

In conclusion, since $\widetilde{f}$ satisfies (1.1)-(1.4) with $q = 4$ and $p \in (2\sqrt{n}, n)$ if $n \geq 5$, its convexity assures the existence of $W^{1,p}_{\mathrm{loc}}$-local minimizers of the corresponding functional $\widetilde{\mathscr{F}}$, which, in view of Theorem 1.1, are locally Hölder continuous.

Lemma 5.1. *There exists a function $\varphi : \mathbb{R}^n \to [0, 2]$, $\varphi \in C^\infty_c(B_3)$, with $\varphi(\underline{0}) = 2$,*

$$D^2\varphi(\underline{0}) = -2\,\mathrm{Id}_{n\times n}$$

and

$$\sup_{x\in\mathbb{R}^n}\ \sup_{\eta\in\mathbb{R}^n,\ |\eta|\leq 1}|\langle D^2\varphi(x)\eta, \eta\rangle| = \sup_{\eta\in\mathbb{R}^n,\ |\eta|\leq 1}|\langle D^2\varphi(\underline{0})\eta, \eta\rangle| = 2.$$

Proof. Define $\phi : \mathbb{R} \to [0, \infty)$,

$$\phi(t) := \begin{cases} (t+2)^2 & \text{if } t \in [-2, -1] \\ 2 - t^2 & \text{if } t \in (-1, 1) \\ (t-2)^2 & \text{if } t \in [1, 2] \\ 0 & \text{elsewhere.} \end{cases}$$

We have that $\phi \in C^{1,1}(\mathbb{R})$, $\phi \in C^\infty(\mathbb{R} \setminus \{-2, -1, 1, 2\})$ and

$$\max\Big\{|\phi''(t)|, \Big|\frac{\phi'(t)}{t}\Big|\Big\} \leq 2 \qquad \forall t \in \mathbb{R} \setminus \{-2, -1, 0, 1, 2\}. \tag{5.18}$$

Let us define $\Phi : \mathbb{R}^n \to [0, 2]$, by $\Phi(x) := \phi(|x|)$. Then $\Phi \in C^2(\mathbb{R}^n \setminus \{x : |x| \in \{1, 2\}\})$,

$$D\Phi(\underline{0}) = \underline{0}, \qquad D\Phi(x) = \phi'(|x|)\frac{x}{|x|} \qquad \text{if } |x| \neq 0,$$

and

$$D^2\Phi(x) = \frac{\phi'(|x|)}{|x|}\,\mathrm{Id}_{n\times n} + \left(\phi''(|x|) - \frac{\phi'(|x|)}{|x|}\right)\frac{x}{|x|}\otimes\frac{x}{|x|} \qquad \text{if } |x| \neq 0,1,2$$

and $D^2\Phi(\underline{0}) = \phi''(0)\,\mathrm{Id}_{n\times n}$.

In particular, $\phi''(0)$ is the only eigenvalue of $D^2\Phi(\underline{0})$. Moreover, we claim that if $|x| \neq 0$ then the eigenvalues of $D^2\Phi(x)$ are $\frac{\phi'(|x|)}{|x|}$ and $\phi''(|x|)$. Indeed, if $|x| \neq 0$,

$$D^2\Phi(x)v = \frac{\phi'(|x|)}{|x|}v$$

for every $v \in \mathbb{R}^n$, $v \perp \frac{x}{|x|}$; moreover, if $|x| \neq 0, 1, 2$,

$$D^2\Phi(x)x = \phi''(|x|)x$$

Therefore, using (5.18), if $|x| \notin \{0,1,2\}$

$$\sup_{\eta\in\mathbb{R}^n,\,|\eta|\le1} |\langle D^2\Phi(x)\eta,\eta\rangle| \le \max\left\{|\phi''(|x|)|, \frac{|\phi'(|x|)|}{|x|}\right\} \le 2,$$

that, taking into account that $\sup_{|\eta|\le1} |\langle D^2\Phi(\underline{0})\eta,\eta\rangle| = |\phi''(0)| = 2$, implies

$$\operatorname*{ess\text{-}sup}_{x\in\mathbb{R}^n} \sup_{\eta\in\mathbb{R}^n,\,|\eta|\le1} |\langle D^2\Phi(x)\eta,\eta\rangle| = 2 = |\phi''(0)|.$$

Let us now consider a family of positive radial symmetric mollifiers $\rho_\varepsilon \in C_c^\infty(B_\varepsilon)$, $\varepsilon \in (0,1)$, such that $\int_{\mathbb{R}^n} \rho_\varepsilon(x)\,dx = 1$. Consider $\Phi_\varepsilon := \Phi \star \rho_\varepsilon$. It is easy to check that $\Phi_\varepsilon \in C_c^\infty(B_{2+\varepsilon})$ and, since $\Phi \in C^{1,1}(\mathbb{R}^n)$, for all $x \in \mathbb{R}^n$ we get

$$D^2\Phi_\varepsilon(x) = (D^2\Phi \star \rho_\varepsilon)(x).$$

Moreover, for all $x,\ \eta \in \mathbb{R}^n$ it holds

$$\langle D^2\Phi_\varepsilon(x)\eta,\eta\rangle = \int_{\mathbb{R}^n} \langle D^2\Phi(x-y)\eta,\eta\rangle\rho_\varepsilon(y)\,dy\,,$$

then we have

$$\sup_{\eta\in\mathbb{R}^n,\,|\eta|\le1} |\langle D^2\Phi_\varepsilon(x)\eta,\eta\rangle| \le \int_{\mathbb{R}^n} \sup_{\eta\in\mathbb{R}^n,\,|\eta|\le1} |\langle D^2\Phi(x-y)\eta,\eta\rangle|\rho_\varepsilon(y)\,dy \le 2.$$

Since $D^2\Phi(x) = -2\,\mathrm{Id}_{n\times n}$ if $|x| < 1$, there exists $\varepsilon_0 \in (0,1)$ small enough so that $D^2\Phi_{\varepsilon_0}(\underline{0}) = -2\,\mathrm{Id}_{n\times n}$. Thus,

$$\sup_{x\in\mathbb{R}^n} \sup_{\eta\in\mathbb{R}^n,\,|\eta|\le1} |\langle D^2\Phi_{\varepsilon_0}(x)\eta,\eta\rangle| = \sup_{\eta\in\mathbb{R}^n,\,|\eta|\le1} |\langle D^2\Phi_{\varepsilon_0}(\underline{0})\eta,\eta\rangle| = 2.$$

The conclusion then follows on setting $\varphi := \Phi_{\varepsilon_0}$. □

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