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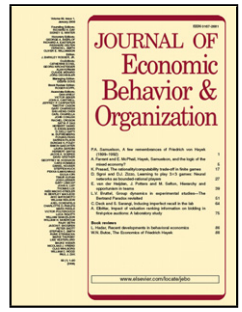

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Abstract

In the framework of a vertically differentiated mixed duopoly, with uncovered market and costless quality choice, we study the existence of a price equilibrium when a welfare-maximizing public firm producing low quality goods competes against a profit-maximizing private firm producing high quality goods. We show that a price equilibrium exists if the quality spectrum is wide enough *vis à vis* a measure of the convexity of the distribution of the consumers' willingness to pay, and that such equilibrium is unique if this sufficient condition is tightened. Log-concavity of the income distribution is inconsistent with the existence of equilibrium.

JEL classification: D43, L13, L51.

Keywords: price equilibrium, vertical differentiation, mixed duopoly

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Price equilibrium and willingness to pay in a vertically differentiated mixed duopoly

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Abstract

In the framework of a vertically differentiated mixed duopoly, with uncovered market and zero costs, we study the existence of a price equilibrium when a welfare-maximizing public firm producing low quality goods competes against a profit-maximizing private firm producing high quality goods. We show that a price equilibrium exists if the quality spectrum is wide enough *vis à vis* a measure of the convexity of the distribution of the consumers' willingness to pay, and that such equilibrium is unique if this sufficient condition is tightened. Logconcavity of the income distribution is inconsistent with the existence of equilibrium.

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1 Introduction

Mixed oligopolies can be observed in many countries and sectors, as mixed industries in advanced economies became particularly relevant in the last decades, following extensive privatization programmes of public monopolies in the 80's and 90's.[1] In mixed industries (e.g. public utilities, transportation, telecommunication, energy, postal services, education, health care, etc.) public firms compete with private firms in price, quantity and the quality of goods. It is frequently argued that public firms supply goods or services, the quality of which is lower than that provided by private firms: e.g., such is allegedly the case in many countries for education and health care, or in transportation and postal services. To be sure, the idea that public firms consistently supply lower quality can be challenged on empirical grounds – indeed, cases can even be found where the same industry is characterized by public firms supplying higher or lower quality, depending on the country or the sector one looks at (Epple & Romano 1998, Jofre-Bonet 2000, Sanjo 2009, Cremer & Maldonado 2013). However, cases where public firms do offer lower quality are many, and the literature of mixed oligopolies usually relies on such an assumption (see, e.g., Ishibashi & Kaneko 2008).[2]

A number of papers address the question of why this should be so, in the framework of a welfare-maximizing public firm competing with a profit-

[1]In Europe several public utilities such as telecommunication, electricity, gas retailing, and postal services became mixed markets as private firms were allowed to compete with public firms. The same happened with many previously public industries, such as airlines, railways, energy, steel, banking, broadcasting, life insurance, health care, and education. On the main privatization programmes and the relevance of mixed industries see e.g., Cuervo & Villalonga (2000), Megginson & Netter (2001).

[2]Examples of mixed industries where public firms allegedly provide low quality are transport services (Dodgson & Katsoulacos 1988), postal services (Mizutani & Uranishi 2003), telecommunication (Ros 1999), and financial services (Barros & Modesto 1999). It should perhaps also be noticed that, according to Blackorby & Donaldson (1988) and Besley & Coate (1991), if the quality differential between public and private sectors is justified by a concern for accessibility, the quality offered by the public firm should be sufficiently low to make accessibility effective.

maximizing private one;[3] however, the answer they provide is usually sought by assuming away any role for the distribution of the willingness to pay across consumers: either because the crucial feature of uncovered market is ruled out, or because – while allowing for uncovered markets – the standard, uniform-distribution model of vertical differentiation is used.[4] This is somewhat surprising on at least two counts: at a very general level, most informal arguments justifying the very existence of public firms competing with private firms rely on distributional concerns about inequality and providing the poor with access to goods and services; and, more to the point at the analytical level, it is in general well known that the distribution of the willingness to pay affects the firms' equilibrium choices and can in principle affect the very existence of equilibria (Grandmont 1993, Anderson et al. 1997).

