

Alma Mater Studiorum Università di Bologna
Archivio istituzionale della ricerca

Addendum to: "The Bolzano-Weierstrass theorem is the jump of weak König's lemma" [Ann. Pure Appl. Logic 163 (6) (2012) 623-655]

This is the final peer-reviewed author's accepted manuscript (postprint) of the following publication:

Published Version:

Vasco Brattka, Andrea Cettolo, Guido Gherardi, Alberto Marcone, Matthias Schröder (2017). Addendum to: "The Bolzano-Weierstrass theorem is the jump of weak König's lemma" [Ann. Pure Appl. Logic 163 (6) (2012) 623-655]. ANNALS OF PURE AND APPLIED LOGIC, 168(8), 1605-1608 [10.1016/j.apal.2017.04.004].

Availability:

This version is available at: <https://hdl.handle.net/11585/597811> since: 2020-07-02

Published:

DOI: <http://doi.org/10.1016/j.apal.2017.04.004>

Terms of use:

Some rights reserved. The terms and conditions for the reuse of this version of the manuscript are specified in the publishing policy. For all terms of use and more information see the publisher's website.

This item was downloaded from IRIS Università di Bologna (<https://cris.unibo.it/>).
When citing, please refer to the published version.

(Article begins on next page)

This is the final peer-reviewed accepted manuscript of:

V. Brattka, A. Cettolo, G. Gherardi, A. Marcone and M. Schröder (2017) Addendum to “The Bolzano-Weierstrass Theorem is the jump of weak König’s Lemma”. *Annals of Pure and Applied Logic*, vol 168, no. 8, pp. 1605-1608, DOI: 10.1016/j.apal.2017.04.004

final published version is available online at: <http://dx.doi.org/10.1016/j.apal.2017.04.004>

Rights / License:

The terms and conditions for the reuse of this version of the manuscript are specified in the publishing policy. For all terms of use and more information see the publisher's website.

This item was downloaded from IRIS Università di Bologna (<https://cris.unibo.it/>)

When citing, please refer to the published version.

Addendum to: “The Bolzano–Weierstrass theorem is the jump of weak König’s lemma” [Ann. Pure Appl. Logic 163 (6) (2012) 623–655]

Vasco Brattka^{a,b,*}, Andrea Cettolo^c, Guido Gherardi^d, Alberto Marcone^c,
Matthias Schröder^e

^a *Department of Mathematics & Applied Mathematics, University of Cape Town, South Africa*

^b *Faculty of Computer Science, Universität der Bundeswehr München, Germany*

^c *Dipartimento di Scienze Matematiche, Informatiche e Fisiche, Università di Udine, Italy*

^d *Dipartimento di Filosofia e Comunicazione, Università di Bologna, Italy*

^e *Fachbereich Mathematik, Universität Darmstadt, Germany*

A B S T R A C T

The purpose of this addendum is to close a gap in the proof of [1, Theorem 11.2], which characterizes the computational content of the Bolzano–Weierstraß Theorem for arbitrary computable metric spaces.

In [1, Theorem 11.2] it is stated that $\text{BWT}_X \equiv_{\text{sW}} K'_X$ holds for all computable metric spaces X . Here BWT_X denotes the Bolzano–Weierstraß Theorem, K'_X denotes the jump of compact choice and \equiv_{sW} stands for strong Weihrauch equivalence. We refer the reader to [1] for the definition of all notions that are not defined here.

While the reduction $\text{BWT}_X \leq_{\text{sW}} K'_X$ was proved correctly in [1], the proof provided for $K'_X \leq_{\text{sW}} \text{BWT}_X$ contains a gap and is only correct for the special case of compact X as it stands. This fact was pointed out by one of us (M. Schröder) and is due to the fact that in general the closure of $L_X^{-1}(K)$ is not compact. We close this gap in this addendum.

We start with a lemma that shows that compact sets given in $\mathcal{K}'_-(X)$ are effectively totally bounded in a particular sense. By $\mathcal{O}(X)$ we denote the set of open subsets of X , represented as complements of elements

E-mail addresses: Vasco.Brattka@cca-net.de (V. Brattka), Andrea.Cettolo@spes.uniud.it (A. Cettolo), Guido.Gherardi@unibo.it (G. Gherardi), Alberto.Marcone@uniud.it (A. Marcone), Matthias.Schroeder@cca-net.de (M. Schröder).

of $\mathcal{A}_-(X)$, i.e., p is a name of an open set U if and only if it is a ψ_- -name of the closed set $X \setminus U$. We call an open ball $B(a, r)$ *rational*, if a is a point of the dense subset of X (that is used to define the computable metric space X) and $r \geq 0$ is a rational number.

