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Higher integrability for minimizers of asymptotically convex integrals with discontinuous coefficients

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HIGHER INTEGRABILITY FOR MINIMIZERS OF ASYMPTOTICALLY CONVEX INTEGRALS WITH DISCONTINUOUS COEFFICIENTS

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This paper is dedicated to Nicola Fusco, who directed us to the theory of regularity in the Calculus of Variations, on the occasion of his 60th birthday.

ABSTRACT. We study the local regularity of vectorial minimizers of integral functionals with standard p-growth. We assume that the non-homogeneous densities are uniformly convex and have a radial structure, with respect to the gradient variable, only at infinity. Under a $W^{1,n}$-Sobolev dependence on the spatial variable of the integrand, n being the space dimension, we show that the minimizers have the gradient locally in L^q for every $q > p$. As a consequence, they are locally α-Hölder continuous for every $\alpha < 1$.

1. INTRODUCTION

In this paper we study the regularity of vectorial local minimizers of functionals with irregular integrands in the x-variable and only asymptotically convex with respect to the gradient variable. In order to state our result precisely, we introduce right now our hypotheses. We will consider

$$\mathcal{F}(u; \Omega) := \int_\Omega f(x, Du) \, dx, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n > 2$, is a bounded open set, $u : \Omega \to \mathbb{R}^N$, $N > 1$, is a Sobolev map and $f : \Omega \times \mathbb{R}^{nN} \to [0, +\infty)$ is a Carathéodory function convex with respect to the second variable.

As it is well known since the famous example by De Giorgi [13] (see also [39], [40], [45]), in order to avoid the irregularity phenomena peculiar of the vectorial minimizers, the dependence of the energy density on the modulus of the gradient variable is necessary. We shall assume it only *at infinity*, i.e., for large values of the gradient variable ξ. Precisely:

(A1) there exist $\tilde{R} > 0$ and a function $\tilde{f} : \Omega \times [\tilde{R}, +\infty) \to [0, +\infty)$ such that

$$f(x, \xi) = \tilde{f}(x, |\xi|), \tag{1.2}$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^{nN} \setminus B_{\tilde{R}}(0)$.

The integrand f will satisfy the so-called *p-growth condition*, that is

(A2) there exist an exponent $p > 1$ and constants $c_1, c_2, L > 0$ such that

$$c_1 |\xi|^p - c_2 \leq f(x, \xi) \leq L(1 + |\xi|)^p,$$

for a.e. $x \in \Omega$ and $\xi \in \mathbb{R}^{nN}$.

The usual *p-uniform convexity* will be assumed only *at infinity*. More precisely, we shall suppose that $\xi \to f(x, \xi) \in C^2(\mathbb{R}^{nN} \setminus B_{\tilde{R}}(0))$ and

(A3) there exists $\nu > 0$ such that

$$\langle D_{\xi\xi}f(x,\xi)\lambda, \lambda \rangle \geq \nu (1 + |\xi|)^{p-2}|\lambda|^2,$$

for a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^{nN} \setminus B_{\tilde{R}}(0)$ and for every $\lambda \in \mathbb{R}^{nN}$.

Note that, since f is C^2 with respect to the gradient variable outside the ball $B_{\tilde{R}}(0)$, the assumption in (A3) is equivalent to the C^2- asymptotic convexity introduced in [5].

Also the bound on the second order derivatives in the gradient variable will be required only at infinity. Indeed, we shall assume that

(A4) there exists $L_1 > 0$ such that

$$|D_{\xi\xi}f(x,\xi)| \leq L_1(1 + |\xi|)^{p-2},$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^{nN} \setminus B_{\tilde{R}}(0)$.

We now introduce the main property of our energy density. As already mentioned, we will not ask a regular dependence of f on the x-variable. Indeed, the function $x \to D_\xi f(x, \xi)$ will be required to be weakly differentiable for every $\xi \in \mathbb{R}^{nN} \setminus B_{\tilde{R}}(0)$ and it will be assumed that

(A5) there exists a non-negative function $k \in L^n_{\text{loc}}(\Omega)$ such that

$$|D_{\xi x}f(x,\xi)| \leq k(x)(1 + |\xi|)^{p-1},$$

for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^{nN} \setminus B_{\tilde{R}}(0)$.

If we take into account that many properties are requested only *at infinity*, a model functional in our setting is

$$\mathcal{F}(u; \Omega) := \int_\Omega a(x)|Du|^p \, dx, \qquad p > 1$$

with $a \in W^{1,n}_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$, $a \geq 1$.

Let us now briefly discuss our assumptions.

We begin noting that the counterpart of the uniform p-convexity at infinity, in case of *non-smooth* functions f is that there exists $\nu > 0$ such that

$$\frac{1}{2}[f(x,\xi_1) + f(x,\xi_2)] \geq f\left(x, \frac{\xi_1 + \xi_2}{2}\right) + \nu(1 + |\xi_1|^2 + |\xi_2|^2)^{\frac{p-2}{2}}|\xi_1 - \xi_2|^2, \qquad (1.3)$$

for a.e. $x \in \Omega$ and for every $\xi_1, \xi_2 \in \mathbb{R}^{nN} \setminus B_{\tilde{R}}(0)$ endpoints of a segment contained in the complement of $B_{\tilde{R}}(0)$. This property was introduced by Fonseca, Fusco and Marcellini in [18] in order to study the existence of real valued minimizers of some non convex variational problems. Already in such a paper, the condition (1.3) has been revealed sufficient for the Lipschitz regularity of local minimizers. Since then, the regularity properties of local minimizers of $\mathcal{F}(u; \Omega)$ when the integrand f is p-uniformly convex at infinity with respect to the the gradient variable have been widely investigated assuming a dependence on the x variable through a smooth function (see [4], [9], [10], [17], [20], [21], [22], [23], [24], [44]).

On the other hand, many regularity results are available when the integrand f is uniformly convex with respect to the last variable but discontinuous with respect to x. In particular, when the dependence on x is given through a coefficient belonging to a suitable Sobolev class, the higher differentiability as well as the partial Hölder continuity of the gradient of the local minimizers have been obtained in [28], [32], [41], [42], [43]. More recently, the regularity of the solutions of some parabolic systems with Sobolev coefficients has been faced in [30]. We refer to [27], [29] for the case of functionals with variable exponents growth condition.

The aim of this paper is to show that the Sobolev dependence on the x-variable expressed by the assumption (A5) is sufficient to prove that, under a p-convexity condition only at infinity, the local minimizers u of $\mathcal{F}$ have the gradient locally in L^q for every $q > p$ and, therefore, that they are locally Hölder continuous for every exponent $0 < \alpha < 1$.

More precisely, taking into account that, without loss of generality, we can assume $\tilde{R} = 1$, our main result is the following.

Theorem 1.1. *Let $f : \Omega \times \mathbb{R}^{nN} \to [0, +\infty)$ be a Carathéodory function such that $f = f(x, \xi)$ is convex and C^2 with respect to the last variable and satisfies the assumptions (A1)-(A5), with $p > 1$. Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a local minimizer of (1.1). Then u is locally α-Hölder continuous for all $\alpha \in (0, 1)$. Moreover, for all $q > p$ and for all $B_\rho(x_0) \subset B_R(x_0) \Subset \Omega$ we have that*

$$\left[\fint_{B_\rho(x_0)} |Du|^q \, dx \right]^{\frac{1}{q}} \leq C \left[\fint_{B_R(x_0)} (1 + f(x, Du)) \, dx \right]^{\frac{1}{p}},$$

where $C = C(n, N, p, q, L, L_1, \nu, c_1, c_2, \rho, R)$.

We remark that Hölder continuity results for any exponent α strictly less than 1, are not uncommon when the integrands depend on x. We refer to [8] and [42] for examples of not locally Lipschitz continuous minimizers, but Hölder continuous for every exponent.

It is worth pointing out that we were inspired by a regularity result proved in [14] for functionals satisfying the non-standard (p, q)-growth conditions. We recall that the theory of regularity of minimizers in this framework of non-standard growths was started by Marcellini, see [36], [37], see also Acerbi-Fusco [2], Boccardo-Marcellini-Sbordone [3], Fusco-Sbordone [25], [26], and later widely investigated by many authors and in different settings of applicability. For more details and references on this subject we refer to [38].