In this paper we focus on the existence of a short-run price equilibrium in a vertically differentiated mixed duopoly with uncovered market, to confirm that the distribution of the willingness to pay affects equilibria. We assume costless production, which allows us to concentrate upon the relevant features of demand and hence the distribution of the willingness to pay; and we model a mixed duopoly as a case where a welfare-maximizing, low-quality producing public firm competes against a profit-maximizing, high-quality producing private firm – which in principle might provide a first

[3]For an overview of the theory of mixed oligopoly see De Fraja & Delbono (1990) and for standard models of mixed oligopoly see, e.g., Harris & Wiens (1980), De Fraja & Delbono (1989), Grilo (1994), Barros & Martinez-Giralt (2002), Cantos-S. & Moner-Colonques (2006), Cremer & Maldonado (2013).

[4]Thus, e.g., Ishibashi & Kaneko (2008) use the Hotelling model to argue that the in a duopoly equilibrium the public firm would supply the lower quality, and the private firm the higher (in fact, higher than is efficient) quality level. On the other hand, Delbono et al. (1996) use the standard uncovered market model to show that an equilibrium where the public (private) firm chooses the low (high) quality exists, though an equilibrium with inverted quality allocations also exists, and market segmentation is exogenous (also, this is a framework where it is problematic to find analytical solutions).

step in addressing the important general question of comparing the overall performance of 'mixed' vs 'pure' oligopolies within vertically differentiated markets.[5] In this framework, we show that for a price equilibrium to exist the distribution of the willingness to pay cannot be logconcave, and that sufficient conditions for existence and uniqueness place a lower bound on the (given) quality spectrum – a lower bound which is higher, the higher the given convexity bound on the income distribution.

The paper is organized as follows. Section 2 presents the model and the general framework of mixed duopoly with vertical differentiation; Section 3 gives the solution for a market price equilibrium and discusses existence and uniqueness; Section 4 presents an example where the consumers' willingness to pay is supposed to be distributed as a Pareto distribution, while some concluding remarks are gathered in Section 5.

2 The model

We start from a standard model of duopoly competition with vertical differentiation, uncovered market and costless quality choice, as developed by Mussa & Rosen (1978), Shaked & Sutton (1982) and Tirole (1988). There are two competing firms, $i = H, L$, playing a non-cooperative game on price. Each firm i produces a good of quality $s_i \in \{s_H, s_L\}$, where $0 < s_L < s_H < \infty$ and $\Delta = s_H - s_L > 0$ denotes the quality differential. We crucially assume that L is a public firm producing low quality goods, while H is a profit-maximizing firm producing high quality goods; production costs are normalized to zero.[6] The firms' profits are $\Pi_i = p_i D_i$, where

[5] As we argue in subsection 3.2, the focus on short-run price equilibria with given qualities makes it easier to think of marginal costs as constant.

[6] This amounts to marginal costs being constant and independent of quality (see our discussion in Section 3.2). Notice that in this framework a high-quality producing public firm would serve the whole market at a price equal to marginal cost, and no profit

p_i and D_i, $i = H, L$, denote prices and demands: higher quality s_H sells at a price p_H, and lower quality s_L at a price p_L.

Each consumer is identified by her marginal willingness to pay for quality, θ, and has a utility $U_i(\theta) = \theta s_i - p_i$ if she buys a unit of good from firm i, and 0 otherwise. The marginal consumer, who is indifferent between buying the high and the low quality, has utility $U_H(\theta) = U_L(\theta)$, and is accordingly identified by $\theta_H = (p_H - p_L)/\Delta$; the marginal consumer who is indifferent between purchasing the low quality commodity and nothing at all has utility $U_L(\theta) = 0$, and is identified by $\theta_L = p_L/s_L$.[7] Clearly, θ_L and θ_H denote the positions of these marginal consumers along the "income" scale: for later reference, it is useful to derive the price elasticities of θ_L and θ_H, which are given by $\varepsilon_H = \frac{\partial \theta_H}{\partial p_H}\frac{p_H}{\theta_H} = \frac{p_H}{p_H - p_L} > 1$ and $\varepsilon_L = \frac{\partial \theta_H}{\partial p_L}\frac{p_L}{\theta_H} = \frac{-p_L}{p_H - p_L} < 0$, such that $\varepsilon_H + \varepsilon_L = 1$.

