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# FABER-KRAHN AND LIEB-TYPE INEQUALITIES FOR THE COMPOSITE MEMBRANE PROBLEM

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ABSTRACT. The classical Faber-Krahn inequality states that, among all domains with given measure, the ball has the smallest first Dirichlet eigenvalue of the Laplacian. Another inequality related to the first eigenvalue of the Laplacian has been proved by Lieb in 1983 and it relates the first Dirichlet eigenvalues of the Laplacian of two different domains with the first Dirichlet eigenvalue of the intersection of translations of them. In this paper we prove the analogue of Faber-Krahn and Lieb inequalities for the composite membrane problem.

## 1. INTRODUCTION

The composite membrane problem is an eigenvalue optimization problem that received a considerable attention starting from the works of Chanillo and al. [3–6, 16]. In physical terms the problem can be stated as follows: build a membrane of prescribed shape and mass using materials of varying densities, in such a way that the basic frequency is the smallest possible. As shown in [3] and [4] the composite membrane problem can be considered as a special instance of a more general eigenvalue optimization problem, which we are going to introduce in  $\mathbb{R}^n$ ,  $n \geq 2$ , keeping in mind that the physically relevant case is  $n = 2$ .

Let  $\Omega \subset \mathbb{R}^n$  be a non-empty open, bounded and connected set with Lipschitz boundary  $\partial\Omega$ . For every  $A \in [0, |\Omega|]$ , we denote

$$\mathcal{D} := \{D \subset \Omega : D \text{ measurable set, } |D| = A\}$$

the class of *admissible sets* and for any set  $D \in \mathcal{D}$ , let  $\chi_D$  be its characteristic function. For every  $\alpha > 0$  and  $D \in \mathcal{D}$  we consider

$$\begin{cases} -\Delta u + \alpha \chi_D u = \lambda u & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

The lowest eigenvalue of this problem is denoted  $\lambda_\Omega(\alpha, D)$ . The eigenvalue optimization problem given by the infimum of  $D \mapsto \lambda_\Omega(\alpha, D)$  can be also considered, i.e.,

$$\Lambda_\Omega(\alpha, A) := \inf_{D \in \mathcal{D}} \lambda_\Omega(\alpha, D). \quad (1.2)$$

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A variational characterization of  $\Lambda_\Omega(\alpha, A)$  is given by

$$\Lambda_\Omega(\alpha, A) = \inf_{0 \neq u \in H_0^1(\Omega), D \in \mathcal{D}} \frac{\int_\Omega |\nabla u|^2 + \alpha \int_\Omega \chi_D u^2}{\int_\Omega u^2}.$$

Any minimizer  $D$  in (1.2) is called an *optimal configuration* for the data  $(\Omega, \alpha, A)$ . If moreover  $u$  satisfies (1.1) then  $(u, D)$  is called an *optimal pair*. Due to the variational characterization of (1.2), changing  $D$  by a set of measure zero does not affect  $\Lambda_\Omega(\alpha, D)$  nor  $u$ . Therefore we consider sets  $D$  that differs by a null-set as equal.

In this context, uniqueness of optimal pairs is a delicate issue and cannot concern the function  $u$ . Indeed, if  $(u, D)$  is an optimal pair, then for every constant  $c \neq 0$ , also  $(cu, D)$  is an optimal pair of the same problem. Therefore, it is interesting to look only for uniqueness of optimal configurations  $D$ . Nevertheless, this cannot be expected in general, as shown in [3, Theorem 7], but on balls there is a unique optimal configuration, see [3, Corollary 5]. Moreover, if  $u$  solves (1.1) then, denoted by  $u^*$  ( $u_\star$ ) its Schwarz decreasing (increasing) symmetrization, by  $\Omega^*$  the ball centered at the origin and volume  $|\Omega^*| = |\Omega|$  and by  $D_\star$  the set defined through its characteristic function,

$$\chi_{D_\star}(x) := (\chi_D)_\star(x), \quad x \in \Omega^*, \quad (1.3)$$

then  $(u^*, D_\star)$  is an optimal pair of (1.2) on  $\Omega^*$ , see the proof of [3, Theorem 4]. We stress that quite simple computations show that the set  $D_\star$  is an annuli containing the boundary, see [3, Corollary 5] and Remark 2.7.

When  $\alpha = 0$  (or  $A = 0$ ) a solution  $u \in H_0^1(\Omega)$  of (1.1) is an eigenfunction of  $-\Delta$ , and  $\lambda(\Omega)$  is the first Dirichlet eigenvalue of  $-\Delta$ . It is well known, see e.g. [9, 14], that the balls minimize  $\Omega \mapsto \lambda(\Omega)$  among all the sets of given measure. This result is known in the literature as *Faber-Krahn inequality*. We refer to [10, 13] for a proof and related results. In particular, since  $\lambda(\Omega)|\Omega|^{2/n}$  is invariant under dilation, we have that

$$\lambda(\Omega) \geq \beta_n |\Omega|^{-2/n}, \quad (1.4)$$

where  $\beta_n$  is the lowest eigenvalue of a ball of unit volume.

In this note we address a similar problem for the problem (1.2) with suitable constraints on the parameter  $\alpha > 0$ . To state our result, we need to define the constant  $\bar{\alpha}_\Omega(A)$ . Given  $\Omega$  and  $A \in [0, |\Omega|)$ , there exists a unique positive number, denoted by  $\bar{\alpha}_\Omega(A)$ , such that

$$\Lambda_\Omega(\bar{\alpha}_\Omega(A), A) = \bar{\alpha}_\Omega(A),$$

see Section 2. Notice that in the interval  $(0, \bar{\alpha}_\Omega(A))$ , where  $\alpha$  takes its values, problem (1.2) and the composite membrane problem are in one-to-one correspondence. We refer to Section 2 for more details.

