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Two Monetary Models with Alternating Markets

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We present a thought-provoking study of two monetary models: the cash-in-advance and the Lagos and Wright (2005) models. The different ap-proaches to modeling money—reduced form versus explicit role—induce neither fundamental theoretical nor quantitative differences in results. Given conformity of preferences, technologies, and shocks, both models re-duce to equilibrium difference equations that coincide unless price dis-tortions are *differentially imposed* on cash prices, across models. Equal distortions support equally large welfare costs of inflation. Performance differences stem from unequal assumptions about the pricing mechanism that governs cash transactions, not the differential modeling of the monetary exchange process.

THE QUESTION "WHAT'S THE BEST APPROACH to modeling money?" is one of those that economists have struggled with for many years and is yet unsettled. Three decades ago, some viewed the overlapping generations framework as the only satisfactory approach to modeling money (Kareken and Wallace 1980), whereas others saw merits from placing real balances in the utility function and noted that such a device could be used to unify several results in the literature (McCallum 1983, Feenstra 1986). Today, there is a debate about the framework proposed in Lagos and Wright (2005) (henceforth, LW) in relation to reduced-form models of money.

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Advocates of the LW model emphasize that the role of money is made explicit (Williamson and Wright 2010) in contrast with reduced-form models such as those imposing cash-in-advance (CIA) constraints (e.g., Lucas 1980, 1982, 1984, Lucas and Stockey 1983). This distinction, it is argued, is theoretically appealing and can make a significant difference for quantitative results, especially for the welfare cost of inflation (LW, p. 464). Yet there are design similarities with the CIA framework. In both models, a key assumption is agents synchronously alternate between a centralized market (CM) and a decentralized market (DM) (Lucas 1984, p. 20, LW, p. 481). Moreover, in both models consumption utility depends on where purchases are settled, and asset trading decisions (money balances' adjustments, in particular) are made *before* a random shock is observed (Lucas 1984, pp.10–11, LW, pp. 462–66).

These considerations have raised several questions among monetary economists. Are there differences in the main equilibrium equations of these two theoretical platforms? If so, what model features lead to disparities in theoretical results? And do the models generally produce dissimilar quantitative results? We offer some answers by presenting what we find when we juxtapose the models' main equations and quantitative implications for the welfare cost of inflation. We do so by laying out the CIA framework following Lucas (1984), which has an explicit and transparent description of the physical environment. Then, we report the main mathematical relationships describing monetary equilibrium allocations in LW and discuss how the assumption of Nash bargaining in cash trades induces a price distortion that depends on the seller's bargaining power parameter. We thus place the two frameworks on equal footing—in terms of preferences, technologies, and shocks—and illustrate a way to introduce price distortions in the CIA model without altering its fundamental structure. Finally, we derive the equations describing monetary equilibrium allocations in the CIA model.

Our analysis focuses on stationary equilibrium, which is the focus of the LW literature. We find that the equations characterizing stationary equilibrium in LW when sellers have no bargaining power coincide with the equations that characterize stationary competitive equilibrium in the CIA model. This also holds when sellers *do* have some bargaining power, when the price distortion from Nash bargaining is replicated in the other model. For illustrative purposes, this distortion is implemented via a tax on cash revenues (equivalently, a sales tax on cash purchases). Such correspondence between equations immediately extends outside of steady state, if sellers have no bargaining power and workers have isoelastic preferences; otherwise, a one-to-one mapping between the equations cannot be immediately established outside of steady state. Hence, there may exist dynamical equilibria which are not the same in the two models. Before concluding, we propose a quantitative illustration showing that the welfare costs of inflation in the CIA model match those in LW.

The main insight is that the two models reduce to a single difference equation. The equations correspond if the price distortion in one model is matched in the other model, and in that case one cannot distinguish one model from the other based on their quantitative performance. The differences in the models' main equations reduce to differences in the pricing mechanism imposed in cash trades. To the extent that the trading mechanism is not considered an integral part of the model, or a primitive, this is evidence that the pricing mechanism assumed to govern cash transaction is the source of quantitative and theoretical differences, not the structure of the model itself (e.g., the explicit description of trade interactions).

Overall, the analysis offers a pedagogical lesson in the quest for the "best approach to modeling money." It provides a unique perspective on the similarities in the performance of two models of money that are often perceived as being very different. In addition, it helps a reader to more deeply understand how to put to use these models; in particular, it suggests that one does not need to go through the heavier machinery of LW for many research questions.[1]

1. A CIA MODEL

We present a compact version of the model in Lucas (1984), a general-equilibrium incomplete markets model that introduces money imposing CIA constraints. The model adopts the convention—also found in LW (2005)—that agents periodically alternate between a CM and a DM.