Besides the (p, q)-growth condition, Eleuteri, Marcellini and Mascolo in [14] deal with an integrand f satisfying (1.2) for all $\xi \in \mathbb{R}^{nN}$, but satisfying the assumption (A5) with a function k belonging to L^r, with $r > n$. They proved that the local minimizers of the integrals are locally Lipschitz continuous if $\frac{q}{p} < 1 + \frac{1}{n} - \frac{1}{r}$ (note that this assumption on the gap $\frac{q}{p}$ is sharp as shown in [16] and [19]). See also the papers by Colombo and Mingione [6], [7] for related results. We also cite the paper [34] by Kristensen and Mingione, see also Kuusi and Mingione [35], for other interesting regularity results for vector valued minimizers under weak properties in the x-dependence of the energy density.

Even though the proof of Theorem 1.1 relies on analogous arguments to those in [14], it seems that under the weaker assumption $k \in L^n_{\mathrm{loc}}(\Omega)$ the local Lipschitz continuity of the minimizers cannot be achieved even if $p = q$.

As in [14], we shall combine a suitable a priori estimate for the gradient of the local minimizers with an approximation procedure. As a common feature, we shall establish the a priori estimate for smooth minimizers and then we will show that such a priori estimate is preserved in passing to the limit.

Note that, in order to face new difficulties arising from our weaker assumptions on the function k, to obtain the apriori estimate we shall use a double iteration procedure that will allow us to reabsorb certain integrals involving the gradient with a critical integrability exponent. Moreover, in our approximation procedure will have to take into account the lack of convexity of the integrand for small values of the gradient.

Remark 1.2. *When f satisfies the assumptions of Theorem 1.1, but is not convex in the last variable, the local minimizers of $\mathcal{F}$ still satisfies the thesis of the previous theorem, provided*

$$\mathrm{sc}^- \mathcal{F}(w; \Omega) = \int_\Omega f^{**}(x, Dw) \, dx, \tag{1.4}$$

where $\mathrm{sc}^- \mathcal{F}$ *is the relaxed functional of* $\mathcal{F}$ *and* $\xi \mapsto f^{**}(x, \xi)$ *is the convex envelope of* $f(x, \cdot)$. *In fact,* f^{**} *fulfills (A1), (A2), (A3), (A4) with suitable constants* ν *and* R. *Moreover,* $f = f^{**}$ *in* $\Omega \times B_{\overline{R}}$ *for some* $\overline{R} \geq \tilde{R}$ *(see e.g. Theorem 2.5 (iv) in [9]) therefore (A5) holds for* f^{**}, *with* $\overline{R}$ *replacing* $\tilde{R}$. *Since* $\mathrm{sc}^- \mathcal{F}$ *is the relaxed functional, if* u *is a local minimizer of* $\mathcal{F}$, *then* u *is a local minimizer of* $\mathrm{sc}^- \mathcal{F}$, *too. As said, the integrand* f^{**} *satisfies the assumptions of Theorem 1.1; thus Theorem 1.1 applies and we get that the gradient of* u *is in* $L_{\mathrm{loc}}^q(\Omega)$ *for every finite* $q > p$.

The plan of the paper is the following: in Section 2 we fix the notations and collect preliminary results; in Section 3 we establish the a priori estimate and in Section 4 we give the proof of the main result.

2. PRELIMINARIES

In this section we recall some standard definitions and collect several lemmas that we shall use to establish our main result.

First of all, we recall that $u \in W_{\mathrm{loc}}^{1,p}(\Omega, \mathbb{R}^N)$ is a *local minimizer* of $\mathcal{F}$ in (1.1) if

$$\mathcal{F}(u; \mathrm{supp}\, \varphi) \leq \mathcal{F}(u + \varphi; \mathrm{supp}\, \varphi)$$

for any $\varphi \in W^{1,p}(\Omega, \mathbb{R}^N)$ with $\mathrm{spt}\, \varphi \Subset \Omega$.

We will follow the usual convention and denote by C a general constant that may vary on different occasions, even within the same line of estimates. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses or subscripts. All the norms we use on $\mathbb{R}^n$, $\mathbb{R}^N$ and $\mathbb{R}^{nN}$ will be the standard Euclidean ones and denoted by $|\cdot|$ in all cases. In particular, for matrices $\xi, \eta \in \mathbb{R}^{nN}$ we write $\langle \xi, \eta \rangle := \mathrm{trace}(\xi^T \eta)$ for the usual inner product of ξ and η, and $|\xi| := \langle \xi, \xi \rangle^{\frac{1}{2}}$ for the corresponding euclidean norm.

In what follows, $B_r^n(x)$ will denote the ball in $\mathbb{R}^n$ centered at x of radius r. Analogously, $B_r^{nN}(\xi)$ is a ball in $\mathbb{R}^{nN}$. If no confusion may arise we shall omit the superscripts related to the space dimension and the dependence on the center.

The following lemma finds an important application in the so called hole-filling method. Its proof can be found for example in [33, Lemma 6.1] .

Lemma 2.1. *Let* $h : [r, R_0] \to \mathbb{R}$ *be a non-negative bounded function and* $0 < \vartheta < 1$, $A, B \geq 0$ *and* $\beta > 0$. *Assume that*

$$h(s) \leq \vartheta h(t) + \frac{A}{(t-s)^\beta} + B,$$

for all $r \leq s < t \leq R_0$. *Then*

$$h(r) \leq \frac{cA}{(R_0 - r)^\beta} + cB,$$

where $c = c(\vartheta, \beta) > 0$.

We shall need the following

Lemma 2.2. *Let* $u \in W_{\mathrm{loc}}^{1,p}(\Omega, \mathbb{R}^N)$ *be a local minimizer of*

$$\mathcal{F}(w; \Omega) = \int_\Omega f(x, Dw(x))\, dx,$$

where $f : \Omega \times \mathbb{R}^{nN} \to [0, +\infty)$ *is a Carathéodory function, convex with respect to the last variable, satisfying (A2) and (A3). Suppose that there exist* $B_r(x_0) \Subset \Omega$ *and* $v \in u + W_0^{1,1}(B_r(x_0), \mathbb{R}^N)$ *such that* $\mathcal{F}(u; B_r(x_0)) = \mathcal{F}(v; B_r(x_0))$. *Then there exists* $R_0(p, \nu, \tilde{R}, L) > \tilde{R}$ *such that the Lebesgue measure of the set*

$$\{x \in B_r(x_0) : |Du(x) + Dv(x)| > 2R_0 \ \ and \ \ |Du(x) - Dv(x)| > 0\} \tag{2.1}$$

is zero.

For the proof we refer to [9]. We conclude recalling a regularity result for minimizers of functionals whose integrands are smooth with respect to the x-variable (see e.g. [31] for the case $p \geq 2$ and [1] for the case $1 < p < 2$; see also [15] and, in particular, Theorem 1.1 in [9] as far as the Lipschitz continuity of the local minimizers is concerned).

Lemma 2.3. *Let $u \in W_{\text{loc}}^{1,p}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional at (1.1), with $f : \Omega \times \mathbb{R}^{nN} \to [0, +\infty)$, $f \in C^2(\Omega \times \mathbb{R}^{nN})$. Assume that there exists $p > 1$ such that for every $x \in \Omega$ and every $\xi, \lambda \in \mathbb{R}^{nN}$,*

$$c_1|\xi|^p - c_2 \leq f(x, \xi) \leq L(1 + |\xi|)^p,$$

$$\nu (1 + |\xi|)^{p-2}|\lambda|^2 \leq \langle D_{\xi\xi}f(x, \xi)\lambda, \lambda \rangle,$$

$$|D_{\xi\xi}f(x, \xi)| \leq L_1 (1 + |\xi|)^{p-2},$$

$$|D_{\xi x}f(x, \xi)| \leq K(1 + |\xi|)^{p-1},$$

with positive constants c_1, c_2, L, L_1, ν, K.

Then $u \in W_{\text{loc}}^{2,a}(\Omega)$, with $a = \min\{2, p\}$ and $(1 + |Du|^2)^{\frac{p-2}{2}}|D^2u|^2 \in L_{\text{loc}}^1(\Omega)$. If, moreover, (A1) holds, then $u \in W_{\text{loc}}^{1,\infty}(\Omega)$ and $u \in W_{\text{loc}}^{2,2}(\Omega)$.

3. THE A PRIORI ESTIMATE

The following a priori estimate is the key tool for our main result. Its proof follows the one of [14, Proposition 3.1], but it is drastically different in the second step were we have to take into account the weaker assumption on the function $k(x)$ appearing in (A5).