Normalizing the consumers' population to 1 and assuming that the willingness to pay θ is continuously distributed over some nonnegative support $\Theta \subseteq \mathbb{R}_+$, we define the density function $f(\theta)$ such that the implied cumulative distribution is $F : \Theta \to [0, 1]$. Using primes to denote derivatives, it is convenient for our purposes to define also the following elasticities:

$$\eta(\theta) = \frac{\theta f(\theta)}{1 - F(\theta)}, \tag{1}$$

$$\pi(\theta) = \lim_{h \to 0} \frac{d \log\left(\frac{1}{\mu}\int_y^{\theta+h} x f(x) dx\right)}{d \log \theta} = 1 + \frac{\theta f'(\theta)}{f(\theta)}, \tag{2}$$

where definition (1) is the (positive) elasticity of $1 - F(\theta)$ and definition (2) is the Esteban elasticity of the density $f(\theta)$.[8] We use these definitions to

maximizing firm could produce low-quality goods in equilibrium.

[7]These are the basic features of the standard vertical differentiation model (Mussa & Rosen 1978); as is well known, the marginal willingness to pay θ can be looked at as a proxy for income (Gabszewicz & Thisse 1979).

[8]Esteban (1986) defines the function $\pi(\cdot)$ as per our definition (2) and shows that it

gather our basic assumptions on F in the following assumption:

Assumption 1 *The distribution F is such that:*

(a) the lower bound of the support Θ ($\theta_{\min}$ say) obeys $\theta_{\min} = 0 = \eta(\theta_{\min})$, and the upper bound ($\theta_{\max}$ say) is such that $\lim_{\theta \to \theta_{\max}} \eta(\theta) > 1$;

(b) there exists some $\alpha \in (0, 1]$ such that, $\forall \tilde{\alpha} \in [\alpha, 1]$, $(1 + \tilde{\alpha}) \eta(\theta) + \pi(\theta) - 1 > 0$, $\forall \, \theta \in \Theta$;

(c) let $\tilde{\theta}$ be the smallest value such that $\eta(\cdot) = 1$: then there exists a (unique) value θ°, $0 < \theta^\circ \leq \tilde{\theta}$ such that $\pi(\theta^\circ) = 0$, and such that $\pi(\theta^\circ) > 0$ for $\theta < \theta^\circ$ and $\pi(\theta^\circ) < 0$ for $\theta > \theta^\circ$.

Assumption 1(a) implies that at equilibrium the market cannot be completely covered, and that the value of θ at which $\eta(\theta) = 1$ (which is pivotal in what follows) lies strictly within Θ. Assumption 1(b) implies that $(1 - F)^{-\alpha}$ is a convex function, which in turn limits in some way the convexity of the relationship between the size of the covered market, $1 - F(\cdot)$, and the consumers' willingness to pay θ.[9] It should be stressed that by excluding the extreme value $\alpha = 0$ we are ruling out log-concavity, while α being finite rules out the uniform distribution, which one would get as $\alpha \to -\infty$ (e.g., Caplin & Nalebuff (1991), p. 3). The same assumption also implies that:

$$\eta(\theta) + \pi(\theta) > 1 - \alpha \eta(\theta), \tag{3}$$

stands in a one-to-one relationship with the underlying density $f(\cdot)$: accordingly, it gives an alternative representation of the density itself, which in some circumstances may be useful, especially so as some regularity features are apparently supported by empirical evidence. See (Benassi & Chirco 2006) for the relationship between the Esteban elasticity and stochastic dominance, and Majumder & Chakravarty (1990) for some related empirical evidence.

[9]Indeed, it is easily seen that $\frac{d^2}{d\theta^2}(1-F)^{-\alpha} = \frac{\alpha}{\theta} \frac{f(\theta)}{(1-F(\theta))^{\alpha+1}} [(1+\alpha)\eta(\theta) + \pi(\theta) - 1] > 0$. Following Caplin & Nalebuff (1991), we can say that function $(1-F)$ is ρ-concave (with $\rho = -\alpha < 0$), which is equivalent to saying that $-(1-F)^{\alpha}$ is concave. Moreover, a ρ-concave function is also $\tilde{\rho}$-concave for all $\tilde{\rho} < \rho$, which means that what is true for a given $\alpha \in (0, 1]$ is also true for every $\tilde{\alpha} > \alpha$ included in the same interval.

which in turn means that η is monotonically increasing for $\eta(\theta) \leq 1$, i.e. over $\left[0, \tilde{\theta}\right]$. Also, this places a restriction on the function $\pi(\cdot)$ to the effect that, for $\eta(\theta) \leq 1$, i.e. over $\left[0, \tilde{\theta}\right]$,

$$\pi(\theta) > -\alpha \geq -1. \tag{4}$$

All this should clarify Assumption 1(c), which rules that the function π (surely positive by condition (3) as θ nears zero) changes sign only once within $\left[0, \tilde{\theta}\right]$.[10]