Time is denoted $t = 0, 1, \ldots$ There is a constant population composed of a continuum of *ex ante* homogeneous infinitely lived agents. Their preferences are defined over nonstorable produced goods and labor. Each agent owns equal shares in a representative firm that produces goods using the concave technology F; labor is the only factor of production. In a period, traders alternate synchronously between CM and DM. Each period is divided into two subperiods, say, morning and afternoon. The DM is open in the morning, whereas the CM is open in the afternoon. It is assumed that some morning trades must be settled immediately with the exchange of money (cash trades) whereas others can be settled in the afternoon (credit trades). Goods purchased with cash are distinct from goods purchased on credit, called goods 1 and 2, respectively. Money is injected through lump-sum transfers by a central bank.

Let s_t be a shock, drawn at the start of t from a time-invariant set, which affects agents' ability to consume and produce cash goods; $\{s_t\}_{t=0}^{\infty}$ is a path of shocks, $S^t = (s_1, \ldots, s_t)$ is a history of shocks (from the set of all possible histories) known before period t trading, $f^t(S^t)$ is the density of S^t. Neither F nor the money supply process depend on S^t. Events on t evolve as follows (timeline variants are possible).

Morning of t (DM). The shock s_t is observed. Agents and firms trade goods 1 and 2, and labor. Agents hold $M_t(S^{t-1})$ money and buy $c_{1t}(S^t)$ goods in exchange for money (cash goods), buy $c_{2t}(S^t)$ goods on credit (credit goods), supply $h_t(S^t)$ labor to the firm on credit. The firm demands $h_t^F(S^t)$ labor and supplies $F(h_t^F(S^t))$ goods. Nominal spot prices are $p_{jt}(S^t)$, $j = 1, 2$, the nominal wage is $w_t(S^t)$; given profit maximization (see Supporting Information online at the publisher's website, henceforth SI) we have

$$p_{1t}(S^t) = p_{2t}(S^t) = p_t(S^t) \qquad \text{with} \qquad p_t(S^t)F'\left(h_t^F(S^t)\right) = w_t(S^t) \text{ for all } t, S^t. \quad (1)$$

1. We thank Christian Zimmerman for making this point in his NEP-DGE blog.

Afternoon of t (CM). DM credit trades (morning of t) are settled: firms pay wages and dividends (from DM profits); agents pay for credit goods. The central bank retires the old money supply $\bar{M}_{t-1}$ and issues a new supply $\bar{M}_t$ through lump-sum transfers Θ_t to agents. In a financial market, agents trade state-contingent claims to money delivered in the CM of $t+1$, and exit t holding $M_{t+1}(S^t)$ money.

The initial money supply is $\bar{M} \geq 0$. Let $q_t(S^t)$ be the date-0 price of a claim to one dollar delivered in the CM of t, contingent on S^t (state-contingent nominal bond). In the CM of t, the central bank issues $\bar{M}_{t+1}$ money, valued at $q_t(S^t)$ in date-0 prices, and retires it in the CM of $t+1$, when the expected value of money is $\int q_{t+1}(S^{t+1})ds_{t+1}$. Lump-sum transfers Θ_t are valued at $q_t(S^t)$. The date-0 central bank's budget constraint is

$$\bar{M} = \sum_{t=0}^{\infty} \int \left\{ \bar{M}_{t+1} \left[q_t(S^t) - \int q_{t+1}(S^{t+1}) ds_{t+1} \right] - \Theta_t q_t(S^t) \right\} dS^t.$$

Equivalently, the flow constraint $\bar{M}_{t+1} - \bar{M}_t = \Theta_t$ for all t, S^t identify monetary policy.

Agents who contract on date 0 maximize the expected utility

$$\sum_{t=0}^{\infty} \beta^t \int U(c_{1t}(S^t), c_{2t}(S^t), h_t(S^t)) f^t(S^t) dS^t,$$

where U is a real-valued function, C^2 in each argument, strictly increasing in c_j, decreasing in h, and concave. Agents choose sequences of state-contingent consumption, labor, and money holdings $c_{1t}(S^t)$, $c_{2t}(S^t)$, $h_t(S^t)$, and $M_{t+1}(S^t)$, subject to two types of constraints. First, CIA constraints

$$p_{1t}(S^t) c_{1t}(S^t) \leq M_t(S^{t-1}) \text{ for all } t \text{ and } S^t,$$

where $M_t(S^{t-1})$ are money balances held at the start of t, brought in from the CM of $t-1$, when the shock s_t was not yet realized. Given this uncertainty, money may be held to conduct transactions and for precautionary reasons.