Theorem 3.1. *Let $u \in W_{\text{loc}}^{1,p}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}$ at (1.1) with the integrand f satisfying the assumptions (A1)-(A5). If $u \in W_{\text{loc}}^{2,\min\{2,p\}}(\Omega; \mathbb{R}^N) \cap W_{\text{loc}}^{1,\infty}(\Omega; \mathbb{R}^N)$ and $(1 + |Du|^2)^{\frac{p-2}{2}}|D^2u|^2 \in L_{\text{loc}}^1(\Omega)$, then, for every exponent $q \geq p$ and for all $x_0 \in \Omega$, there exist a radius $R_q = R_q(n, N, p, q, \nu, k, L, L_1, x_0)$ and a ball $B_{R_q}(x_0) \Subset \Omega$ such that the following estimate*

$$\left[\int_{B_\rho(x_0)} |Du|^q \, dx \right]^{\frac{1}{q}} \leq C \left[\int_{B_R(x_0)} (1 + f(x, Du)) \, dx \right]^{\frac{1}{p}},$$

holds for every ρ and R, with $0 < \rho < R < R_q$, and with a positive constant $C = C(n, N, p, q, L, L_1, \nu, c_1, c_2, \rho, R)$.

Proof. We shall divide the proof into three steps.

Step 1. Our first aim is to prove that, for every $\gamma \geq 0$ and for every cut off function $\eta \in C_0^\infty(\Omega)$, the following integral estimate holds

$$\int_\Omega \eta^2 (1 + (|Du| - 1)_+)^{\gamma-2}(|Du| - 1)_+)^2|Du|^{p-2}|D^2u|^2 \, dx$$

$$\leq C(\gamma + 1)^2 \int_\Omega \eta^2 k^2(x)(1 + (|Du| - 1)_+)^{\gamma+p} \, dx$$

$$+ C \int_\Omega |D\eta|^2 (1 + (|Du| - 1)_+)^{\gamma+p} \, dx, \tag{3.1}$$

with $C = C(n, N, L, L_1, \nu, p)$.

Since u is a local minimizer of the functional $\mathcal{F}(u, \Omega)$, then u satisfies the Euler's system

$$\int_\Omega \sum_{i,\alpha} f_{\xi_i^\alpha}(x, Du)\varphi_{x_i}^\alpha(x)\, dx = 0 \qquad \forall \varphi \in C_0^\infty(\Omega; \mathbb{R}^N),$$

and, using the second variation, for every $s = 1, \ldots, n$ it holds

$$\int_\Omega \left\{ \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du)\varphi_{x_i}^\alpha u_{x_s x_j}^\beta + \sum_{i,\alpha} f_{\xi_i^\alpha x_s}(x, Du)\varphi_{x_i}^\alpha \right\} dx = 0 \qquad \forall \varphi \in C_0^\infty(\Omega; \mathbb{R}^N). \quad (3.2)$$

Fix $s = 1, \ldots, n$, an exponent $\gamma \geq 0$, a cut off function $\eta \in C_0^\infty(\Omega)$ and consider the function

$$\varphi^\alpha := \eta^2 u_{x_s}^\alpha (1 + (|Du| - 1)_+)^{\gamma - 2}((|Du| - 1)_+)^2 \quad \alpha = 1, \ldots, N.$$

Therefore, denoting

$$\Phi(t) := (1 + t)^{\gamma - 2} t^2,$$

we have

$$\begin{aligned}
\varphi_{x_i}^\alpha =\, &2\eta \eta_{x_i} u_{x_s}^\alpha \Phi((|Du| - 1)_+) \\
&+ \eta^2 u_{x_s x_i}^\alpha \Phi((|Du| - 1)_+) + \eta^2 u_{x_s}^\alpha \Phi'((|Du| - 1)_+)[(|Du| - 1)_+]_{x_i}.
\end{aligned}$$

Thanks to our assumptions on the minimizer u, through a standard density argument, we can use φ as test function in the equation (3.2), thus getting

$$\begin{aligned}
0 = &\int_\Omega 2\eta \Phi((|Du| - 1)_+) \sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du)\eta_{x_i} u_{x_s}^\alpha u_{x_s x_j}^\beta\, dx \\
&+ \int_\Omega \eta^2 \Phi((|Du| - 1)_+) \sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du) u_{x_s x_i}^\alpha u_{x_s x_j}^\beta\, dx \\
&+ \int_\Omega \eta^2 \Phi'((|Du| - 1)_+) \sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du) u_{x_s}^\alpha u_{x_s x_j}^\beta [(|Du| - 1)_+]_{x_i}\, dx \\
&+ \int_\Omega 2\eta \Phi((|Du| - 1)_+) \sum_{i,s,\alpha} f_{\xi_i^\alpha x_s}(x, Du)\eta_{x_i} u_{x_s}^\alpha\, dx \\
&+ \int_\Omega \eta^2 \Phi((|Du| - 1)_+) \sum_{i,s,\alpha} f_{\xi_i^\alpha x_s}(x, Du) u_{x_s x_i}^\alpha\, dx \\
&+ \int_\Omega \eta^2 \Phi'((|Du| - 1)_+) \sum_{i,s,\alpha} f_{\xi_i^\alpha x_s}(x, Du) u_{x_s}^\alpha [(|Du| - 1)_+]_{x_i}\, dx \\
=: &\, I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \quad (3.3)
\end{aligned}$$

Let us now estimate the integrals I_j, $j = 1, \ldots, 6$.

ESTIMATE OF I_1

By (A3) and (A4), we can estimate the integral I_1 by the use of Cauchy-Schwartz and Young's inequalities as follows

$$|I_1| \leq 2 \int_\Omega \Phi((|Du| - 1)_+) \left\{ \sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du)\eta_{x_i} u_{x_s}^\alpha \eta_{x_j} u_{x_s}^\beta \right\}^{\frac{1}{2}} \left\{ \eta^2 \sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du) u_{x_s x_i}^\alpha u_{x_s x_j}^\beta \right\}^{\frac{1}{2}}$$

$$\leq C(\varepsilon, L_1) \int_\Omega |D\eta|^2 \Phi((|Du| - 1)_+)|Du|^p \, dx$$

$$+ \varepsilon \int_\Omega \eta^2 \Phi((|Du| - 1)_+) \sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du) u_{x_s x_i}^\alpha u_{x_s x_j}^\beta \, dx, \tag{3.4}$$

where $\varepsilon > 0$ will be chosen later.

ESTIMATE OF I_3

Since

$$f_{\xi_i^\alpha \xi_j^\beta}(x, \xi) = \left(\frac{\tilde{f}_{tt}(x, |\xi|)}{|\xi|^2} - \frac{\tilde{f}_t(x, |\xi|)}{|\xi|^3} \right) \xi_i^\alpha \xi_j^\beta + \frac{\tilde{f}_t(x, |\xi|)}{|\xi|} \delta_{\xi_i^\alpha \xi_j^\beta}$$

and

$$[(|Du| - 1)_+]_{x_i} = (|Du|)_{x_i} = \frac{1}{|Du|} \sum_{\alpha,s} u_{x_i x_s}^\alpha u_{x_s}^\alpha \qquad \text{a.e. in } \{|Du| \geq 1\} \tag{3.5}$$

then, for a.e. $x \in \{|Du| \geq 1\}$,

$$\sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du) u_{x_s}^\alpha u_{x_s x_j}^\beta [(|Du| - 1)_+]_{x_i}$$

$$= \left(\frac{\tilde{f}_{tt}(x, |Du|)}{|Du|^2} - \frac{\tilde{f}_t(x, |Du|)}{|Du|^3} \right) \sum_{i,j,s,\alpha,\beta} u_{x_s}^\alpha u_{x_s x_j}^\beta u_{x_i}^\alpha u_{x_j}^\beta [(|Du| - 1)_+]_{x_i}$$

$$+ \frac{\tilde{f}_t(x, |Du|)}{|Du|} \sum_{i,s,\alpha} u_{x_s}^\alpha u_{x_s x_i}^\alpha [(|Du| - 1)_+]_{x_i}$$

$$= \left(\frac{\tilde{f}_{tt}(x, |Du|)}{|Du|} - \frac{\tilde{f}_t(x, |Du|)}{|Du|^2} \right) \sum_{i,s,\alpha} u_{x_s}^\alpha (|Du|)_{x_s} u_{x_i}^\alpha (|Du|)_{x_i}$$