Since we look for the Nash equilibrium of the game, we first have to determine the demand functions faced by firms L and H: $D_H = 1 - F(\theta_H)$, $D_L = F(\theta_H) - F(\theta_L)$, where $F(\theta_j)$ represents the fraction of consumers with a taste parameter less than θ_j, $j = L, H$ (Tirole 1988). The corresponding profit functions are given by $\Pi_H = p_H D_H$, $\Pi_L = p_L D_L$. Finally, we define the social welfare function as the sum of the consumers' surplus: $W = s_H \int_{\theta_H}^{\infty} \theta f(\theta)\, d\theta + s_L \int_{\theta_L}^{\theta_H} \theta f(\theta)\, d\theta$, and crucially assume that the public firm sets the price of low quality goods p_L to maximize the social welfare function W.[11]

[10] This is the case with many widely used distributions, such as the Gamma and Pareto distributions.

[11] In principle, however, if one looks at the public firm's objective as justified in terms of the median voter theorem, and the income distribution is asymmetric, the policy makers may be driven to look at the marginal willingness to pay of the median consumer, instead of that of the average consumer (as required by social welfare maximization). For alternative models of mixed oligopoly with non-welfare-maximizing behavior, see, e.g., Fershtman (1990), Cremer et al. (1991), Barros (1995), Estrin & De Meza (1995). In particular, to analyze mixed oligopoly equilibria when the firms' objectives are endogenous, De Donder & Roemer (2009) study a vertically differentiated mixed market where one firm is profit-maximizing while the other maximizes revenues, but one firm becomes welfare-maximizing when the government takes a participation in it.

3 Price equilibrium

In this Section, we take up Nash equilibria in prices: we first study existence, and then enquire about uniqueness.[12]

3.1 Existence of the price equilibrium

Given the price p_L set by the public firm on the 'low-quality' goods, p_H is charged by firm H maximizing its profit Π_H. The corresponding first order conditions (FOCs) in terms of elasticity are given by:

$$\eta\left(\theta_H\right)\varepsilon_H = 1, \tag{5}$$

which implies $\eta\left(\theta_H\right) < 1$. The second order conditions (SOCs) can be similarly characterized in elasticity terms as:

$$2\eta\left(\theta_H\right) + \pi\left(\theta_H\right) > 0, \tag{6}$$

which implies, given that $\eta\left(\theta_H\right) < 1$ by condition (5), the necessary condition $\pi\left(\theta_H\right) > -1$, consistently with (4).[13]

To set the price p_L, the public firm maximizes the social welfare W. The corresponding FOCs are:

$$\frac{\eta\left(\theta_H\right)}{\eta\left(\theta_L\right)} = \frac{1 - F\left(\theta_L\right)}{1 - F\left(\theta_H\right)} > 1, \tag{7}$$

[12] Given a quality pair (s_H, s_L), existence and uniqueness can clearly be established with reference to a (or the) pair of marginal consumers along the 'income' scale, (θ_H^*, θ_L^*), as it will be $p_H^* = \theta_H^*\left(s_H - s_L\right) + \theta_L^* s_L$, and $p_L = \theta_L^* s_L$.

[13] The FOCs and SOCs for firm H can be written out as: $\frac{\partial \Pi_H}{\partial p_H} = 1 - F\left(\theta_H\right) - \frac{p_H}{\Delta} f\left(\theta_H\right) = 0$; $\frac{\partial^2 \Pi_H}{\partial p_H^2} = -2f\left(\theta_H\right) - \frac{p_H}{\Delta} f'\left(\theta_H\right) < 0$, from which (5) and (6) can easily be derived by using definitions (1) and (2).

from which $\theta_H > \theta_L$ implies $\eta(\theta_H) > \eta(\theta_L)$. The SOCs are given by:

$$(1 - \eta(\theta_H))\,\pi(\theta_H) + \eta(\theta_H)\,\pi(\theta_L) > 0, \tag{8}$$

which again are set in elasticity terms.[14]

As a result, at a price equilibrium for given s_H and s_L, p_H and p_L are identified by the twin FOCs (5) and (7), such that the twin SOCs (6) and (8) hold.