The second constraint is the date-0 nominal intertemporal budget constraint

$$\begin{aligned} &\sum_{t=0}^{\infty} \int \left\{ q_t(S^t) \left[p_{1t}(S^t) c_{1t}(S^t) + p_{2t}(S^t) c_{2t}(S^t) - w_t(S^t) h_t(S^t) - M_t(S^{t-1}) \right.\right. \\ &\left.\left. + M_{t+1}(S^t) - \Theta_t \right] \right\} dS^t \leq \Pi + \bar{M}. \end{aligned}$$

Sources of funds are $\bar{M}$ initial money holdings (initial liabilities of the central bank) and the firm's nominal value Π. The date-0 present value of net expenditure is calculated using the price of money delivered in the CM of t (see SI). Letting $\mu_t(S^t)$ be the Kühn–Tucker multiplier on the CIA constraint on t, and omitting the arguments from U, in an interior optimum the first order conditions (FOCs) for all t, S^t are (see SI):

$$\begin{aligned}
c_{1t}(S^t) &: \beta^t U_1 f^t(S^t) - \lambda p_{1t}(S^t) q_t(S^t) - \mu_t(S^t) p_{1t}(S^t) = 0 \\
& \quad p_{1t}(S^t) c_{1t}(S^t) \leq M_t(S^{t-1}) \\
c_{2t}(S^t) &: \beta^t U_2 f^t(S^t) - \lambda p_{2t}(S^t) q_t(S^t) = 0 \\
h_t(S^t) &: \beta^t U_3 f^t(S^t) + \lambda w_t(S^t) q_t(S^t) = 0 \\
M_{t+1}(S^t) &: -\lambda q_t(S^t) + \lambda \textstyle\int q_{t+1}(S^{t+1}) ds_{t+1} + \int \mu_{t+1}(S^{t+1}) ds_{t+1} = 0.
\end{aligned} \tag{2}$$

Given (1) we get

$$\frac{-U_3}{U_2} = F'(h_t(S^t); S^t) \qquad \text{and} \qquad \frac{U_1}{U_2} = \frac{\lambda q_t(S^t) + \mu_t(S^t)}{\lambda q_t(S^t)} \text{ for all } t, S^t. \tag{3}$$

Fix t and S^t. The (reciprocal of the) nominal risk-free interest rate on a bond sold in the CM of t is $\frac{1}{1+r_t(S^t)}$. This is the price of a claim to money bought on $t = 0$ delivered in the CM of $t + 1$ conditional on S^t (but not on s_{t+1}) divided by the price of a claim to money delivered in the CM of t conditional on S^t:

$$\begin{aligned}
\frac{1}{1 + r_t(S^t)} &:= \frac{\int q_{t+1}(S^{t+1}) ds_{t+1}}{q_t(S^t)} \\
&= \frac{\lambda \int q_{t+1}(S^{t+1}) ds_{t+1}}{\lambda \int q_{t+1}(S^{t+1}) ds_{t+1} + \int \mu_{t+1}(S^{t+1}) ds_{t+1}}.
\end{aligned} \tag{4}$$

From (3), the interest rate makes agents indifferent between buying money or risk-free bonds in the CM of t.[2] With cash the agent can buy either cash or credit goods in $t + 1$; by holding bonds, he can *only* buy credit goods, as bonds mature in the afternoon of $t + 1$. So the interest rate compensates agents for the bond's illiquidity, which is why μ_{t+1} appears in the denominator of (4). Substituting $q_t(S^t) = (1 + r_t(S^t)) \int q_{t+1}(S^{t+1}) ds_{t+1}$ in the last line of (2), we get

$$\begin{aligned}
(1 + r_t(S^t)) \int q_{t+1}(S^{t+1}) ds_{t+1} = &\int q_{t+1}(S^{t+1}) ds_{t+1} \\
&+ \frac{1}{\lambda} \int \mu_{t+1}(S^{t+1}) ds_{t+1}.
\end{aligned}$$

Agents must be indifferent between buying an illiquid bond or holding money. The expected benefit from buying a risk-free bond in the CM of t that pays one dollar in

2. The second step in (4) comes from the last line in (2). No arbitrage requires that expenditures in $t = 0$ are equivalent. Agents can spend $q_t(S^t) \frac{1}{1+r_t(S^t)}$ to buy $\frac{1}{1+r_t(S^t)}$ delivered on t conditional on S^t and then reinvest on t the receipts in a risk-free bond to get 1 good on $t + 1$. Alternatively, agents can spend $\int q_{t+1}(S^{t+1}) ds_{t+1}$ on $t = 0$ to have one unit on $t + 1$, given S^t.

the CM of $t + 1$ is $(1 + r_t(S^t)) \quad q_{t+1}(S^{t+1})ds_{t+1}$. Money has lower expected value $\int q_{t+1}(S^{t+1})ds_{t+1}$, but provides the liquidity premium $\frac{1}{\lambda} \int \mu_{t+1}(S^{t+1})ds_{t+1}$ because, unlike the bond, cash can be spent in the DM of $t + 1$ to buy cash goods.

2. JUXTAPOSING THE TWO MODELS

To compare LW and the CIA model, we utilize the feature that in monetary equilibrium the LW model can be reduced to a single difference equation (LW, p. 469).