**Theorem 1.1.** *Given a ball  $\Omega^*$  centered at the origin,  $A \in (0, |\Omega^*|)$  and  $\alpha \in (0, \bar{\alpha}_{\Omega^*}(A))$ , then*

$$\Lambda_{\Omega^*}(\alpha, A) \leq \Lambda_\Omega(\alpha, A)$$

*for every open and bounded connected set  $\Omega \subset \mathbb{R}^n$  with Lipschitz boundary, with  $|\Omega| = |\Omega^*|$ . Moreover, the equality holds if and only if  $\Omega = \Omega^*$  up to translations.*

To prove Theorem 1.1 we adapt to our setting the proof of the classical Faber-Krahn inequality due to Kesavan [12], which relies on a well known result by Talenti, see Theorem 2.3 in Section 2. We stress that, in our situation, we need to pay attention to the presence

of the infimum among all the sets  $D \subset \Omega$  of given measure  $|D| = A$ . The crucial ingredient of the proof is Proposition 3.3, which provides an explicit expression of the Schwarz symmetrization of a function on which we apply the above mentioned theorem of Talenti.

Recently, in [7, 8], some of the results proved for the composite problem in [3] have been extended to the fourth-order case of the composite plate problem. It would be interesting to address Faber-Krahn-type problems in that context as well.

Our second result is a Lieb-type inequality for the intersection of two composite membranes, see Theorem 1.2. To state it precisely, we have to introduce the following notation: for every non-empty set  $\Omega \subset \mathbb{R}^n$  and for every  $x \in \mathbb{R}^n$ , we denote the *translated of  $\Omega$  by  $x$*  as

$$\Omega_x := \Omega + x = \{y \in \mathbb{R}^n : y = z + x, z \in \Omega\}. \quad (1.5)$$

Our result extends to the case  $\alpha > 0$  in (1.1) a result proved by Lieb [15, Theorem 1] for the first Dirichlet eigenvalue of  $-\Delta$ . Roughly speaking, Lieb's Theorem states that given two open and bounded sets  $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ , if we denote by  $\lambda_{\Omega_1}, \lambda_{\Omega_2}$  be the lowest eigenvalue of  $-\Delta$  with Dirichlet boundary conditions, then there exists a set  $\Sigma \subseteq \mathbb{R}^n$ ,  $|\Sigma| > 0$ , such that for a.e.  $x \in \Sigma$ , it is  $|\Omega_1 \cap \Omega_{2,x}| > 0$ , and

$$\lambda_{\Omega_1 \cap \Omega_{2,x}} < \lambda_{\Omega_1} + \lambda_{\Omega_2}, \quad \text{for every } x \in \Sigma.$$

Even more, since the map  $x \mapsto \lambda_{\Omega_1 \cap \Omega_{2,x}}$  is upper semicontinuous, the set  $\Sigma$  is open.

Our result in this context reads as follows:

**Theorem 1.2.** *Let  $\Omega_1, \Omega_2 \subset \mathbb{R}^n$  be two open connected and bounded sets in  $\mathbb{R}^n$  with Lipschitz boundary. Fixed  $A_i \in [0, |\Omega_i|]$  and  $\alpha_i > 0$ ,  $i \in \{1, 2\}$ , let  $D_1$  and  $D_2$  be optimal configurations for  $(\Omega_1, \alpha_1, A_1)$  and  $(\Omega_2, \alpha_2, A_2)$ , respectively. Then there exists a set  $\Sigma \subseteq \mathbb{R}^n$ ,  $|\Sigma| > 0$ , such that for a.e.  $x \in \Sigma$  it is  $|\Omega_1 \cap \Omega_{2,x}| > 0$  and, for every  $\alpha \in (0, \alpha_1 + \alpha_2]$  and for every  $A \in [0, |D_1 \cap D_{2,x}|]$ , we have*

$$\Lambda_{\Omega_1 \cap \Omega_{2,x}}(\alpha, A) < \Lambda_{\Omega_1}(\alpha_1, A_1) + \Lambda_{\Omega_2}(\alpha_2, A_2).$$

We point out that both in Theorem 1.1 and in Theorem 1.2 we required the boundaries of  $\Omega, \Omega_1$  and  $\Omega_2$  to be Lipschitz continuous. Even if this assumption does not explicitly play a role along the proofs of our results, we must ask for it because it guarantees the existence of optimal pairs, see [3, Theorem 1]. Notice that this regularity on the sets is not necessary neither for the Faber-Krahn inequality nor the Lieb inequality for the Laplacian  $-\Delta$ .

We finally remark that sort of reversed Faber-Krahn and Lieb inequality have been recently proved for the first eigenvalue of a degenerate operator called *truncated Laplacian*, see [2].

The paper is organized as follows: in Section 2 we recall basic facts on Steiner/Schwarz rearrangements and the composite membrane problem studied in [3]. In Section 3 we prove a few technical results that are needed in Section 4, where we prove Theorem 1.1. Finally, in Section 5 we prove Theorem 1.2.

## 2. PRELIMINARIES

The first part of this section is devoted to a brief summary of the definitions and the basic properties of Steiner and Schwarz rearrangements needed for the proof of Theorem 1.1. We refer to the monographs [11, 13] and the references therein for a more comprehensive introduction to the subject. The second part of this section contains part of the results proved in [3] on the composite membrane problem.

Let  $\Omega \subset \mathbb{R}^n$  be a measurable set with finite  $n$ -dimensional Lebesgue measure  $|\Omega| < +\infty$ . We denote by  $\Omega^\star$  the open ball centered at the origin and measure  $|\Omega^\star| = |\Omega|$ . We also denote by  $\omega_n$  the measure of the unit ball.

Let  $u : \Omega \rightarrow \mathbb{R}$  be a measurable function. The *distribution function*  $\mu_u : \mathbb{R} \rightarrow \mathbb{R}$  of  $u$  is defined as

$$\mu_u(\tau) := |\{x \in \Omega : u(x) > \tau\}|, \quad (2.1)$$

whose image is in  $[0, |\Omega|]$ . To simplify the notation, in the following we will write  $\{u > \tau\}$  in place of  $\{x \in \Omega : u(x) > \tau\}$ , and similarly for (sub-)level sets.