Before enquiring about the existence of equilibrium, it is perhaps worth stressing that – irrespective of our assumptions on the distribution of the willingness to pay and indeed justifying them – the basic framework we are using (though indeed quite standard) is inconsistent with a logconcave distribution of the consumers' willingness to pay – including the limit case of the uniform distribution. Intuitively, this is so because of the way a marginal changes in prices affect the positions of the marginal consumers. An increase in p_L pushes the marginal consumers nearer each other, by shifting linearly one to the right (θ_L) and the other to the left (θ_H) – that is, the set of middle-class consumers patronizing low-quality gets smaller, and that of the high-income consumers patronizing high quality gets larger: since this has opposite effects on overall welfare, the latter is maximized when the marginal contribution to welfare of enlarging the set of high-quality consumers is equal to the marginal cost of pricing out the poor. This however (a) requires that the income density falls sharply enough as we move from θ_L to θ_H, and (b) has to be consistent with the high-quality firm maximizing its profits. The latter obviously calls for the price elasticity of demand

[14] The FOCs and SOCs for the public firm are respectively $\frac{\partial W}{\partial p_L} = \theta_H f(\theta_H) - \theta_L f_L(\theta_L) = 0$, and $\frac{\partial^2 W}{\partial p_L^2} = -\frac{f(\theta_H)}{\Delta}\pi(\theta_H) - \frac{f(\theta_L)}{s_L}\pi(\theta_L) < 0$. In equilibrium, the latter is equivalent to condition (8), as can be seen by multiplying through by $p_L > 0$, substituting for $\varepsilon_L = 1 - \varepsilon_H$, and taking advantage of the FOCs (5) and (7).

for high-quality be one: given the structure of preferences (such that a small increase in p_H has a big effect on the location of the high-quality marginal consumer: $\varepsilon_H > 1$), this in turn dictates that $\eta(\theta_H) < 1$ as from (5). Logconcavity, by constraining the relationship between $\eta(\cdot)$ and $\pi(\cdot)$ as defined in (1) and (2), is inconsistent with both requirements holding at once: if the distribution is logconcave, high-quality demand being sufficiently rigid is inconsistent with the density falling rapidly enough around θ_H, which under logconcavity would mean high demand elasticity from the marginal high-quality consumer.[15]

We can now state the following proposition on the existence of a price equilibrium.

Proposition 1 *Let (s_H, s_L) be a given pair of qualities, such that $0 < s_L < s_H < \infty$, and let $k = s_L/\Delta$ such that $\eta(\theta^\circ) < \frac{1}{1+k}$. Then under Assumption 1 a price equilibrium exists.*

Proof. See Appendix 6A ■

Proposition 1 establishes that a price equilibrium exists, if some constraints are satisfied concerning the distribution of the willingness to pay *vis à vis* the quality spectrum being offered. First notice that welfare maximization by firm L leads to θ_H lying on a downward portion of the density $f(\theta)$. Indeed, the FOCs (7) boil down to $\theta_L f(\theta_L) = \theta_H f(\theta_H)$: analytically, this is inconsistent with both marginal consumers being on an upward sloping portion of the density itself, while economically it amounts to the marginal gain

[15]Log-concavity amounts to the constraint $\pi(\theta) > 1 - \eta(\theta)$ for all θ, such that $\eta(\theta_H) < 1$ is inconsistent with $\pi(\theta_H) < 0$. On the other hand, $\pi(\theta_H)$ has to be negative, if welfare has to be maximized. This condition, which dictates that the density should be sufficiently (and negatively) steep around θ_H, can be seen by observing that $\theta f(\theta)$ is the marginal contribution to social welfare of the consumers whose willingness to pay is θ, and that its derivative is $f(\theta)\pi(\theta)$: the former cannot be increasing around θ_H if FOCs (7) is to be satisfied (see also footnote 13.

in welfare due to a marginal increase in p_L being nil. In other words, welfare maximization leads to an 'aggressive' behaviour by the low quality (public) firm which expands output, driving the 'high-quality' indifferent consumer (identified by θ_H) towards the right tail of the distribution.[16] This in turn accounts for our Assumption 1(a) ruling out complete market coverage, as this would imply $\theta_H f(\theta_H) = 0$, which is inconsistent with firm H maximizing its profits.[17] It also accounts for Assumption 1(b), which rules out log-concavity: as already remarked, log-concave distributions (as well as the uniform distribution) are inconsistent with a price equilibrium of this game.