Lemma 1. *Let X be a computable metric space. Consider the multivalued function $F_X : \subseteq \mathcal{K}'_-(X) \rightrightarrows \mathcal{O}(X)^\mathbb{N}$ with $\text{dom}(F_X) = \{K \in \mathcal{K}'_-(X) : K \neq \emptyset\}$ and such that, for each $K \neq \emptyset$, we have $(U_n)_n \in F_X(K)$ if and only if the following conditions hold for each $n \in \mathbb{N}$:*

- (1) U_n is a union of finitely many rational open balls of radius $\leq 2^{-n}$,
- (2) $K \subseteq U_n$.

Then F_X is computable.

Proof. Let X be a computable metric space and let $K \subseteq X$ be a nonempty compact set. Let $\langle p_i \rangle_i$ be a κ'_- -name of K . This means that $p := \lim_{i \rightarrow \infty} p_i$ is a κ_- -name for K and, in particular, for each $n \in \mathbb{N}$:

- $p_i(n)$ is a name for a finite set of rational open balls for each $i \in \mathbb{N}$,
- there exists $k \in \mathbb{N}$ such that the finite set of rational balls given by $p_k(n)$ covers K and $p_k(n) = p_i(n)$ for all $i \geq k$.

We also have that $\{p(n) : n \in \mathbb{N}\}$ is a set of names of all finite covers of K by rational open balls. We want to build a sequence of open sets $(U_n)_n$ such that (1) and (2) hold. We describe how to construct a name of a generic open set U_n for $n \in \mathbb{N}$. We start at stage 0 with $U_n = \emptyset$. At each stage $s = \langle m, i \rangle$ that the computation reaches, we focus on the balls $B(a_0, r_0), \dots, B(a_l, r_l)$ given by $p_i(m)$ and we check whether $r_0, \dots, r_l \leq 2^{-n}$. If this is not true, then we go to stage $s + 1$. Otherwise, if the condition is met, we add these balls to the name of U_n and we check whether $p_i(m) = p_{i+1}(m)$. If this is the case we add again $B(a_0, r_0), \dots, B(a_l, r_l)$ to the name of U_n . We repeat this operation as long as we find the same open balls given by $p_j(m)$ for $j > i$. If we find $p_i(m) \neq p_j(m)$ for some $j > i$, then the computation goes to stage $s + 1$.

We claim that, for each n , there exists a stage in which the computation goes on indefinitely. Consider, in fact, $\{B(a_0, r_0), \dots, B(a_l, r_l)\}$, a finite rational cover of K with $r_0, \dots, r_l \leq 2^{-n}$, which exists by a simple argument using the compactness of K . Since $\langle p_i \rangle_i$ is a κ'_- -name of K , there exists a minimum $\langle m, i \rangle$ such that:

- $p_i(m)$ is a name for the cover $\{B(a_0, r_0), \dots, B(a_l, r_l)\}$,
- $p_i(m) = p_j(m)$ for each $j > i$.

If the algorithm reaches stage $s = \langle m, i \rangle$, then it is clear that the computation goes on indefinitely within this stage. If the algorithm never reaches stage s , then necessarily it already stopped at a previous stage. In both cases our claim is true.

Finally, since we built the name of U_n by adding only balls of radius $\leq 2^{-n}$ and since the computation stabilizes at a finite stage, it is clear that conditions (1) and (2) are met. \square

We note that even though the open sets U_n constructed in the previous proof are finite unions of rational open balls, the algorithm does not provide a corresponding rational cover in a finitary way. It rather provides an infinite list of rational open balls that is guaranteed to contain only finitely many distinct rational balls. This is a weak form of effective total boundedness and the best one can hope for, given that the input is represented by the jump of κ_- .

The following lemma shows that sequences that we choose in $\text{range}(F_X)$ in a particular way give rise to totally bounded sets.

Lemma 2. *Let X be a metric space and let $U_n \subseteq X$ be a finite union of balls of radius $\leq 2^{-n}$ for each $n \in \mathbb{N}$. Let $(x_n)_n$ be a sequence in X with $x_n \in \bigcap_{i=0}^n U_i$. Then $\{x_n : n \in \mathbb{N}\}$ is totally bounded.*

Proof. We obtain $\{x_n : n \in \mathbb{N}\} \subseteq \bigcap_{i=0}^{\infty} \left(U_i \cup \bigcup_{n=0}^{i-1} B(x_n, 2^{-i}) \right)$ and the set on the right-hand side is clearly totally bounded. Hence the set on the left-hand side is totally bounded and so is its closure. \square

We mention that it is well known that a subset of a metric space is totally bounded if and only if any sequence in it has a Cauchy subsequence [2, Exercise 4.3.A (a)].