The *decreasing Steiner rearrangement*  $u^\sharp : [0, |\Omega|] \rightarrow \mathbb{R}$  of  $u$  is defined as

$$u^\sharp(s) := \begin{cases} \text{ess sup } u, & s = 0, \\ \inf \{\tau : \mu_u(\tau) < s\}, & 0 < s \leq |\Omega|. \end{cases} \quad (2.2)$$

The *increasing Steiner rearrangement*  $u_\sharp : [0, |\Omega|] \rightarrow \mathbb{R}$  of  $u$  is defined as

$$u_\sharp(s) := \begin{cases} \text{ess sup } u, & s = |\Omega|, \\ \inf \{\tau : |\{u < \tau\}| > s\}, & 0 \leq s < |\Omega|. \end{cases}$$

*Remark 2.1.* The function  $u^\sharp$  is left-continuous, see [13, Proposition 1.1.1].

The *decreasing Schwarz symmetrization*  $u^\star : \Omega^\star \rightarrow \mathbb{R}$  of  $u$  is defined as

$$u^\star(x) := u^\sharp(\omega_n \|x\|^n), \quad x \in \Omega^\star, \quad (2.3)$$

and the *increasing Schwarz symmetrization*  $u_\star : \Omega^\star \rightarrow \mathbb{R}$  of  $u$  is defined as

$$u_\star(x) := u_\sharp(\omega_n \|x\|^n), \quad x \in \Omega^\star.$$

It follows from the previous definitions that the increasing and decreasing Steiner rearrangements are related as follows:

$$u^\sharp(s) = -(-u)_\sharp(s) \quad \text{and} \quad u_\sharp(s) = -(-u)^\sharp(s), \quad \text{for a.e. } s \quad (2.4)$$

and the analogous relations hold for the Schwarz symmetrizations as well.

We recall also the *Hardy-Littlewood inequality*, which follows combining [13, Corollary 1.4.1, Equation (1.3.3)] and (2.4):

**Proposition 2.2.** *Let  $f \in L^p(\Omega)$  and  $g \in L^q(\Omega)$  with  $\frac{1}{p} + \frac{1}{q} = 1$ . Then*

$$\int_{\Omega} f(x)g(x) dx \geq \int_{\Omega^\star} f_\star(x)g^\star(x) dx.$$

Due to its importance in the proof of Theorem 1.1, we recall here the following result due to Talenti, see [17]:

**Theorem 2.3.** *Let  $\Omega \subset \mathbb{R}^n$  be an open set and let  $\Omega^\star$  be the ball centered at the origin and measure  $|\Omega^\star| = |\Omega|$ . Let  $f \in L^2(\Omega)$  and let  $f^\star$  be its Schwarz symmetrization. If  $u \in H_0^1(\Omega)$  is a weak solution of*

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

*and  $v \in H_0^1(\Omega^\star)$  is a weak solution of*

$$\begin{cases} -\Delta v = f^\star, & \text{in } \Omega^\star, \\ v = 0, & \text{on } \partial\Omega^\star, \end{cases}$$

*then  $v(x) \geq u^\star(x)$  for almost every  $x \in \Omega^\star$ .*

*Remark 2.4.* The equality case in the above theorem is particularly interesting because it yields rigidity results. In particular, it holds that for  $f \geq 0$ , if  $u^* = v$  a.e. then  $\Omega$  must be a ball. We refer to [1, Theorem 1] and [13, Proposition 3.2.2] for a proof.

We end this section with a brief recap on the optimization eigenvalue problem (1.2). We refer to [3] for proofs and more results. The next theorem condensates parts of the content of [3, Theorem 1, Theorem 2, Proposition 10].

**Theorem 2.5** (Chanillo-Grieser-Imai-Kurata-Ohnishi, [3]). *For any  $\alpha > 0$  and  $A \in [0, |\Omega|]$  there exists an optimal pair  $(u, D)$  with  $u$  positive. Moreover, every optimal pair satisfies the following properties:*

- (i)  $u \in H^2(\Omega) \cap C^{1,\delta}(\Omega) \cap C^\gamma(\overline{\Omega})$ , for some  $\gamma > 0$  and every  $\delta < 1$ .
- (ii) There exists a positive number  $t = t(A, \Omega, u) > 0$  such that

$$D = \{x \in \Omega : u(x) \leq t\}.$$

- (iii)  $D$  contains a tubular neighborhood of the boundary  $\partial\Omega$  of  $\Omega$ .
- (iv)  $\Lambda_\Omega(\alpha, \cdot)$  is strictly increasing for fixed  $\alpha > 0$ , and the function  $\Lambda_\Omega(\cdot, A)$  is strictly increasing for fixed  $A > 0$ . Moreover,  $\Lambda_\Omega(\alpha, A) - \alpha$  is strictly decreasing in  $\alpha$  for any fixed  $A \in (0, |\Omega|)$ .
- (v) Given  $A \in [0, |\Omega|]$ , there exists a unique positive number  $\overline{\alpha}_\Omega(A) > 0$  such that

$$\Lambda_\Omega(\overline{\alpha}_\Omega(A), A) = \overline{\alpha}_\Omega(A).$$

- (vi) If  $\Omega$  is simply connected and  $\alpha < \overline{\alpha}_\Omega(A)$ , then  $D$  is connected.