2.1 The Main Equilibrium Equation in LW

Agents in LW alternate between DM and CM. The DM opens and DM goods are traded; then the CM opens and CM goods are traded (timing can be reversed). CM markets are Walrasian; DM trade is pairwise with Nash bargaining and an agent has equal probability $\delta \leq 1/2$ (our notation—see SI) to buy or to sell using money, so the ratio of buyers to sellers is one (assuming no barter). Let

$$U(c_1, c_2, h_1, h_2) = u_1(c_1) - \eta(h_1) + u_2(c_2) - h_2, \tag{5}$$

where h_1, h_2 and c_1, c_2 are, respectively, labor effort and consumption in DM and CM, u_1, u_2, η are C^2, strictly increasing, u_1 and u_2 are concave, η is convex, $u_1(0) = \eta(0) = 0$. Finally, $c_j^* \in \mathbb{R}_{++}$ for $j = 1, 2$ exist such that $u_1'(c_1^*) = \eta'(c_1^*)$ and $u_2'(c_2^*) = 1$ with $u_2(c_2^*) > c_2^*$, and $u_1'(0) = \infty$ is usually imposed for equilibrium existence (LW, p. 472).

Consider monetary equilibrium. On each t consumption of CM goods satisfies

$$u_2'(c_2) = 1. \tag{6}$$

Let $\theta \in (0, 1]$ denote the buyer's bargaining power. From LW, equation (17), $p_{1t}c_{1t} = M_t$, where DM consumption satisfies

$$\frac{1}{p_{2t}} = \frac{\beta}{p_{2,t+1}} \left[\delta u_1'(c_{1,t+1}) \frac{1}{z'(c_{1,t+1}; \theta)} + 1 - \delta \right], \tag{7}$$

with $p_{2t} = \frac{M_t}{z(c_{1t};\theta)}$. Using LW Equations (8) and omitting the time subscript

$$z(c_1; \theta) := \frac{\theta \eta(c_1) u_1'(c_1) + (1 - \theta) u_1(c_1) \eta'(c_1)}{\theta u_1'(c_1) + (1 - \theta) \eta'(c_1)}.$$

Equations (6) and (7) determine equilibrium consumption in LW.

The LW literature's focus is stationary equilibrium when money grows at constant rate $\gamma \geq \beta$, and consumption and real money balances are constant. Here, the inflation rate is γ, $r_t = r = \frac{\gamma}{\beta} - 1$ and the LW model reduces to the equation

$$\frac{u_1'(c_1)}{z'(c_1;\theta)} = 1 + \frac{r}{\delta}. \tag{8}$$

Bargaining introduces distortions relative to competitive pricing. The ratio $\frac{u_1'(c_1)}{z'(c_1;\theta)}$ is the marginal benefit from spending a dollar, which varies with the bargaining parameter θ. This ratio becomes $\frac{u_1'(c_1)}{p_1/p_2}$, with $\frac{p_1}{p_2} = \eta'(c_1) \leq z'(c_1;\theta)$, when $\theta = 1$ or under competitive pricing. Hence, we capture the bargaining price distortion using

$$\psi(c_1, \theta) := \frac{\eta'(c_1)}{z'(c_1;\theta)},$$

where $\psi(c_1, 1) = 1$ (no distortion) and $\psi(c_1, \theta) < 1$ for $\theta < 1$ (see SI). Also, when $\theta < 1$ multiple $c_1 > 0$ may satisfy (8), but additional assumptions guarantee uniqueness; see Rocheteau and Wright (2005). As noted by a referee, that paper makes also evident the impact of the Nash bargaining price distortion: it develops an LW variant with participation costs for DM sellers, showing that $r = 0$ yields the first best under competitive search when prices are posted, but never under bargaining, even if $\theta = 1$.

2.2 *Model Consistency*

To present a meaningful comparison, preferences, technologies, and shocks in the CIA model must conform to those in LW. This logical coherence is achieved as follows.

Technologies. $F(h) = h$ as in LW. Because the marginal product of labor is fixed and independent of S^t, it is convenient (and without loss in generality) to interpret production of goods 1 and 2 as occurring in two batches. The firm chooses h_{jt}^F (labor demand for good $j = 1, 2$) and c_{jt}^F (supply) to solve

$$\begin{aligned}&\text{Maximize}: \sum_{t=0}^{\infty} q_t(S^t)\left[p_{1t}(S^t)c_{1t}^F + p_{2t}(S^t)c_{2t}^F - w_{1t}(S^t)h_{1t}^F - w_{2t}(S^t)h_{2t}^F\right]\\&\text{subject to}: c_{2t}^F = h_{2t}^F \text{ and } c_{1t}^F = h_{1t}^F.\end{aligned}$$

Substituting the constraints, the FOCs are

$$p_{jt}(S^t) - w_{jt}(S^t) = 0 \text{ for all } t \text{ and } j = 1, 2. \tag{9}$$

Prices equal marginal cost and profits are zero ($\Pi = 0$).