Now we use the previous two lemmas to complete the proof of [1, Theorem 11.2]. Within the proof we use the canonical completion \hat{X} of a computable metric space. It is known that this completion is a computable metric space again and that the canonical embedding $X \hookrightarrow \hat{X}$ is a computable isometry that preserves the dense sequence [3, Lemma 8.1.6]. We will identify X with a subset of \hat{X} via this embedding.

Theorem 3 ([1, Theorem 11.2]). $\text{BWT}_X \equiv_{\text{sw}} K'_X$ for all computable metric spaces X .

Proof. The reduction $\text{BWT}_X \leq_{\text{sw}} K'_X$ has been proved in [1], so we focus on the reduction $K'_X \leq_{\text{sw}} \text{BWT}_X$. Let (X, d, α) be a computable metric space and let $K \subseteq X$ be a nonempty compact set given by a κ'_- -name $\langle p_i \rangle_i$. We want to compute a point of K using BWT_X . The idea is to define a sequence $(x_n)_n$ in X , working within the completion \hat{X} of X and using the open sets built in Lemma 1, such that $\overline{\{x_n : n \in \mathbb{N}\}}$ is compact in X .

It is clear that K is a compact subset of \hat{X} and that $\langle p_i \rangle_i$ can be considered as a κ'_- -name for K in \hat{X} . We consider the map

$$\mathsf{L}_{\hat{X}} : \hat{X}^{\mathbb{N}} \rightarrow \mathcal{A}'_-(\hat{X}), (x_n)_n \mapsto \{x \in \hat{X} : x \text{ is a cluster point of } (x_n)_n\}.$$

By [1, Corollary 9.5] $\mathsf{L}_{\hat{X}}^{-1}$ is computable and hence $\mathsf{L}_{\hat{X}}^{-1}(K)$ yields a sequence $(z_m)_m$ in \hat{X} whose cluster points are exactly the elements of K .

Let $F_{\hat{X}}$ be the multivalued function defined in Lemma 1. We can compute a sequence $(U_n)_n \in F_{\hat{X}}(K)$. Since $\{z_m : m \in \mathbb{N}\}$ is not compact (and hence not in $\text{dom}(\text{BWT}_X)$) in general, we refine it recursively to a sequence $(y_n)_n$ using $(U_n)_n$ in the following way: for each $n \in \mathbb{N}$, $y_n := z_{m_n}$ for the first m_n that we find with $z_{m_n} \in U_0 \cap \dots \cap U_n$ and such that $m_i < m_n$ for all $i < n$. Note that we can always find such a y_n , since $U_0 \cap \dots \cap U_n$ covers K which is the set of cluster points of $(z_m)_m$. Clearly every cluster point of $(y_n)_n$ is also a cluster point of $(z_m)_m$, hence it belongs to K .

Recall now that $(y_n)_n$ is a sequence of points in \hat{X} and that we want a sequence $(x_n)_n$ in X in order to apply BWT_X . We compute $(x_n)_n$ as follows: for each $n \in \mathbb{N}$, x_n is the first element that we find in the dense subset $\text{range}(\alpha)$ such that $d(x_n, y_n) < 2^{-n}$ and $x_n \in U_0 \cap \dots \cap U_n$, where d also denotes the extension of the metric to \hat{X} . By density of X in \hat{X} such an x_n always exists and it is clear that the cluster points of $(x_n)_n$ and those of $(y_n)_n$ are the same in \hat{X} .

Now $A := \overline{\{x_n : n \in \mathbb{N}\}}$ is totally bounded in X by Lemma 2 and hence every sequence in A has a Cauchy subsequence, which has a limit in \hat{X} , since \hat{X} is complete. By construction of $(x_n)_n$ the limit of such a subsequence is in K and hence in X . Thus every sequence in A has a subsequence that converges in X and hence A is compact in X .

Finally, we can obtain an element of K by applying BWT_X to $(x_n)_n$. \square

References

- [1] Vasco Brattka, Guido Gherardi, Alberto Marcone, The Bolzano–Weierstrass theorem is the jump of weak König’s lemma, *Ann. Pure Appl. Logic* 163 (2012) 623–655.
- [2] Ryszard Engelking, *General Topology*, Sigma Ser. Pure Math., vol. 6, Heldermann, Berlin, 1989.
- [3] Klaus Weihrauch, *Computable Analysis*, Springer, Berlin, 2000.