*Remark 2.6.* The positive number  $t > 0$  appearing in (ii) is defined as

$$t = t(A, \Omega, u) := \sup\{s : |\{u < s\}| < A\}.$$

We stress that the number  $\overline{\alpha}_\Omega(A)$  appearing in (v) is well defined due to (iv) of Theorem 2.5. An immediate consequence of (iv) and (v) is that

$$\Lambda_\Omega(\alpha, A) - \alpha > 0, \quad \text{for every } \alpha < \overline{\alpha}_\Omega(A). \quad (2.5)$$

*Remark 2.7.* If  $D \subseteq \mathbb{R}^n$ , then

$$(\chi_D)_\#(s) := \inf\{\tau : |\{\chi_D < \tau\}| > s\}.$$

Therefore, if  $D \subset \Omega \subseteq \mathbb{R}^n$ ,  $D, \Omega$  bounded sets,  $|\Omega \setminus D| > 0$ , then it is easy to prove that

$$(\chi_D)_\#(s) = \begin{cases} 0 & \text{if } s \in [0, |\Omega \setminus D|) \\ 1 & \text{if } s \in [|\Omega \setminus D|, |\Omega|] \end{cases}$$

Taking into account that

$$(\chi_D)_*(x) = (\chi_D)_\#(\omega_n \|x\|^n),$$

we get

$$D_\star = B\left(0, \left(\frac{|\Omega|}{\omega_n}\right)^{1/n}\right) \setminus B\left(0, \left(\frac{|\Omega \setminus D|}{\omega_n}\right)^{1/n}\right) = \Omega^\star \setminus (\Omega \setminus D)^\star.$$

Since  $(\Omega \setminus D)^\star$  is a ball, then

$$|\Omega^\star \setminus \overline{(\Omega \setminus D)^\star}| = |\Omega^\star \setminus (\Omega \setminus D)^\star| = |D_\star| = A. \quad (2.6)$$

In other words, the set  $\Omega^\star \setminus \overline{(\Omega \setminus D)^\star}$  is an admissible set for problem (1.2) with  $\Omega$  replaced by  $\Omega^\star$ .

## 3. TECHNICAL LEMMAS

In this section we prove several technical results needed to prove Theorem 1.1. Let  $\Omega \subset \mathbb{R}^n$  be an open and bounded connected set with Lipschitz boundary. Let  $\alpha \in (0, \bar{\alpha}_\Omega(A))$ , and let  $(u, D)$  be an optimal pair which realizes the double infimum in (1.2). To simplify the notation, in this section we will denote  $\Lambda_\Omega(\alpha, A)$  by  $\Lambda$ .

We recall that by Theorem 2.5 (ii) there exists  $t > 0$  such that

$$D = \{x \in \Omega : u(x) \leq t\}. \quad (3.1)$$

This implies

$$\mu_u(t) = |\Omega \setminus D|. \quad (3.2)$$

Moreover, since  $\Lambda - \alpha > 0$ , see (2.5), we also get

$$\{x \in \Omega : (\Lambda - \alpha)u(x) > (\Lambda - \alpha)t\} = \Omega \setminus D.$$

We now introduce the function  $U : \Omega \rightarrow \mathbb{R}$  as

$$U(x) := (\Lambda - \alpha\chi_D(x))u(x), \quad x \in \Omega. \quad (3.3)$$

**Lemma 3.1.** *The distribution function  $\mu_U$  of  $U$  is*

$$\mu_U(\tau) = \begin{cases} \mu_u\left(\frac{\tau}{\Lambda}\right), & \text{if } \tau \geq \Lambda t, \\ |\Omega \setminus D|, & \text{if } (\Lambda - \alpha)t \leq \tau \leq \Lambda t, \\ \mu_u\left(\frac{\tau}{\Lambda - \alpha}\right), & \text{if } \tau \leq (\Lambda - \alpha)t. \end{cases} \quad (3.4)$$

*Proof.* By its definition in (2.1), the distribution function  $\mu_U$  of  $U$  is

$$\mu_U(\tau) = |\{x \in \Omega : U(x) > \tau\}|.$$

By (3.1) and the definition of  $U$  the following properties easily follows:

$$\{x \in D : U(x) > \tau\} = \{x \in D : u(x) > \frac{\tau}{(\Lambda - \alpha)}\} = \emptyset \quad \forall \tau \geq (\Lambda - \alpha)t, \quad (3.5)$$

$$\begin{aligned} \{x \in \Omega : U(x) > \tau\} &= \{x \in \Omega \setminus D : U(x) > \tau\} \\ &= \{x \in \Omega \setminus D : \Lambda u(x) > \tau\} \quad \forall \tau \geq (\Lambda - \alpha)t. \end{aligned} \quad (3.6)$$

Moreover,

$$\{x \in \Omega \setminus D : U(x) \leq \tau\} = \{x \in \Omega \setminus D : \Lambda u(x) \leq \tau\} = \emptyset \quad \forall \tau \leq \Lambda t \quad (3.7)$$

and, equivalently,

$$\{x \in \Omega \setminus D : U(x) > \tau\} = \{x \in \Omega \setminus D : \Lambda u(x) > \tau\} = \Omega \setminus D \quad \forall \tau \leq \Lambda t. \quad (3.8)$$

We consider three possible cases.

**Case I:**  $\tau \geq \Lambda t$ .

By the assumption on  $\tau$  and (3.1)

$$\{x \in D : \Lambda u(x) > \tau\} = \emptyset,$$

therefore by (3.6) we get

$$\{x \in \Omega : U(x) > \tau\} = \{x \in \Omega \setminus D : \Lambda u(x) > \tau\} = \{x \in \Omega : \Lambda u(x) > \tau\}.$$

This shows that  $\mu_U(\tau) = \mu_u\left(\frac{\tau}{\Lambda}\right)$ .

**Case II:**  $(\Lambda - \alpha)t \leq \tau \leq \Lambda t$ .

By (3.5) and (3.8) we get

$$\{x \in \Omega : U(x) > \tau\} = \Omega \setminus D.$$



Therefore  $\mu_U(\tau) = |\Omega \setminus D|$ . On the other hand, by (3.2), we know that  $\mu_u(t) = |\Omega \setminus D|$  as well.