Preferences and shocks. s_t is an i.i.d. shock such that in each t a randomly drawn portion $\delta \in (0, 1)$ of agents desires good 1 and produces it. Hence,

$$f^t(S^t) = f^t(s_t; S^{t-1}) = f(s_t) f^{t-1}(S^{t-1}) \text{ for all } t \geq 0,$$

where f denotes the distribution of the date-t shock. Here, $s_t = (s_t^i)_{\text{all } i}$ where

$$s_t^i = \begin{cases} 1 & \text{with probability } \delta \\ 0 & \text{with probability } 1-\delta \end{cases} \quad \text{for all } t \geq 0 \text{ and all agents } i,$$

where $s_t^i = 0$ means that agent i neither derives utility from consuming good 1 nor can produce it. For any agent i, the marginal probabilities are thus $\int f(s_t) \mathbf{1}_{\{s_t^i=0\}} ds_t = 1 - \delta$ and $\int f(s_t) \mathbf{1}_{\{s_t^i=1\}} ds_t = \delta$.

Assume preferences (5), where h_{jt}^i is labor supplied by agent i to produce good $j = 1, 2$. For agent i on date t, we have

$$U(c_{1t}, c_{2t}, h_{1t}, h_{2t}) = \left[u_1\left(c_{1t}^i\right) - \eta\left(h_{1t}^i\right)\right] \mathbf{1}_{\{s_t^i=1\}} + u_2\left(c_{2t}^i\right) - h_{2t}^i. \tag{10}$$

Price distortion. A parsimonious way to match the bargaining price distortion is to introduce a proportional tax either on sales or purchases of cash goods. For example, a share $1 - \tau$ of revenue from cash-sales—taken as given—must be rebated back to the firm's owners, lump sum. For mnemonic ease, we call τ a "cash-revenue tax," which distorts the relative price of cash and credit goods, without altering the model's structure or equilibrium concept. The firm's problem is unchanged: we simply substitute $p_{1t}\tau c_{1t}^F$ for $p_{1t}c_{1t}^F$, so the marginal condition for cash goods becomes $p_{1t}\tau = w_{1t}$ and $\frac{p_{1t}}{p_{2t}} = \frac{w_{1t}}{w_{2t}} \times \frac{1}{\tau}$. Because buyers spend $p_{1t}c_{1t}$ and sellers receive $p_{1t}\tau c_{1t}$, we interpret $p_{1t}c_{1t}(1-\tau)$ as a sales tax and $\frac{1}{\tau} - 1$ as the sales tax rate on cash trades. The rationale for introducing τ is *not* to add a (un)realistic feature, but to match the artifactual price distortion in LW where only DM cash trades are bargained.

2.3 The Main Result

We focus on stationary monetary equilibrium.

PROPOSITION 1. *Let the CIA model have preferences, technologies, and shocks in line with LW. Let the LW and CIA models be parameterized by θ and τ, respectively. If $\tau = \psi(c_1, \theta)$, then the equations characterizing stationary competitive monetary equilibrium in the CIA model coincide with equations (6) and (8), which characterize stationary monetary equilibrium in LW.*

To prove it, we derive the monetary equilibrium equations of the CIA model. Consider a generic agent i. On date 0, he can spend $q_t(S^t)$ to buy a claim to one unit of money delivered in the afternoon of t, contingent on the history S^t. Let q_t be the price of money delivered on t *unconditional* on S^t (a risk-free discount bond). No arbitrage requires equal expenditures, that is, $q_t = \int q_t(S^t)dS^t$. It also implies[3]

$$q_t(S^t) = q_t f^t(S^t).$$

To keep the discussion focused, suppose $\tau = 1$ (no price distortion). The problem of agent i is as Section 1 but we substitute $q_t(S^t) = q_t f^t(S^t)$, U from (10), separate the labor choices for each production batch, and set $\Pi = 0$ in the intertemporal budget constraint.[4] Agent i maximizes

$$\begin{aligned} L^i := & \sum_{t=0}^{\infty} \beta^t \int U(c_{1t}(S^t), c_{2t}(S^t), h_{1t}(S^t), h_{2t}(S^t)) f^t(S^t) dS^t + \lambda \bar{M} \\ & - \lambda \sum_{t=0}^{\infty} \int q_t f^t(S^t) \{[p_{1t}(S^t)c_{1t}(S^t) + p_{2t}(S^t)c_{2t}(S^t) - w_{1t}(S^t)h_{1t}(S^t) \\ & - w_{2t}(S^t)h_{2t}(S^t) - M_t(S^{t-1}) + M_{t+1}(S^t) - \Theta_t]\} dS^t \\ & + \sum_{t=0}^{\infty} \int \mu_t(S^t)[M_t(S^{t-1}) - p_{1t}(S^t)c_{1t}(S^t)] dS^t, \end{aligned} \tag{11}$$

choosing sequences $c_{1t}(S^t), c_{2t}(S^t), h_{1t}(S^t), h_{2t}(S^t), M_{t+1}(S^t)$. FOCs, for all t, S^t, are