$$+ \frac{\tilde{f}_t(x, |Du|)}{|Du|} \sum_{i,s,\alpha} u_{x_s}^\alpha u_{x_s x_i}^\alpha (|Du|)_{x_i}$$

$$= \left(\frac{\tilde{f}_{tt}(x, |Du|)}{|Du|} - \frac{\tilde{f}_t(x, |Du|)}{|Du|^2} \right) \sum_\alpha \left(\sum_i u_{x_i}^\alpha (|Du|)_{x_i} \right)^2$$

$$+ \tilde{f}_t(x, |Du|)|D(|Du|)|^2. \tag{3.6}$$

Thus,

$$I_3 = \int_\Omega \eta^2 \Phi'((|Du| - 1)_+) \left\{ \left(\frac{\tilde{f}_{tt}(x, |Du|)}{|Du|} - \frac{\tilde{f}_t(x, |Du|)}{|Du|^2} \right) \sum_\alpha \left(\sum_i u_{x_i}^\alpha (|Du|)_{x_i} \right)^2 \right.$$

$$\left. + \tilde{f}_t(x, |Du|)|D(|Du|)|^2 \right\} dx.$$

Now, if we use the Cauchy-Schwartz inequality, we have

$$\sum_\alpha \left(\sum_i u_{x_i}^\alpha (|Du|)_{x_i} \right)^2 \leq |Du|^2 |D(|Du|)|^2$$

and observing that

$$\Phi'((|Du| - 1)_+) \geq 0 \quad \text{and} \quad \tilde{f}_t(x, |Du|) \geq 0,$$

we conclude

$$I_3 \geq \int_\Omega \eta^2 \Phi'((|Du| - 1)_+) \frac{\tilde{f}_{tt}(x, |Du|)}{|Du|} \sum_\alpha \left(\sum_i u_{x_i}^\alpha (|Du|)_{x_i} \right)^2 dx \geq 0. \tag{3.7}$$

ESTIMATE OF I_4

By using the assumption (A5) we obtain

$$
\begin{aligned}
|I_4| &\leq 2^{p-1} \int_\Omega \eta \Phi((|Du| - 1)_+) k(x) |Du|^{p-1} \sum_{i,s,\alpha} |\eta_{x_i} u_{x_s}^\alpha| \, dx \\
&\leq 2^{p-1} \int_\Omega \eta |D\eta| k(x) \Phi((|Du| - 1)_+) |Du|^p \, dx.
\end{aligned} \tag{3.8}
$$

ESTIMATE OF I_5

Using the assumption (A5) and Young's inequality we have that

$$
\begin{aligned}
|I_5| &\leq 2^{p-1} \int_\Omega \eta^2 \Phi((|Du| - 1)_+) k(x) |Du|^{p-1} |D^2 u| \, dx \\
&\leq \varepsilon \int_\Omega \eta^2 \Phi((|Du| - 1)_+) |Du|^{p-2} |D^2 u|^2 \, dx \\
&\quad + C_\varepsilon \int_\Omega \eta^2 \Phi((|Du| - 1)_+) k^2(x) |Du|^p \, dx,
\end{aligned} \tag{3.9}
$$

where $\varepsilon \in (0, 1)$ will be chosen later.

ESTIMATE OF I_6

Using the assumption (A5) and (3.5), we get

$$
\begin{aligned}
|I_6| &\leq 2^{p-1} \int_\Omega \eta^2 \Phi'((|Du| - 1)_+) k(x) |Du|^{p-1} |Du| \, |D((|Du| - 1)_+)| \, dx \\
&\leq C \int_\Omega \eta^2 \Phi'((|Du| - 1)_+) k(x) |Du|^p |D^2 u| \, dx.
\end{aligned} \tag{3.10}
$$

Now, since

$$\Phi'(t) = (\gamma t + 2) t (1 + t)^{\gamma - 3},$$

it follows that

$$t\Phi'(t) = t(\gamma t + 2) t (1 + t)^{\gamma - 3} = \frac{\gamma t + 2}{1 + t} \Phi(t) \leq 2(1 + \gamma)\Phi(t) \qquad \text{for all } t \geq 0. \tag{3.11}$$

Using the notation $c_\gamma := 2(1 + \gamma) > 0$, multiplying and dividing the integrand in the right hand side of (3.10) by $c_\gamma^{-1/2}(\delta + (|Du| - 1)_+)^{1/2}$, with $0 < \delta < 1$ to be chosen later, we have

$$
\begin{aligned}
|I_6| &\leq \int_\Omega \eta^2 \left\{ \frac{1}{c_\gamma} \Phi'((|Du| - 1)_+)(\delta + (|Du| - 1)_+) |Du|^{p-2} |D^2 u|^2 \right\}^{\frac{1}{2}} \\
&\quad \times \left\{ C^2 c_\gamma \Phi'((|Du| - 1)_+) k^2(x)(\delta + (|Du| - 1)_+)^{-1} |Du|^{p+2} \right\}^{\frac{1}{2}} dx \\
&\leq c_\gamma C_\varepsilon \int_\Omega \eta^2 \Phi'((|Du| - 1)_+) k^2(x) \, (\delta + (|Du| - 1)_+)^{-1} \, |Du|^{p+2} \, dx \\
&\quad + \frac{\varepsilon}{c_\gamma} \int_\Omega \eta^2 \Phi'((|Du| - 1)_+) \, (\delta + (|Du| - 1)_+) \, |Du|^{p-2} |D^2 u|^2 \, dx \,,
\end{aligned} \tag{3.12}
$$

where we used Young's inequality. In order to estimate the last integral in (3.12) we note that

$$\delta + (|Du| - 1)_+ \leq 2(|Du| - 1)_+ \qquad \text{a.e. in } \{|Du| \geq 2\},$$

and, since $\Phi'(t)$ is increasing, also that

$$\Phi'((|Du| - 1)_+) \leq \Phi'(2)\chi_{\{|Du|\geq 1\}} \qquad \text{a.e. in } \{|Du| \leq 2\}.$$

Therefore, by virtue of (3.11), we have

$$\frac{\varepsilon}{c_\gamma} \int_\Omega \eta^2 \Phi'((|Du| - 1)_+) \left(\delta + (|Du| - 1)_+\right) |Du|^{p-2}|D^2u|^2\, dx$$

$$\leq \quad 2\frac{\varepsilon}{c_\gamma} \int_{|Du|\geq 2} \eta^2 \Phi'((|Du| - 1)_+)(|Du| - 1)_+|Du|^{p-2}|D^2u|^2\, dx$$

$$+ \quad \frac{\varepsilon}{c_\gamma} \int_{|Du|\leq 2} \eta^2 \Phi'((|Du| - 1)_+)(|Du| - 1)_+|Du|^{p-2}|D^2u|^2\, dx$$

$$+ \quad \frac{\varepsilon\,\delta}{c_\gamma} \int_{|Du|\leq 2} \eta^2 \Phi'((|Du| - 1)_+)|Du|^{p-2}|D^2u|^2\, dx$$

$$\leq \quad 2\varepsilon \int_\Omega \eta^2 \Phi((|Du| - 1)_+)|Du|^{p-2}|D^2u|^2\, dx$$

$$+ \quad \frac{\varepsilon\,\delta}{c_\gamma} \int_{1\leq|Du|\leq 2} \eta^2 \Phi'(2)|Du|^{p-2}|D^2u|^2\, dx$$

$$\leq \quad 2\varepsilon \int_\Omega \eta^2 \Phi((|Du| - 1)_+)|Du|^{p-2}|D^2u|^2\, dx$$

$$+ \quad \frac{\varepsilon\,\delta}{2} \int_{|Du|\geq 1} \eta^2 \Phi(2)|Du|^{p-2}|D^2u|^2\, dx. \tag{3.13}$$

Inserting the estimate (3.13) in (3.12), we obtain that

$$|I_6| \quad \leq \quad 2\varepsilon \int_\Omega \eta^2 \Phi((|Du| - 1)_+)|Du|^{p-2}|D^2u|^2\, dx$$

$$+ \quad \frac{\varepsilon\,\delta}{2} \int_{|Du|\geq 1} \eta^2 \Phi(2)|Du|^{p-2}|D^2u|^2\, dx$$

$$+ \quad c_\gamma C_\varepsilon \int_\Omega \eta^2 \Phi'((|Du| - 1)_+)k^2(x)\left(\delta + (|Du| - 1)_+\right)^{-1}|Du|^{p+2}\, dx. \tag{3.14}$$