**Case III:**  $\tau < (\Lambda - \alpha)t$ .

We have

$$\{x \in \Omega : U(x) > \tau\} = \{x \in \Omega : \tau < U(x) \leq (\Lambda - \alpha)t\} \cup \{x \in \Omega : U(x) > (\Lambda - \alpha)t\}. \quad (3.9)$$

By (3.6), (3.8) and (3.1)

$$\begin{aligned} \{x \in \Omega : U(x) > (\Lambda - \alpha)t\} &= \{x \in \Omega \setminus D : U(x) > (\Lambda - \alpha)t\} \\ &= \Omega \setminus D = \{x \in \Omega : (\Lambda - \alpha)u(x) > (\Lambda - \alpha)t\}. \end{aligned} \quad (3.10)$$

Let us now consider the first set at the right hand side of (3.9). By (3.7)

$$\begin{aligned} \{x \in \Omega : \tau < U(x) \leq (\Lambda - \alpha)t\} &= \{x \in D : \tau < U(x) \leq (\Lambda - \alpha)t\} \\ &= \{x \in D : \tau < (\Lambda - \alpha)u(x) \leq (\Lambda - \alpha)t\}. \end{aligned}$$

On the other hand, from the characterization of  $D$ , see (3.1), it follows that

$$\{x \in \Omega \setminus D : \tau < (\Lambda - \alpha)u(x) \leq (\Lambda - \alpha)t\} \subseteq \{x \in \Omega \setminus D : u(x) \leq t\} = \emptyset.$$

Thus, we get

$$\{x \in \Omega : \tau < U(x) \leq (\Lambda - \alpha)t\} = \{x \in \Omega : \tau < (\Lambda - \alpha)u(x) \leq (\Lambda - \alpha)t\}. \quad (3.11)$$

Combining (3.9), (3.10) and (3.11), we get the desired result.  $\square$

**Lemma 3.2.** *The decreasing Steiner rearrangement  $U^\sharp$  of  $U$  is*

$$U^\sharp(s) = (\Lambda - \alpha \chi_{(|\Omega \setminus D|, |\Omega|]}(s)) u^\sharp(s). \quad (3.12)$$

*Proof.* By definition,

$$0 \leq \mu_U(\tau) \leq |\Omega| \quad \forall \tau \in \mathbb{R}.$$

Let us first consider the case  $s = 0$ . By (2.2) and (3.3),

$$U^\sharp(0) = \text{ess sup } U = \Lambda \text{ess sup } u = \Lambda u^\sharp(0).$$

Thus, the equality (3.12) holds for  $s = 0$ . Let us now consider the case  $s \in (0, |\Omega \setminus D|]$ . Keeping in mind (3.4) and the fact that  $\mu_u(t) = |\Omega \setminus D|$ , we have that

$$\mu_U(\Lambda t) = |\Omega \setminus D| \geq s. \quad (3.13)$$

Therefore, by (2.2), (3.4) and (3.13) we have

$$\begin{aligned} U^\sharp(s) &:= \inf\{\tau : \mu_U(\tau) < s\} \\ &= \inf\{\tau > \Lambda t : \mu_U(\tau) < s\} \\ &= \inf\{\tau > \Lambda t : |\{x \in \Omega : u(x) > \frac{\tau}{\Lambda}\}| < s\} \\ &= \Lambda \inf\{\sigma > t : \mu_u(\sigma) < s\}, \end{aligned} \quad (3.14)$$

where in the last equality we used the change of variable  $\sigma = \frac{\tau}{\Lambda}$ .

Now, notice that, by (3.1) and  $s \leq |\Omega \setminus D|$ ,

$$\mu_u(\sigma) \geq \mu_u(t) = |\{x \in \Omega : u(x) > t\}| = |\Omega \setminus D| \geq s \quad \forall \sigma \leq t.$$

Therefore

$$\{\sigma > t : \mu_u(\sigma) < s\} = \{\sigma : \mu_u(\sigma) < s\}$$

that implies

$$\inf\{\sigma > t : \mu_u(\sigma) < s\} = \inf\{\sigma : \mu_u(\sigma) < s\} = u^\sharp(s).$$

Combined with (3.14), this gives

$$U^\sharp(s) = \Lambda u^\sharp(s), \quad \forall s \in (0, |\Omega \setminus D|]$$

and (3.12) follows.

Let us finally consider the case  $s \in (|\Omega \setminus D|, |\Omega|]$ . The assumption on  $s$  and Lemma 3.1 imply

$$\mu_U((\Lambda - \alpha)t) = |\Omega \setminus D| < s,$$

therefore

$$\inf\{\tau : \mu_U(\tau) < s\} = \inf\{\tau \leq (\Lambda - \alpha)t : \mu_U(\tau) < s\}.$$

This fact, together with the change of variable  $\gamma = \frac{\tau}{\Lambda - \alpha}$ , gives

$$\begin{aligned} U^\sharp(s) &= \inf\{\tau \leq (\Lambda - \alpha)t : \mu_U(\tau) < s\} \\ &= (\Lambda - \alpha) \inf\{\gamma \leq t : \mu_U((\Lambda - \alpha)\gamma) < s\}. \end{aligned} \quad (3.15)$$

By Lemma 3.1

$$\mu_u(\gamma) = \mu_U((\Lambda - \alpha)\gamma) \quad \forall \gamma \leq t.$$

Therefore

$$\inf\{\gamma \leq t : \mu_U((\Lambda - \alpha)\gamma) < s\} = \inf\{\gamma \leq t : \mu_u(\gamma) < s\}. \quad (3.16)$$

Since by (3.2)

$$\mu_u(t) = |\Omega \setminus D| < s,$$

then

$$\inf\{\gamma \leq t : \mu_u(\gamma) < s\} = \inf\{\gamma : \mu_u(\gamma) < s\}. \quad (3.17)$$

Collecting (3.15), (3.16) and (3.17) we get

$$U^\sharp(s) = (\Lambda - \alpha)u^\sharp(s)$$

and (3.12) follows. This concludes the proof.  $\square$

**Proposition 3.3.** *Let  $U : \Omega \rightarrow \mathbb{R}$  be the measurable function defined in (3.3). Then  $U^\star : \Omega^\star \rightarrow \mathbb{R}$  is well defined and*

$$U^\star(x) = \left( \Lambda - \alpha \chi_{\Omega^\star \setminus \overline{(\Omega \setminus D)^\star}} \right) u^\star(x).$$