$$\begin{aligned} c_{1t}(S^t) : \; & \beta^t u_1'(c_{1t}(S^t)) f^t(S^t) - \lambda p_{1t}(S^t) q_t f^t(S^t) \\ & - \mu_t(S^t) p_{1t}(S^t) = 0 \text{ for } s_t^i = 1 \\ & p_{1t}(S^t)c_{1t}(S^t) \le M_t(S^{t-1}), \\ c_{2t}(S^t) : \; & \beta^t u_2'(c_{2t}(S^t)) - \lambda p_{2t}(S^t) q_t = 0, \\ h_{1t}(S^t) : \; & -\beta^t \eta'(h_{1t}(S^t)) + \lambda w_{1t}(S^t) q_t = 0, \text{ for } s_t^i = 1, \\ h_{2t}(S^t) : \; & -\beta^t + \lambda w_{2t}(S^t) q_t = 0, \\ M_{t+1}(S^t) : \; & \lambda q_t f^t(S^t) = \lambda q_{t+1} f^t(S^t) + \int \mu_{t+1}(S^{t+1}) ds_{t+1}. \end{aligned} \tag{12}$$

The last line is derived using $q_{t+1} f^{t+1}(S^{t+1}) = q_{t+1} f(s_{t+1}) f^t(S^t)$ and noticing that $\int q_{t+1} f(s_{t+1}) f^t(S^t) ds_{t+1} = q_{t+1} f^t(S^t)$ because $\int f(s_{t+1}) ds_{t+1} = 1$ by definition.

From $-\beta^t + \lambda w_{2t}(S^t) q_t = 0$, we have that w_{2t} is independent of S^t and therefore, using the firm's optimality conditions, p_{2t} is independent of S^t. Because $-\beta^t +$

3. If $q_t(S^t) < q_t f^t(S^t)$, then $q_t(\tilde{S}^t) > q_t f^t(\tilde{S}^t)$ for some other state $\tilde{S}^t$ because $\int f^t(S^t) dS^t = 1$. In this case, the agent could make large profits with zero net investment by (i) purchasing claims that pay in state S^t at a cheap price $q_t(S^t)$, while selling risk-free claims at price q_t; and (ii) selling claims that pay in state $\tilde{S}^t$ at a steep price $q_t(\tilde{S}^t)$, while buying risk-free claims at price q_t. Thus noncontingent claims would not be traded at price q_t, which is a contradiction.

4. In competitive equilibrium, the firm makes zero profits and because $\tau = 1$ agents get no rebate on cash purchases. Therefore, the value of holding the firm, Π, must be zero.

$\lambda w_{2t} q_t = 0$ and $w_{2t} = p_{2t}$ (from the firm's problem), the optimal choice of credit goods in (12) satisfies $\beta^t u_2'(c_{2t}(S^t)) = \lambda p_{2t} q_t$; this implies

$$u_2'(c_{2t}(S^t)) = 1 \text{ for all } t,\, S^t,$$

so $c_{2t}(S^t) = c_2$ for all t, S^t and all agents i. This coincides with (6).

Consider cash goods in monetary equilibrium. Their consumption is heterogeneous because if $s_t^i = 0$ for agent i, then $c_{1t}^i(S^t) = 0$; this also implies $\mu_t(S^t) = 0$ for agent i because the cash constraint does not bind. Now consider $s_t^i = 1$. We prove (see SI) that if an agent desires to consume cash goods, then the quantity consumed is independent of the history of shocks S^t and of the identity i.

Lemma 1. *Consider any agent i and let $s_t^i = 1$. In competitive monetary equilibrium:*

(i) If $\mu_t(S^t) = 0$, then $c_{1t}(S^t) = c_1$ for all t, S^t, with $\frac{u_1'(c_1)}{\eta'(c_1)} = 1$.
(ii) If $\mu_t(S^t) > 0$, then $c_{1t}(S^t) = \frac{M_t}{p_{1t}} = c_{1t}$ for all t, S^t, where c_{1t} satisfies

$$\frac{\beta}{p_{2,t+1}}\left[\delta u_1'(c_{1,t+1})\frac{1}{\eta'(c_{1,t+1})} + 1 - \delta\right] - \frac{1}{p_{2t}} = 0 \text{ for all } t, \tag{13}$$

with $p_{2t} = \frac{M_t}{\eta'(c_{1t})c_{1t}}$.

On date t, not everyone consumes cash goods ($c_{1t}^i = 0$ when $s_t^i = 0$) but those who do, consume a quantity c_{1t}, independent of the history of shocks. Because U is linear in h_2, everyone saves the same amount of money $M_t(S^{t-1}) = M_t$ on $t-1$, there is a degenerate distribution of money, and prices are history independent. If $\mu_t = 0$, then $u_1' = \eta'$ and the agent consumes the efficient quantity $c_{1t} = c_1^*$. Otherwise, $u_1' > \eta'$ and $c_{1t} = \frac{M_t}{p_{1t}} < c_1^*$ (first and third equations in (12) with $p_{1t} = w_{1t}$).