Since the equality (3.3) can be written as follows

$$I_2 + I_3 = -I_1 - I_4 - I_5 - I_6\,,$$

by virtue of (3.7), we get

$$I_2 \leq |I_1| + |I_4| + |I_5| + |I_6|$$

and therefore, recalling the estimates (3.4), (3.8), (3.9) and (3.14), we get

$$(1 - \varepsilon) \int_\Omega \eta^2 \Phi((|Du| - 1)_+) \sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du)u_{x_s x_i}^\alpha u_{x_s x_j}^\beta\, dx$$

$$\leq C_\varepsilon \int_\Omega |D\eta|^2 \Phi((|Du| - 1)_+)|Du|^p\, dx$$

$$+ 2^{p-1} \int_\Omega (\eta|D\eta|)k(x)\Phi((|Du| - 1)_+)|Du|^p\, dx$$

$$+ C_\varepsilon \int_\Omega \eta^2 \Phi((|Du| - 1)_+)k^2(x)|Du|^p\, dx$$

$$+3\varepsilon \int_\Omega \eta^2 \Phi((|Du|-1)_+)|Du|^{p-2}|D^2u|^2 \, dx$$

$$+\frac{\varepsilon \, \delta}{2} \int_{|Du|\geq 1} \eta^2 \Phi(2)|Du|^{p-2}|D^2u|^2 \, dx$$

$$+c_\gamma C_\varepsilon \int_\Omega \eta^2 \Phi'((|Du|-1)_+)k^2(x)\,(\delta + (|Du|-1)_+)^{-1}\,|Du|^{p+2} \, dx. \tag{3.15}$$

By the assumption (A3), the integral in the left hand side of (3.15) can be estimated as

$$\int_\Omega \eta^2 \Phi((|Du|-1)_+)\sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x,Du)u_{x_s x_i}^\alpha u_{x_s x_j}^\beta \, dx \geq \nu \int_\Omega \eta^2 \Phi((|Du|-1)_+)|Du|^{p-2}|D^2u|^2 \, dx.$$

Hence

$$(1-\varepsilon)\nu \int_\Omega \eta^2 \Phi((|Du|-1)_+)|Du|^{p-2}|D^2u|^2 \, dx$$

$$\leq \quad 3\varepsilon \int_\Omega \eta^2 \Phi((|Du|-1)_+)|Du|^{p-2}|D^2u|^2 \, dx$$

$$+ \quad C\int_\Omega |D\eta|^2 \Phi((|Du|-1)_+)|Du|^p \, dx + C\int_\Omega \eta|D\eta|k(x)\Phi((|Du|-1)_+)|Du|^p \, dx$$

$$+ \quad C\int_\Omega \eta^2 \Phi((|Du|-1)_+)k^2(x)|Du|^p \, dx + \frac{\varepsilon\delta}{2}\int_{|Du|\geq 1} \eta^2 \Phi(2)|Du|^{p-2}|D^2u|^2 \, dx$$

$$+ \quad C(1+\gamma)\int_\Omega \eta^2 \Phi'((|Du|-1)_+)k^2(x)\,(\delta + (|Du|-1)_+)^{-1}\,|Du|^{p+2} \, dx,$$

with $C = C(n,N,p,\nu,L,L_1,\varepsilon)$. Choosing $\varepsilon = \frac{\nu}{6+\nu}$, we can reabsorb the first integral in the right hand side of the previous inequality by the left hand side, thus getting

$$\int_\Omega \eta^2 \Phi((|Du|-1)_+)|Du|^{p-2}|D^2u|^2 \, dx$$

$$\leq \quad C\int_\Omega |D\eta|^2 \Phi((|Du|-1)_+)|Du|^p \, dx + C\int_\Omega \eta|D\eta|k(x)\Phi((|Du|-1)_+)|Du|^p \, dx$$

$$+ \quad C\int_\Omega \eta^2 \Phi((|Du|-1)_+)k^2(x)|Du|^p \, dx + C\delta\int_{|Du|\geq 1} \eta^2 \Phi(2)|Du|^{p-2}|D^2u|^2 \, dx$$

$$+ \quad C(1+\gamma)\int_\Omega \eta^2 \Phi'((|Du|-1)_+)k^2(x)\,(\delta + (|Du|-1)_+)^{-1}\,|Du|^{p+2} \, dx. \tag{3.16}$$

Using (3.11), we get

$$\Phi'((|Du|-1)_+)\,(\delta + (|Du|-1)_+)^{-1}$$

$$= \Phi'((|Du|-1)_+)((|Du|-1)_+)^{-1}\frac{(|Du|-1)_+}{\delta + (|Du|-1)_+}$$

$$\leq 2(\gamma+1)\Phi((|Du|-1)_+)((|Du|-1)_+)^{-2}$$

$$\leq 2(\gamma+1)(1+(|Du|-1)_+)^{\gamma-2} \qquad \text{a.e. in } \{|Du|>1\}$$

where we used also that

$$\frac{(|Du|-1)_+}{\delta + (|Du|-1)_+} \leq 1 \quad \forall \, \delta > 0$$

and the definition of Φ. Hence, taking into account that $\Phi'(0) = 0$, we obtain that

$$\int_\Omega \eta^2 \Phi'((|Du| - 1)_+) k^2(x) \left(\delta + (|Du| - 1)_+\right)^{-1} |Du|^{p+2} \, dx$$
$$\leq \quad C(1 + \gamma) \int_\Omega \eta^2 k^2(x)(1 + (|Du| - 1)_+)^{\gamma+p} \, dx. \tag{3.17}$$

Inserting (3.17) in (3.16), we obtain

$$\int_\Omega \eta^2 \Phi((|Du| - 1)_+)|Du|^{p-2}|D^2 u|^2 \, dx$$
$$\leq \quad C \int_\Omega |D\eta|^2 \Phi((|Du| - 1)_+)|Du|^p \, dx + C \int_\Omega \eta |D\eta| k(x) \Phi((|Du| - 1)_+)|Du|^p \, dx$$
$$+ \quad C \int_\Omega \eta^2 \Phi((|Du| - 1)_+) k^2(x)|Du|^p \, dx + C\delta \int_{|Du| \geq 1} \eta^2 \Phi(2)|Du|^{p-2}|D^2 u|^2 \, dx$$
$$+ \quad C(\gamma + 1)^2 \int_\Omega \eta^2 k^2(x)(1 + (|Du| - 1)_+)^{\gamma+p} \, dx. \tag{3.18}$$

Since

$$\int_\Omega \eta |D\eta| k(x) \Phi((|Du| - 1)_+)|Du|^p \, dx$$
$$= \int_\Omega \left\{ \eta^2 k^2(x) \Phi((|Du| - 1)_+)|Du|^p \right\}^{\frac{1}{2}} \left\{ |D\eta|^2 \Phi((|Du| - 1)_+)|Du|^p \right\}^{\frac{1}{2}} \, dx$$
$$\leq \frac{1}{2} \int_\Omega \eta^2 k^2(x)(1 + (|Du| - 1)_+)^{\gamma+p} \, dx + \frac{1}{2} \int_\Omega |D\eta|^2 (1 + (|Du| - 1)_+)^{\gamma+p} \, dx,$$

and noting that, by Lemma 3.3 in [14], $|Du|^{p-2}|D^2 u|^2 \chi_{\{|Du| \geq 1\}} \in L^1_{\text{loc}}(\Omega)$, we can let δ go to 0 in (3.18), thus obtaining

$$\int_\Omega \eta^2 \Phi((|Du| - 1)_+)|Du|^{p-2}|D^2 u|^2 \, dx$$
$$\leq C \int_\Omega |D\eta|^2 (1 + (|Du| - 1)_+)^{\gamma+p} \, dx + C(\gamma + 1)^2 \int_\Omega \eta^2 k(x)^2 (1 + (|Du| - 1)_+)^{\gamma+p} \, dx$$

with $C = C(n, N, L, L_1, \nu)$. Estimate (3.1) follows recalling the definition of Φ.

Step 2. In this step we prove that for all $x_0 \in \Omega$ there exists $R_\gamma > 0$ such that $B_{R_\gamma}(x_0) \Subset \Omega$ and for every ρ and R, $0 < \rho < R < R_\gamma$,

$$\fint_{B_\rho(x_0)} \left| (1 + (|Du| - 1)_+) \right|^{\frac{(\gamma+p)n}{n-2}} \, dx \leq \frac{C(\gamma + p)^{\frac{2n}{n-2}}}{(R - \rho)^{\frac{2n}{n-2}}} \left(\fint_{B_R(x_0)} (1 + (|Du| - 1)_+)^{\gamma+p} \, dx \right)^{\frac{n}{n-2}}, \tag{3.19}$$

for some $C = C(n, N, L, L_1, \nu)$.