*Proof.* We have that

$$\{x \in \mathbb{R}^n : \omega_n \|x\|^n \leq |\Omega \setminus D|\} = B\left(0, \left(\frac{|\Omega \setminus D|}{\omega_n}\right)^{1/n}\right) = \overline{(\Omega \setminus D)^\star}.$$

On the other hand,

$$\begin{aligned} \{x \in \mathbb{R}^n : |\Omega \setminus D| < \omega_n \|x\|^n < |\Omega|\} &= B\left(0, \left(\frac{|\Omega|}{\omega_n}\right)^{1/n}\right) \setminus B\left(0, \left(\frac{|\Omega \setminus D|}{\omega_n}\right)^{1/n}\right) \\ &= \Omega^\star \setminus \overline{(\Omega \setminus D)^\star}. \end{aligned}$$

By (2.3),

$$U^\star(x) = U^\sharp(\omega_n \|x\|^n), \quad x \in \Omega^\star,$$

where  $U^\sharp$  has been explicitly determined in Proposition 3.2. Therefore,

$$U^\star(x) = \begin{cases} \Lambda u^\star(x), & x \in \overline{(\Omega \setminus D)^\star}, \\ (\Lambda - \alpha)u^\star(x), & x \in \Omega^\star \setminus \overline{(\Omega \setminus D)^\star}, \end{cases}$$

and this closes the proof.  $\square$

## 4. PROOF OF THEOREM 1.1

In this section we prove Theorem 1.1. To simplify the notation and since  $\alpha$  and  $A$  are fixed, we will write  $\Lambda_\Omega$  in place of  $\Lambda_\Omega(\alpha, A)$ .

*Proof of Theorem 1.1.* The proof consists of two steps. The first one is to show that

$$\Lambda_{\Omega^*} \leq \Lambda_\Omega, \quad (4.1)$$

for every  $\Omega \subset \mathbb{R}^n$  such that  $|\Omega| = |\Omega^*|$ . To this aim, let  $(u, D)$  be an optimal pair which realizes  $\Lambda_\Omega$ . This means that

$$\Lambda_\Omega = \frac{\int_\Omega |\nabla u(x)|^2 dx + \alpha \int_\Omega \chi_D(x) u^2(x) dx}{\int_\Omega u^2(x) dx}.$$

Let  $D_\star$  be defined through its characteristic function in (1.3). Since the decreasing Schwarz symmetrization  $u^* \in H_0^1(\Omega^*)$ , and, by definition,  $D_\star \subset \Omega^*$  with  $|D_\star| = |D| = A$ , we have that  $(u^*, D_\star)$  is an admissible pair for (1.2) on  $\Omega^*$ . Moreover, since  $u > 0$  then  $u^* > 0$  and hence we get that

$$(u^*)^2(x) = (u^2)^*(x), \quad \text{for almost every } x \in \Omega^*, \quad (4.2)$$

see [13, Proposition 1.1.4] with  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  being the non-decreasing function

$$\psi(t) := \begin{cases} 0, & \text{if } t \leq 0, \\ t^2, & \text{if } t > 0. \end{cases}$$

Therefore, we have

$$\begin{aligned} \Lambda_{\Omega^*} &= \frac{\int_{\Omega^*} |\nabla u^*(x)|^2 dx + \alpha \int_{\Omega^*} (\chi_D)_*(x) (u^*)^2(x) dx}{\int_{\Omega^*} (u^*)^2(x) dx} \\ &\leq \frac{\int_\Omega |\nabla u(x)|^2 dx + \alpha \int_\Omega \chi_D(x) u^2(x) dx}{\int_\Omega u^2(x) dx} = \Lambda_\Omega, \end{aligned}$$

where in the second inequality we used (4.2), Proposition 2.2 and Pólya-Szegő. This shows that (4.1) holds true.

The second step is to prove that the ball  $\Omega^*$  is the unique minimizer. In other words, we have to prove that

$$\text{if } \Lambda_\Omega = \Lambda_{\Omega^*}, \text{ then } \Omega = \Omega^*.$$

Since we assume  $\Lambda_\Omega = \Lambda_{\Omega^*}$ , we can simply write  $\Lambda$  omitting the dependance on the set. Let us consider an optimal pair  $(u, D)$  of the double minimization problem (1.2) on  $\Omega$ . Clearly, they satisfy the second order Euler-Lagrange equation associated to (1.2), i.e.

$$\begin{cases} -\Delta u = (\Lambda - \alpha \chi_D)u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where we stress that the right-hand side  $(\Lambda - \alpha \chi_D)u \in L^2(\Omega)$ .

Let us now consider the function  $v \in H_0^1(\Omega^*)$  which solves the following auxiliary boundary value problem,

$$\begin{cases} -\Delta v = [(\Lambda - \alpha \chi_D)u]^*, & \text{in } \Omega^*, \\ v = 0, & \text{on } \partial\Omega^*. \end{cases}$$

A direct application of Theorem 2.3 implies that

$$u^*(x) \leq v(x), \quad \text{for almost every } x \in \Omega^*. \quad (4.3)$$

Now, by Proposition 3.3, we have that  $v$  actually solves

$$\begin{cases} -\Delta v = \left( \Lambda - \alpha \chi_{\Omega^* \setminus \overline{(\Omega \setminus D)^*}} \right) u^*, & \text{in } \Omega^*, \\ v = 0, & \text{on } \partial\Omega^*. \end{cases} \quad (4.4)$$