Using the risk-free interest rate defined in (4), we have

$$\frac{1}{1+r_t} = \frac{\int q_{t+1}(S^{t+1})ds_{t+1}}{q_t(S^t)} = \frac{q_{t+1}f^t(S^t)}{q_t f^t(S^t)} = \frac{\beta}{\pi_t}.$$

The second equality holds because $q_t(S^t) = q_t f^t(S^t)$and $q_{t+1}f^{t+1}(S^{t+1}) = q_{t+1}f(s_{t+1})f^t(S^t)$; hence, $\int q_{t+1}f(s_{t+1})f^t(S^t)ds_{t+1} = q_{t+1}f^t(S^t)$ because $\int f(s_{t+1})ds_{t+1} = 1$. The final step uses $\frac{\beta^t u_2'(c_{2t})}{\lambda p_{2t}} = q_t$ from (12), $u_2'(c_{2t}) = 1$, and the gross inflation rate $\pi_t := \frac{p_{2,t+1}}{p_{2t}}$.

Let $M_{t+1} = \gamma M_t$ and consider stationary equilibrium with $\frac{M_{t+1}}{p_{2,t+1}} = \frac{M_t}{p_{2t}}$, $\frac{p_{2,t+1}}{p_{2t}} = \gamma$ and $r_t = r = \frac{\gamma}{\beta} - 1$ for all t. Equation (13) yields

$$\frac{u_1'(c_1)}{\eta'(c_1)} = \frac{r}{\delta} + 1. \tag{14}$$

The only difference between (14) and (8) is the price distortion. Given linear pricing, the marginal benefit of a dollar spent on cash goods is $\frac{u_1'(c_1)}{p_1/p_2}$ where $\frac{p_1}{p_2} = \eta'(c_1)$.

Now note that equation (14) coincides with (8) when $\theta = 1$, because $z' = \eta'$; intuitively, sellers are price takers in both models. Otherwise, when $\theta < 1$, it does not because $z' > \eta'$, that is, Nash bargaining induces a price distortion. The two equations also coincide if pricing is competitive in the DM—a common assumption in the LW literature (e.g., see Rocheteau and Wright 2005; Berentsen, Camera, and Waller 2007). This is evidence that the two frameworks' differences, in terms of stationary equilibrium allocations, reduce to differences in assumptions about the pricing mechanism that governs those transactions that *must* be settled with the exchange of money. One wonders whether the distortion generated by the Nash bargaining solution can be reproduced by introducing a cash-revenue tax in the CIA model.

Reintroduce the cash-revenue tax parameter $\tau \leq 1$. The agents' problem is (11).[5] The FOCs are in (12), so the model still reduces to the difference equation (13). However, in stationary equilibrium relative prices are $\frac{p_1}{p_2} = \frac{\eta'(h_1)}{\tau}$, so we obtain

$$\frac{u_1'(c_1)}{\eta'(c_1)/\tau} = 1 + \frac{r}{\delta}.$$

This equation coincides with (8) if $\tau = \psi(c_1, \theta)$, which is when the cash-revenue tax in equilibrium reproduces the price distortion induced by Nash bargaining. The lesson is that, in stationary equilibrium, differences in the frameworks' main equations reduce to the price distortion due to bargaining. Such distortion can be replicated in the CIA model with an appropriate "tax" on revenues from cash transactions.

The result partially extends to nonstationary equilibrium.

COROLLARY 1. *If η satisfies $\frac{d \ln \eta(h)}{d \ln h} = \kappa > 0$ and $\theta = 1$, then the equations characterizing nonstationary competitive equilibrium in the CIA model coincide with (6) and (7), which characterize nonstationary equilibrium in LW.*

The result immediately follows from Lemma 1. Rewrite equation (13) as

$$\frac{\eta'(c_{1t})c_{1t}}{M_t} = \beta \frac{\eta'(c_{1,t+1})c_{1,t+1}}{M_{t+1}} \left[\frac{u_1'\left(c_{1,t+1}\right)}{\eta'(c_{1,t+1})}\delta + 1 - \delta \right],$$

and note that it coincides with (7) when $\theta = 1$ and $\frac{d \ln \eta(h)}{d \ln h} = \kappa$, because $p_{2t} = \frac{M_t}{\eta(c_{1t})}$ (because $z(c_1; 1) = \eta(c_1)$) and $\eta'(c_1)c_1 = \kappa\eta(c_1)$. η linear and the common isoelastic formulation $\eta(h) = \frac{h^x}{x}$ for $x > 1$ satisfy $\frac{d \ln \eta(h)}{d \ln h} = \kappa$. The equations characterizing nonstationary allocations coincide when DM goods are priced competitively. This correspondence breaks down when $\theta < 1$; again, the differences hinge on the pricing

5. Π now appears in the budget constraint (as in Section 1). In equilibrium $\Pi = \sum_{t=0}^{\infty} \int q_t f^t(S^t) T_t dS^t$, where the firm's dividend is $T_t = p_{1,t}(1 - \tau)c_{1t}\delta$.