Fix radii $0 < \rho < s < t < R$, with $B_R(x_0) \Subset \Omega$, and a cut off function $\eta \in C_0^\infty(B_t)$ such that $\eta = 1$ on B_s and $|D\eta| \leq \frac{2}{t-s}$. With this choice of η, estimate (3.1) becomes

$$\int_{B_t} \eta^2 (1 + (|Du| - 1)_+)^{\gamma-2}((|Du| - 1)_+)^2 |Du|^{p-2}|D^2 u|^2 \, dx$$
$$\leq \quad C(\gamma + 1)^2 \int_{B_t} \eta^2 k^2(x)(1 + (|Du| - 1)_+)^{\gamma+p} \, dx$$

$$+ \quad \frac{C}{(t-s)^2} \int_{B_t \setminus B_s} (1 + (|Du| - 1)_+)^{\gamma+p} \, dx$$

$$\leq \quad C(\gamma + 1)^2 \left(\int_{B_t} k^n(x) \, dx \right)^{\frac{2}{n}} \left(\int_{B_t} (1 + (|Du| - 1)_+)^{\frac{(\gamma+p)n}{n-2}} \, dx \right)^{\frac{n-2}{n}}$$

$$+ \quad \frac{C}{(t-s)^2} \int_{B_t \setminus B_s} (1 + (|Du| - 1)_+)^{\gamma+p} \, dx \,, \tag{3.20}$$

where we used the assumption $k \in L^n_{\mathrm{loc}}(\Omega)$ and Hölder's inequality with exponents $\frac{n}{2}, \frac{n}{n-2}$. Following [14], we introduce the function

$$G(t) = 1 + \int_0^t (1+s)^{\frac{\gamma+p-4}{2}} s \, ds$$

and we note that

$$\frac{1}{2(\gamma+p)}(1+t)^{\frac{\gamma+p}{2}} \leq G(t) \leq 2(1+t)^{\frac{\gamma+p}{2}}, \qquad\qquad G'(t) = t(1+t)^{\frac{\gamma+p-4}{2}} \,. \tag{3.21}$$

Using the Sobolev imbedding Theorem for the function $\eta G((|Du| - 1)_+)$, we get

$$\left(\int_\Omega \left| \eta G((|Du| - 1)_+) \right|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} \leq c \int_\Omega \left| D(\eta G((|Du| - 1)_+)) \right|^2 dx$$

$$\leq \quad c \int_\Omega |D\eta|^2 \left(G((|Du| - 1)_+) \right)^2 dx + c \int_\Omega \eta^2 \left| G'((|Du| - 1)_+) \right|^2 |D((|Du| - 1)_+)|^2 dx.$$

Using the properties of $G(t)$ at (3.21) in the previous inequality, we obtain

$$\frac{1}{(\gamma+p)^2} \left(\int_\Omega \eta^{\frac{2n}{n-2}} (1 + (|Du| - 1)_+)^{\frac{(\gamma+p)n}{n-2}} \, dx \right)^{\frac{n-2}{n}}$$

$$\leq \quad c \int_\Omega |D\eta|^2 (1 + (|Du| - 1)_+)^{\gamma+p} \, dx$$

$$+ \quad c \int_\Omega \eta^2 (1 + (|Du| - 1)_+)^{\gamma+p-4} ((|Du| - 1)_+)^2 |D((|Du| - 1)_+)|^2 \, dx$$

$$\leq \quad \frac{c}{(t-s)^2} \int_{B_R} (1 + (|Du| - 1)_+)^{\gamma+p} \, dx$$

$$+ \quad c \int_{B_t} \eta^2 (1 + (|Du| - 1)_+)^{\gamma-2} |(|Du| - 1)_+|^2 |Du|^{p-2} |D^2 u|^2 \, dx \,. \tag{3.22}$$

Combining estimates (3.20) and (3.22), we obtain

$$\int_{B_s} (1 + (|Du| - 1)_+)^{\frac{(\gamma+p)n}{n-2}} \, dx$$

$$\leq \quad c(\gamma + p)^{\frac{4n}{n-2}} \left(\int_{B_R} k(x)^n \, dx \right)^{\frac{2}{n-2}} \int_{B_t} (1 + (|Du| - 1)_+)^{\frac{(\gamma+p)n}{n-2}} \, dx$$

$$+ \quad \frac{c(\gamma + p)^{\frac{2n}{n-2}}}{(t-s)^{\frac{2n}{n-2}}} \left(\int_{B_R} (1 + (|Du| - 1)_+)^{\gamma+p} \, dx \right)^{\frac{n}{n-2}} \,. \tag{3.23}$$

By the absolute continuity of the integral we can choose a radius R_γ, which obviously depends also on n, N, p, ν, L, L_1, such that

$$c(\gamma + p)^{\frac{4n}{n-2}} \left(\int_{B_{R_\gamma}} k(x)^n \, dx \right)^{\frac{2}{n-2}} \leq \frac{1}{2} \tag{3.24}$$

so that, if $R < R_\gamma$, estimate (3.23) becomes

$$\int_{B_s} (1 + (|Du| - 1)_+)^{\frac{(\gamma+p)n}{n-2}} \, dx \leq \frac{1}{2} \int_{B_t} (1 + (|Du| - 1)_+)^{\frac{(\gamma+p)n}{n-2}} \, dx$$
$$+ \frac{c(\gamma + p)^{\frac{2n}{n-2}}}{(t - s)^{\frac{2n}{n-2}}} \left(\int_{B_R} (1 + (|Du| - 1)_+)^{\gamma+p} \, dx \right)^{\frac{n}{n-2}}.$$

Since the previous inequality is valid for every radii $\rho < s < t < R < R_\gamma$, we can use the iteration Lemma 2.1, so obtaining (3.19).

Step 3. In this step we conclude the proof.

Fix $q > p$. Let i_* be the unique integer such that $\log_{\frac{n}{n-2}} \frac{q}{p} - 1 \leq i_* < \log_{\frac{n}{n-2}} \frac{q}{p}$. If we define the increasing sequence of exponents p_j, $j \in \{0, 1, \cdots, i_* + 1\}$ by setting

$$p_j := \left(\frac{n}{n-2} \right)^j p,$$

we obtain $p_{i_*} < q \leq p_{i_*+1}$.

For every $j \in \{0, 1, \cdots, i_*\}$ define $\gamma_j := p_j - p$. By Step 2 there exist $R_{\gamma_j} > 0$ and $C > 0$ such that for all $0 < \rho < R < R_{\gamma_j}$ (3.19) holds with γ replaced by γ_j. Note that since R_{γ_j} satisfies (3.24), then we also have that

$$c p_j^{\frac{4n}{n-2}} \left(\int_{B_r} k(x)^n \, dx \right)^{\frac{2}{n-2}} \leq \frac{1}{2}, \tag{3.25}$$

for every $0 < r \leq R_{\gamma_j}$.

Define $R_* := \min\{R_j : j = 1, \cdots, i_*\} > 0$. Fix $0 < \rho < R < R_*$ and let us define the decreasing sequence of radii ρ_j

$$\rho_j := \rho + \frac{R - \rho}{2^j}, \qquad j \in \{0, 1, \cdots, i_*\},$$

so that $\rho_0 = R > \rho_j > \rho_{j+1} > \rho$. By (3.25), with $\sigma = p_j$, and (3.19) we get

$$\left(\int_{B_{\rho_{j+1}}} (1 + (|Du| - 1)_+)^{p_{j+1}} \, dx \right)^{\frac{1}{p_{j+1}}} \leq \frac{C_{p_j}}{(\rho_j - \rho_{j+1})^{\frac{2}{p_j}}} \left(\int_{B_{\rho_j}} (1 + (|Du| - 1)_+)^{p_j} \, dx \right)^{\frac{1}{p_j}} \tag{3.26}$$

where we have chosen $\rho = \rho_{j+1}$, $R = \rho_j$ and $\gamma + p = p_j$. Iterating (3.26), we deduce

$$\left(\int_{B_\rho} ((1 + (|Du| - 1)_+))^{p_{i_*+1}} \, dx \right)^{\frac{1}{p_{i_*+1}}} \leq \tilde{C} \left(\int_{B_R} (1 + (|Du| - 1)_+)^p \, dx \right)^{\frac{1}{p}}$$

where

$$\tilde{C} = \prod_{j=0}^{i_*} \frac{C_{p_j}}{(\rho_j - \rho_{j+1})^{\frac{2}{p_j}}} =: C(n, N, p, q, L, L_1, \nu, \rho, R).$$

Observing that $q \leq p_{i_*+1}$ and by using the left inequality in (A2), the conclusion follows.