Therefore, by (2.5) and (4.3), we have

$$-\Delta v(x) = \left( \Lambda - \alpha \chi_{\Omega^* \setminus \overline{(\Omega \setminus D)^*}}(x) \right) u^*(x) \leq \left( \Lambda - \alpha \chi_{\Omega^* \setminus \overline{(\Omega \setminus D)^*}}(x) \right) v(x)$$

for almost every  $x \in \Omega^*$ . Therefore, multiplying by  $v$  the former inequality and integrating by parts,

$$\frac{\int_{\Omega^*} |\nabla v(x)|^2 dx + \alpha \int_{\Omega^*} \chi_{\Omega^* \setminus \overline{(\Omega \setminus D)^*}}(x) v(x)^2 dx}{\int_{\Omega^*} v(x)^2 dx} \leq \Lambda,$$

for  $v \in H_0^1(\Omega^*)$ . Now, since  $|D_\star| = |\Omega^* \setminus \overline{(\Omega \setminus D)^*}|$ , we get

$$\frac{\int_{\Omega^*} |\nabla v(x)|^2 dx + \alpha \int_{\Omega^*} \chi_{D_\star}(x) v(x)^2 dx}{\int_{\Omega^*} v(x)^2 dx} \leq \Lambda, \quad (4.5)$$

for  $v \in H_0^1(\Omega^*)$  and  $D_\star \subset \Omega^*$  with  $|D_\star| = A$ . Since  $(v, D_\star)$  is an admissible pair, then in (4.5) the equality holds and  $(v, D_\star)$  must be an optimal pair. Therefore  $v$  solves

$$\begin{cases} -\Delta v = (\Lambda - \alpha \chi_{D_\star}) v, & \text{in } \Omega^*, \\ v = 0, & \text{on } \partial\Omega^*. \end{cases} \quad (4.6)$$

By (4.4) and (4.6), subtracting term by term, and keeping in mind (2.6) we get

$$0 = (\Lambda - \alpha \chi_{\Omega^* \setminus \overline{(\Omega \setminus D)^*}}) u^* - (\Lambda - \alpha \chi_{D_\star}) v = (\Lambda - \alpha \chi_{D_\star}) (u^* - v) \quad \text{a.e. in } \Omega^*.$$

The assumption  $\alpha < \bar{\alpha}_\Omega$  implies  $\Lambda - \alpha \chi_{D_\star} > 0$ , see (2.5). Therefore the equality above, together with (4.3), implies

$$v = u^* \quad \text{a.e. in } \Omega^*.$$

By Remark 2.4, this is enough to conclude that  $\Omega = \Omega^*$ . This closes the proof.  $\square$

*Remark 4.1.* We point out that the validity of (4.1) is essentially already contained in [17, Theorem 3] and in the proof of [3, Theorem 4]. We also want to stress that the assumption  $\alpha < \bar{\alpha}_{\Omega^*}(A)$  is needed only in the second step of the proof, because we exploit that  $\Lambda - \alpha > 0$ .

## 5. PROOF OF THEOREM 1.2

In this section we will prove Theorem 1.2, adapting to our setting the Lieb's proof in [15] of the similar inequality for the lowest eigenvalue of  $-\Delta$ .

*Proof of Theorem 1.2.* We know from [3] that for every  $i = 1, 2$ ,  $\Lambda_{\Omega_i}(\alpha_i, A)$  is actually achieved by (at least) one optimal pair  $(v_i, D_i)$ , with  $v_i \in H_0^1(\Omega_i)$  and  $|D_i| = A_i$ . The functions  $v_i$  are uniquely determined, up to a scalar multiple, by  $D_i$ , and may be chosen to be positive in  $\Omega_i$ , see [3]. Without loss of generality, we can assume that

$$\|v_i\|_{L^2(\Omega_i)} = 1. \quad (5.1)$$

Therefore

$$\Lambda_{\Omega_i}(\alpha_i, A_i) = \int_{\Omega_i} |\nabla v_i(x)|^2 dx + \alpha_i \int_{\Omega_i} \chi_{D_i}(x) v_i(x)^2 dx, \quad i \in \{1, 2\}. \quad (5.2)$$

As in [15], from now on and with a slight abuse of notation, we assume the functions  $v_i$ 's to be defined on the whole of  $\mathbb{R}^n$ , with

$$v_i(x) = 0, \quad \text{if } x \notin \Omega_i.$$

For a.e.  $x \in \mathbb{R}^n$  we define  $h_x : \mathbb{R}^n \rightarrow \mathbb{R}$ ,

$$h_x(y) := v_1(y) v_2(y - x).$$

Notice that  $h_x = 0$  a.e. in  $\mathbb{R}^n \setminus (\Omega_1 \cap \Omega_{2,x})$ , see (4.5), and  $h_x \in L^1$ . Since  $v_1, v_2, \nabla_y v_1$  and  $\nabla_y v_2$  are  $L^2$  functions, then  $w_x(y) := \nabla_y v_1(y) v_2(y - x) + v_1(y) \nabla_y v_2(y - x) \in L^1$ ,  $w_x = 0$  a.e. in  $\mathbb{R}^n \setminus (\Omega_1 \cap \Omega_{2,x})$ . Moreover,  $\nabla_y h_x = w_x$  in the sense of distributions. Even more, it holds  $h_x \in H^1(\mathbb{R}^n)$  and  $\nabla_y h_x = w_x \in L^2$ , see [15, p. 445].

Let us consider the functions

$$\begin{aligned} T(x) &:= \int_{\mathbb{R}^n} |\nabla_y h_x(y)|^2 dy + (\alpha_1 + \alpha_2) \int_{\mathbb{R}^n} \chi_{D_1 \cap D_{2,x}}(y) h_x(y)^2 dy \\ &= \int_{\Omega_1 \cap \Omega_{2,x}} |\nabla_y h_x(y)|^2 dy + (\alpha_1 + \alpha_2) \int_{\Omega_1 \cap \Omega_{2,x}} \chi_{D_1 \cap D_{2,x}}(y) h_x(y)^2 dy \end{aligned}$$

and

$$H(x) := \int_{\mathbb{R}^n} h_x(y)^2 dy = \int_{\Omega_1 \cap \Omega_{2,x}} h_x(y)^2 dy.$$

The same computations performed in [15], allows to get that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\nabla_y h_x(y)|^2 dy dx = \int_{\Omega_1} |\nabla v_1|^2 dx + \int_{\Omega_2} |\nabla v_2|^2 dx. \quad (5.3)$$