TABLE 1

QUANTITATIVE COMPARISON WITH LW

Parameter	Case 1	Case 2	Case 3	Case 4	Case 5
$\delta(\equiv \alpha\sigma)$	0.31	0.5	0.5	0.5	0.5
$a(\equiv \eta)$	0.27	0.16	0.30	0.30	0.30
B	2.13	1.97	1.91	1.78	1.78
θ	1	1	0.5	0.343	1
Inflation			Panel 1: Equilibrium c_1		
10%	0.243	0.206	0.143	0.094	0.523
0%	0.638	0.618	0.442	0.296	0.821
−4%	1	1	0.779	0.568	1
Inflation			Panel 2: Average markup		
10%	0	0	{0.056, 0.050}	{0.049, 0.050}	0
0%	0	0	{0.141, 0.123}	{0.123, 0.114}	0
−4%	0	0	{0.213, 0.183}	{0.196, 0.172}	0
Alternative inflation			Panel 3: Welfare cost of 10% inflation		
0%	0.014	0.014	0.032	0.046	0.012
−4%	0.016	0.016	0.042	0.068	0.013

NOTES: The calibration follows LW, table 1. The Parameters column reports our notation (the corresponding LW notation is in parentheses, if different). In each model $c_2 = B$, $\beta^{-1} = 1.04$, and the inflation rate is $\gamma - 1$. We report numbers as a pair {LW, CIA}, only if the numbers differ in the two models.

mechanism assumed to govern transactions that must be settled with money. In this case, there may exist equilibria which are not the same in the two models.

2.4 *Quantitative Comparison: Welfare Cost of Inflation*

To evaluate possible quantitative differences between the CIA and LW model, we adopt the specification in LW table 1, which considers stationary equilibrium and a calibration to annual U.S. data (see SI for details). We find identical welfare costs of inflation in the CIA and LW models, when price distortions are similar.

LW calibrates θ to match the average price markup in U.S. data; the theoretical markup is $\frac{z(c_1;\theta)}{c_1\eta'(c_1)}$, that is, the ratio of the DM good price p_1 to marginal cost (see SI). In the CIA model, we use the calibrated value of θ from LW; the markup is $\frac{p_1}{w_1} = \frac{1}{\tau} \equiv \frac{z'(c_1;\theta)}{\eta'(c_1)}$ when we match the price distortion in LW by setting $\tau = \psi(c_1;\theta)$. Hence, the markups in the two models generally do not coincide.

Table 1 compares results for the CIA and LW model, in five cases. Panel 1 shows that the two models yield identical consumption. Panel 2 shows that average price markups are comparable; given the LW parameter θ, markups increase with inflation in each model. Moreover, if we interpret $\frac{1}{\tau} - 1$ as the sales tax rate on cash trades, then the CIA model does not imply unreasonable average sales tax rates (see SI). Panel 3 shows that the CIA and LW models yield identical welfare cost of inflation.

The CIA model generates the large welfare cost of inflation found in LW, once price distortions are accounted for (cases 3 and 4). This confirms that dissimilarities in the models' quantitative performance hinge on assuming different pricing mechanisms, not the structure of the model or the formulation of money (explicit or reduced form).

3. FINAL COMMENTS

We have examined two monetary models characterized by periodic interactions in CM and DM, as in Lucas (1984), and as in LW (2005). After placing the models on equal footing in terms of preferences, technologies, and shocks, we showed that they reduce to a single equation describing stationary monetary equilibrium. The analysis reveals that the primary source of theoretical and quantitative difference between LW and a reduced-form CIA model is the pricing distortion assumed to operate in some cash transactions, not the structure of the model itself. Differences emerge if the models impose unequal pricing mechanisms on trades that *must* be settled using cash. The equations coincide when sellers have no bargaining power in LW and otherwise differ due to a bargaining price distortion, which can be replicated in the CIA model using a suitable parametric formulation. In this case, the quantitative performance of the models is also comparable.

Our findings rely on altering neither the market structure in LW, nor the equilibrium concept or the fundamental structure of the CIA model. The analysis should not be taken to imply that nothing can be done with one model, which could not be done with the other. For example, a referee noted that though in cash and credit goods models existence of monetary equilibrium depends on curvature conditions for preferences, in some version of the LW model it can also be made to depend on the presence of participation costs for DM sellers (see Rocheteau and Wright 2005). It would indeed be interesting to introduce participation costs in the CIA model, and to comparatively explore situations in which not all goods are consumed. Our analysis can contribute to create scientific consensus in monetary economics, which, in light of the recent discussion in Romer (2015), we view as being topical as well as substantively and methodologically meaningful.

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