$\square$

4. PROOF OF THEOREM 1.1

Before going on with the proof of the main Theorem, we state an approximation result for the energy density f through a sequence of uniformly elliptic integrands having Lipschitz dependence on the x variable.

We recall that, without loss of generality, we assumed that the radius $\tilde{R}$ appearing in the assumptions (A1)-(A5) is equal to 1.

Proposition 4.1. *Let $f : \Omega \times \mathbb{R}^{nN} \to [0, +\infty)$ be a Carathéodory function convex and C^2 with respect to the last variable and satisfying assumptions (A1)-(A5). Fixed an open set $\Omega' \Subset \Omega$, there exists a sequence of C^2-functions $f_h : \Omega' \times \mathbb{R}^{nN} \to [0, +\infty)$, f_h convex in the last variable, such that f_h converges to f pointwise a.e. on Ω' and everywhere in $\mathbb{R}^N$. Moreover, f_h satisfies the following properties:*

($\mathcal{A}1$) *for every $x \in \Omega'$ and $\xi \in \mathbb{R}^{nN} \setminus B_2(0)$, $f_h(x, \xi) = \tilde{f}_h(x, |\xi|)$,*

($\mathcal{A}2$) *there exist $\tilde{L}, \tilde{c}_1, \tilde{c}_2 > 0$, independent of h, such that for all $(x, \xi) \in \Omega' \times \mathbb{R}^{nN}$*
$$\tilde{c}_1 |\xi|^p - \tilde{c}_2 \leq f_h(x, \xi) \leq \tilde{L}(1 + |\xi|)^p,$$

($\mathcal{A}3$) *there exists $\tilde{\nu}$, depending on ν and p, but not on h, such that for every $x \in \Omega'$, $\xi \in \mathbb{R}^{nN} \setminus B_2(0)$ and $\lambda \in \mathbb{R}^{nN}$*
$$\tilde{\nu}(1 + |\xi|)^{p-2} |\lambda|^2 \leq \langle D_{\xi\xi} f_h(x, \xi)\lambda, \lambda \rangle,$$

($\mathcal{A}4$) *there exists $\tilde{L}_1 > 0$, independent of h, such that for all $(x, \xi) \in \Omega' \times \mathbb{R}^{nN} \setminus B_2(0)$*
$$|D_{\xi\xi} f_h(x, \xi)| \leq \tilde{L}_1 (1 + |\xi|)^{p-2},$$

($\mathcal{A}5$) *for every $x \in \Omega'$ and $\xi \in \mathbb{R}^{nN} \setminus B_2(0)$,*
$$|D_{\xi x} f_h(x, \xi)| \leq 2^{p-1} k_h(x)(1 + |\xi|)^{p-1},$$
where $k_h \in C^\infty(\Omega')$ is a non-negative function, such that $k_h \to k$ in $L^n(\Omega')$,

($\mathcal{H}1$) *there exists $\Lambda_h > 0$ such that for every $x \in \Omega'$ and $\xi \in \overline{B_2(0)}$*
$$|D_{\xi x} f_h(x, \xi)| \leq \Lambda_h (1 + |\xi|)^{p-1}.$$

($\mathcal{H}2$) *there exists $\mu_h > 0$ such that for all $(x, \xi) \in \Omega' \times \mathbb{R}^{nN}$ and for all $\lambda \in \mathbb{R}^{nN}$*
$$\mu_h(1 + |\xi|)^{p-2} |\lambda|^2 \leq \langle D_{\xi\xi} f_h(x, \xi)\lambda, \lambda \rangle.$$

($\mathcal{H}3$) *there exists $\sigma_h > 0$ such that for all $(x, \xi) \in \Omega' \times \mathbb{R}^{nN}$*
$$|D_{\xi\xi} f_h(x, \xi)| \leq \sigma_h (1 + |\xi|)^{p-2}.$$

Proof. To prove ($\mathcal{A}1$), ($\mathcal{A}2$) and ($\mathcal{H}2$) we argue as in the proof of Lemma 4.2 in [9], but replacing the function f_k and the exponent q with f and p, respectively. For the sake of completeness, we report the complete proof.

Denote $B_1^n \subset \mathbb{R}^n$ and $B_1^{nN} \subset \mathbb{R}^{nN}$ the unit balls centered in 0. Fixed a positive decreasing sequence $\varepsilon_h \to 0$, with $h \geq h_0$, where h_0 is large enough to have $x + \varepsilon_h y \in \Omega$ for every $x \in \Omega'$ and every $y \in B_1^n$. Without loss of generality we assume $\varepsilon_h \leq \frac{1}{2}$.

Let $\sigma : B_1^n \to [0, +\infty)$ and $\rho : B_1^{nN} \to [0, +\infty)$ be two radially symmetric mollifiers such that $\int_{B_1^n} \sigma = 1$ and $\int_{B_1^{nN}} \rho = 1$. Define $F_h : \Omega' \times \mathbb{R}^{nN} \to [0, +\infty)$,

$$F_h(x, \xi) := \int_{B_1^n} \int_{B_1^{nN}} \sigma(y)\rho(\eta) f(x + \varepsilon_h y, \xi + \varepsilon_h \eta) \, d\eta \, dy$$

and

$$f_h(x, \xi) := F_h(x, \xi) + \varepsilon_h(1 + |\xi|)^p. \tag{4.1}$$

For each h, $f_h \in C^2$, one can easily check that f_h is convex in the last variable and, up to subsequences, f_h converges pointwise a.e. to f in Ω' and everywhere in $\mathbb{R}^N$. Moreover, by (A1), $f_h(x, \cdot)$ is radial with respect to ξ in $\mathbb{R}^{nN} \setminus B_2^{nN}$.

The property ($\mathcal{A}2$) is a trivial consequence of the assumption (A2).

To prove ($\mathcal{A}3$) it is enough to prove that a similar inequality holds for F_h. By (A3)

$$\langle D_{\xi\xi} F_h(x, \xi)\lambda, \lambda\rangle \geq \nu \int_{B_1^n} \int_{B_1^{nN}} \sigma(y)\rho(\eta) \left(1 + |\xi + \varepsilon_h \eta|\right)^{p-2} d\eta\, dy |\lambda|^2.$$

Since

$$(1 + |\xi + \varepsilon_h \eta|)^{p-2} \geq c_p(1 + |\xi|)^{p-2}$$

where $c_p = \min\{\frac{1}{2^{p-2}}, (\frac{3}{2})^{p-2}\}$, we can conclude using the properties of σ and ρ.

To prove property ($\mathcal{A}4$) we observe that for all ξ, with $|\xi| \geq 2$ we have

$$|D_{\xi\xi} f_h(x, \xi)| \leq \int_{B_1^n} \int_{B_1^{nN}} \sigma(y)\rho(\eta) |D_{\xi\xi} f(x + \varepsilon_h y, \xi + \varepsilon_h \eta)| \, d\eta\, dy + \frac{1}{2}|D_{\xi\xi}((1 + |\xi|)^p)|$$

and we conclude using (A4), the inequality

$$(1 + |\xi + \varepsilon_h \eta|)^{p-2} \leq c_p'(1 + |\xi|)^{p-2}$$

and the properties of σ and ρ.

Let us prove ($\mathcal{A}5$). By the assumption (A5), for all $|\xi| > 2$, we have

$$|D_{\xi x} f_h(x, \xi)| = |D_{\xi x} F_h(x, \xi)| \leq \int_{B_1^n} \int_{B_1^{nN}} \sigma(y)\rho(\eta) \, |D_{\xi x} f(x + \varepsilon_h y, \xi + \varepsilon_h \eta)| \, d\eta\, dy$$

$$\leq \int_{B_1^n} \int_{B_1^{nN}} \sigma(y)\rho(\eta) k(x + \varepsilon_h y)(1 + |\xi + \varepsilon_h \eta|)^{p-1} \, d\eta\, dy \leq 2^{p-1} k_h(x)(1 + |\xi|)^{p-1},$$

where

$$k_h(x) := \int_{B_1^n} \sigma(y) k(x + \varepsilon_h y) \, dy$$

is a smooth function, satisfying $k_h \to k$ in $L^n(\Omega')$. Thus, ($\mathcal{A}5$) follows.