Let us now consider

$$W(x) := \int_{\Omega_1 \cap \Omega_{2,x}} \chi_{D_1 \cap D_{2,x}}(y) h_x(y)^2 dy.$$

Since  $\chi_{D_{2,x}}(\cdot) = \chi_{D_2}(\cdot - x)$  and  $D_i \subseteq \Omega_i$ , using also (5.1) we have

$$\begin{aligned} \int_{\mathbb{R}^n} W(x) dx &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_{D_1}(y) \chi_{D_2}(y - x) v_1(y)^2 v_2(y - x)^2 dy dx \\ &= \int_{\mathbb{R}^n} \chi_{D_1}(y) v_1(y)^2 dy \int_{\mathbb{R}^n} \chi_{D_2}(y) v_2(y)^2 dy \\ &\leq \frac{\alpha_1}{\alpha_1 + \alpha_2} \int_{D_1} v_1(y)^2 dy \int_{\Omega_2} v_2(y)^2 dy + \frac{\alpha_2}{\alpha_1 + \alpha_2} \int_{D_2} v_2(y)^2 dy \int_{\Omega_1} v_1(y)^2 dy \\ &= \frac{\alpha_1}{\alpha_1 + \alpha_2} \int_{D_1} v_1(y)^2 dy + \frac{\alpha_2}{\alpha_1 + \alpha_2} \int_{D_2} v_2(y)^2 dy. \end{aligned}$$

Therefore

$$(\alpha_1 + \alpha_2) \int_{\mathbb{R}^n} W(x) dx \leq \alpha_1 \int_{D_1} v_1(y)^2 dy + \alpha_2 \int_{D_2} v_2(y)^2 dy. \quad (5.4)$$

Combining (5.3), (5.4) and (5.2), we obtain

$$\int_{\mathbb{R}^n} T(x) dx \leq \Lambda_{\Omega_1}(\alpha_1, A_1) + \Lambda_{\Omega_2}(\alpha_2, A_2) =: \Theta. \quad (5.5)$$

By (5.1) it holds that

$$\int_{\mathbb{R}^n} H(x) dx = \int_{\mathbb{R}^n} v_1^2(y) dy \int_{\mathbb{R}^n} v_2(x)^2 dx = 1,$$

therefore (5.5) can be rewritten as

$$\int_{\mathbb{R}^n} (T(x) - \Theta H(x)) dx \leq 0.$$

We claim that it is not  $T = \Theta H$  a.e. in  $\mathbb{R}^n$ . Once we have proved this, by the inequality above we get that there exists  $\Sigma \subseteq \mathbb{R}^n$ ,  $|\Sigma| > 0$ , such that  $0 \leq T(x) < \Theta H(x)$  a.e. in  $\Sigma$ . Therefore, denoting

$$A_x := |D_1 \cap D_{2,x}|, \quad (5.6)$$

by definition of  $\Lambda_{\Omega_1 \cap \Omega_{2,x}}(\alpha_1 + \alpha_2, A_x)$

$$\Lambda_{\Omega_1 \cap \Omega_{2,x}}(\alpha_1 + \alpha_2, A_x) \leq \frac{T(x)}{H(x)} < \Theta \quad \text{for a.e. } x \in \Sigma$$

and we conclude by using that  $\Lambda_{\Omega_1 \cap \Omega_{2,x}}(\cdot, A_x)$  and  $\Lambda_{\Omega_1 \cap \Omega_{2,x}}(\alpha, \cdot)$  are strictly increasing functions, see (iv) of Theorem 2.5.

Let us prove the claim, by contradiction. Assume that  $T = \Theta H$  a.e. in  $\mathbb{R}^n$ . Since

$$K(x) := (\chi_{\Omega_1} * \chi_{\Omega_2})(x) = |\Omega_1 \cap \Omega_{2,x}|,$$

is a continuous function with compact support, then for every  $\varepsilon > 0$  there exists a non empty open set  $C_\varepsilon$  such that  $0 < K(x) < \varepsilon$  for  $x \in C_\varepsilon$ . Moreover, the positivity of  $K$  a.e. in  $C_\varepsilon$  and of  $y \mapsto v_1(y)$  and  $y \mapsto v_2(y - x)$  in the open set  $\Omega_1 \cap \Omega_{2,x}$ , imply that  $H(x) > 0$  for a.e.  $x \in C_\varepsilon$ . Therefore,

$$\Lambda_{\Omega_1 \cap \Omega_{2,x}}(\alpha_1 + \alpha_2, A_x) \leq \frac{T(x)}{H(x)} = \Theta \quad \text{for a.e. } x \in C_\varepsilon, \quad (5.7)$$

where  $A_x$  is defined in (5.6).

It is trivial that

$$\Lambda_{\Omega_1 \cap \Omega_{2,x}}(\alpha_1 + \alpha_2, A_x) \geq \inf_{0 \neq u \in H_0^1(\Omega_1 \cap \Omega_{2,x})} \frac{\int_{\Omega_1 \cap \Omega_{2,x}} |\nabla u(y)|^2 dy}{\int_{\Omega_1 \cap \Omega_{2,x}} |u(y)|^2 dy} =: \lambda(\Omega_1 \cap \Omega_{2,x}),$$

where  $\lambda(\Omega_1 \cap \Omega_{2,x})$  is the lowest eigenvalue of  $-\Delta$  in  $\Omega_1 \cap \Omega_{2,x}$  with Dirichlet boundary conditions. Therefore, by (5.7) and by the Faber-Krahn inequality (1.4),

$$\begin{aligned} \Theta &\geq \Lambda_{\Omega_1 \cap \Omega_{2,x}}(\alpha_1 + \alpha_2, A_x) \geq \beta_n |\Omega_1 \cap \Omega_{2,x}|^{-2/n} \\ &= \beta_n K(x)^{-2/n} > \beta_n \varepsilon^{-2/n} \quad \text{for a.e. } x \in C_\varepsilon. \end{aligned}$$

This is impossible for sufficiently small  $\varepsilon$ . This concludes the proof of the claim.  $\square$

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