To prove ($\mathcal{H}1$), we observe that, for all $x \in \Omega'$ and $\xi \in B_2^{nN}$, thanks to assumption (A2), it holds the following

$$|D_{\xi x} f_h(x, \xi)| = |D_{\xi x} F_h(x, \xi)|$$

$$= \varepsilon_h^{-n} \varepsilon_h^{-nN} \left| D_{\xi x} \int_{B_{\varepsilon_h}^n(x)} \int_{B_{\varepsilon_h}^{nN}(\xi)} \sigma\left(\frac{w - x}{\varepsilon_h}\right) \rho\left(\frac{z - \xi}{\varepsilon_h}\right) f(w, z) \, dw\, dz \right|$$

$$\leq C(L, \Omega)\varepsilon_h^{-n-nN-2} \|D_\xi \rho\|_{L^\infty(B_1^{nN})} \|D_x \sigma\|_{L^\infty(B_1^n)} \int_{B_{\varepsilon_h}^{nN}(\xi)} (1 + |z|)(1 + |z|)^{p-1} \, dz$$

$$\leq C(L, \Omega)(1 + |\xi|)^{p-1} \varepsilon_h^{-n-nN-2} \|D_\xi \rho\|_{L^\infty(B_1^{nN})} \|D_x \sigma\|_{L^\infty(B_1^n)} \int_{B_3^{nN}(0)} (1 + |z|) \, dz.$$

Therefore, we get

$$|D_{\xi x} f_h(x, \xi)| \leq C(L, \Omega)\varepsilon_h^{-n-nN-2} 4^{nN+1} \|D\sigma\|_\infty \|D\rho\|_\infty (1 + |\xi|)^{p-1},$$

and ($\mathcal{H}1$) follows.

As far as the property ($\mathcal{H}2$) is concerned, notice that F_h are convex functions, because f is convex, therefore

$$\langle D_{\xi\xi} f_h(x, \xi)\lambda, \lambda\rangle \geq \varepsilon_h \langle D_{\xi\xi}((1 + |\xi|^2)^{\frac{p}{2}})\lambda, \lambda\rangle$$

and we immediately conclude.

Since ($\mathcal{A}4$) holds, the property ($\mathcal{H}3$) follows observing that the second order derivatives with respect to ξ of $f_h(x, \xi)$ are equi-continuous in $\Omega' \times \overline{B_2(0)}$. $\square$

We are now ready to prove our main result.

Proof of Theorem 1.1. Let u be a local minimizer of the functional $\mathcal{F}$ in (1.1). Fixed $\Omega' \Subset \Omega$, let f_h be the sequence of functions given by Proposition 4.1.

Fix a ball $B_r(x_0)$ in Ω', such that $r < R_\gamma$ where R_γ is the radius determined in Theorem 3.1. By virtue of ($\mathcal{A}2$), we have the existence of a weak solution $v_h \in W^{1,p}(B_r(x_0), \mathbb{R}^N)$ of the problem

$$\min \left\{ \mathcal{F}_h(w; B_r(x_0)) := \int_{B_r(x_0)} f_h(x, Dw) \, dx \, : \, w \in u + W_0^{1,p}(B_r(x_0), \mathbb{R}^N) \right\}.$$

From the first inequality in ($\mathcal{A}2$) and the minimality of v_h, there exists $c > 0$, independent of h, such that

$$\int_{B_r(x_0)} |Dv_h|^p \, dx \leq c \int_{B_r(x_0)} (1 + f_h(x, Dv_h)) \, dx \leq c \int_{B_r(x_0)} (1 + f_h(x, Du)) \, dx.$$

By the right inequality in ($\mathcal{A}2$) and the dominated convergence theorem, we have

$$\limsup_{h \to +\infty} \int_{B_r(x_0)} |Dv_h|^p \, dx \leq c \lim_h \int_{B_r(x_0)} (1 + f_h(x, Du)) \, dx \leq c \int_{B_r(x_0)} (1 + f(x, Du)) \, dx.$$

Therefore, up to a subsequence, v_h weakly converges in $W^{1,p}$ to a function $v \in u + W_0^{1,p}(B_r(x_0))$. We now claim that v is a minimizer of $\mathcal{F}(\cdot; B_r(x_0))$. By the convexity of the functions $f_h(x, \cdot)$ and ($\mathcal{A}2$), the sequence $\{\mathcal{F}_h(\cdot; B_r(x_0))\}_h$ Γ-converges to $\mathcal{F}(\cdot; B_r(x_0))$ with respect to the weak $W^{1,p}$-topology induced on $u + W_0^{1,p}(B_r(x_0))$ (see Theorem 5.14 in [12]). Therefore, Corollary 7.20 in [12] implies that v is a minimizer of $\mathcal{F}(\cdot; B_r(x_0))$.

Fix $\rho < r$. We claim that $v \in W^{1,q}(B_\rho(x_0), \mathbb{R}^N)$ for all $q > p$ and that there exists $C > 0$ such that

$$\int_{B_\rho(x_0)} |Dv|^q \, dx \leq C \left[\int_{B_r(x_0)} (1 + f(x, Du)) \, dx \right]^{\frac{q}{p}}. \tag{4.2}$$

Indeed, for all h, f_h satisfies the assumptions of Lemma 2.3, with Ω replaced by $B_r(x_0)$, therefore $v_h \in W_{\text{loc}}^{1,\infty}(B_r(x_0)) \cap W_{\text{loc}}^{2,2}(B_r(x_0))$ and $(1 + |Dv_h|^2)^{\frac{p-2}{2}} |D^2v_h|^2 \in L_{\text{loc}}^1(B_r(x_0))$. We are legitimate to apply the apriori estimate of Theorem 3.1 to f_h and v_h, thus obtaining that $v_h \in W^{1,q}(B_\rho(x_0))$ for all $q > p$ and that the following estimate

$$\int_{B_\rho(x_0)} |Dv_h|^q \, dx \leq C \left[\int_{B_r(x_0)} (1 + f_h(x, Dv_h)) \, dx \right]^{\frac{q}{p}}$$

holds for a positive constant C. Note that, thanks to the definition of f_h in (4.1) and since the constants $\tilde{L}, \tilde{\nu}, \tilde{L}_1$ in ($\mathcal{A}2$)-($\mathcal{A}5$) are independent of h, and since k_h converges to k in $L^n(\Omega')$, then the constant C is independent of h.

It follows that, up to subsequences, v_h weakly converges to v in $W^{1,q}(B_\rho(x_0), \mathbb{R}^N)$ for every fixed $q > p$ and passing to the limit as $h \to +\infty$ in the previous estimate we obtain

$$\int_{B_\rho(x_0)} |Dv|^q, \, dx \leq \liminf_h \int_{B_r(x_0)} |Dv_h|^q \, dx \leq C \left[\int_{B_r(x_0)} (1 + f(x, Du)) \, dx \right]^{\frac{q}{p}}. \tag{4.3}$$

Therefore (4.2) holds true.

We now compare the L^q-norm of Du and Dv.

From Lemma 2.2 there exists $R_0 = R_0(p, \nu, L)$ such that (2.1) holds. Thus,

$$\int_{B_\rho(x_0)} |Du|^q \, dx \leq \int_{B_\rho(x_0) \cap \{x \,:\, |Du+Dv| \leq 2R_0\}} |Du|^q \, dx + \int_{B_\rho(x_0) \cap \{x \,:\, Du=Dv\}} |Du|^q \, dx$$

$$\leq 2^{q-1} \int_{B_\rho(x_0) \cap \{x \,:\, |Du+Dv| \leq 2R_0\}} (|Du + Dv|^q + |Dv|^q) \, dx + \int_{B_\rho(x_0)} |Dv|^q \, dx,$$

which implies that there exists $C > 0$, depending on n, p, q, ν, L, L_1, such that

$$\int_{B_\rho(x_0)} |Du|^q \, dx \leq C \int_{B_\rho(x_0)} (1 + |Dv|^q) \, dx.$$

This inequality, together with (4.3) implies the thesis. The α-Hölder continuity of u, for any exponent $\alpha < 1$, comes from the classical Sobolev-Morrey embedding theorem. $\qquad\square$

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