Secondly, the condition $\eta(\theta^\circ) < \frac{1}{1+k}$, together with Assumption 1(b), implies $k < \alpha$, i.e.

$$\frac{s_H}{s_L} > 1 + \frac{1}{\alpha}, \tag{9}$$

which again is consistent with ruling out log-concavity ($\alpha = 0$), and links the width of the admissible quality spectrum to the degree of convexity of the distribution of the consumers' willingness to pay. In fact, the lower α, the higher the lower bound on the (given) quality differential consistent with the existence of a price equilibrium, while the upper limit case where $\alpha = 1$ yields the constraint $s_H > 2s_L$.[18] Intuitively, this happens because *ceteris paribus* the width of the quality spectrum affects the concavity of the firm's payoff: if the two products are close substitutes, the demand for (say) firm L's product will be very elastic, and indeed too much for firm's L payoff (welfare) function to be well behaved.[19] A minimum quality spread ensures

[16] Notice that, in the 'ordinary' case of both firms being profit-maximizers, both marginal consumers will be on the left of the mode when the density is symmetric and unimodal. See, e.g., Benassi et al. (2006).

[17] Given $p_L = 0$, firm H would set a price p_H such that $\eta(p_H/\Delta) = 1$ so that $\theta_H f(\theta_H) > 0$.

[18] In this case $[1 - F(\theta)]^{-1}$ would be a convex function. Notice that if $1 - F(\theta)$ is log-concave, $[1 - F(\theta)]^{-1}$ is convex, but not viceversa.

[19] Take, e.g., the SOCs for firm L from footnote (9): since $\pi(\theta_H)$ will be negative at equilibrium, this expression cannot be negative if Δ is too small, and more generally, if

that vertical product differentiation survives in equilibrium, and that welfare is not maximized by setting the price equal to marginal cost: the welfare gain associated to complete market coverage is less than the welfare loss associated with lower profits for both the high and the low quality firms.[20]

Finally, from the existence proof reported in Appendix 6A, it turns out that a *necessary* condition for existence is that $\theta_L^* < \theta^\circ < \theta_H^*$: i.e., along the distribution of the willingness to pay, θ° is a sort of pivotal point around which the positions of the marginal consumers arrange themselves. This in turn implies that at equilibrium one necessarily has:

$$\eta(\theta_H) - \eta(\theta_L) > \frac{1}{\frac{s_H}{s_L}\left(\frac{s_H}{s_L} - 1\right)}, \tag{10}$$

which means that the minimum (elasticity) distance between the two marginal consumers (and hence the market for the low quality commodity) is higher, the lower the quality ratio.[21] In some sense there is a trade-off between how steeply demand rises with the willingness to pay going from θ_L to θ_H, and the quality differential: if the latter is low, equilibrium with vertical differentiation requires that 'middle-class' consumers are very willing to pay for even a modest quality premium.

While Assumption 1 and the condition $\eta(\theta^\circ) < \frac{1}{1+k}$ are sufficient to ensure the existence of a price equilibrium, one is naturally interested in looking at the circumstances under which such an equilibrium is unique. Indeed, since we have to rule out log-concavity, we cannot use the well-

products were too close welfare would be a convex function of p_L.

[20]Under this respect, Assumption 1(b) plays a key role, as it amounts *ceteris paribus* to a lower boundary on $\eta(\cdot)$: if the covered market is sufficiently elastic wrt the consumers' willingness to pay, the marginal gain in welfare from a price reduction will be low.

[21]Under our assumptions $\eta(\theta)$ is monotonically increasing in the relevant interval: equation (10) then follows by noting that in equilibrium it must be $\eta(\theta_L) < 1/(1+k)$ and substituting for the definition of k. The income level θ° is such that $\pi(\theta^\circ) = 0$, i.e. the elasticity of the density equals -1.

known result by Caplin & Nalebuff (1991) to the effect that log-concavity implies uniqueness. This is the issue we take up next.

3.2 Uniqueness of the price equilibrium

Our main result on uniqueness is the following:

Proposition 2 *Let the assumptions of Proposition 1 hold, and assume further that:*

(a) $k \leq \frac{\alpha^2}{1-\alpha^2}$, *and*

(b) $\alpha \leq 1/2$,

then the price equilibrium is unique.

Proof. See Appendix 7B ■

Both sufficient conditions can be read as strengthening the looser conditions which ensure existence. Indeed, it is easily seen that condition (a) amounts to:

$$\frac{s_H}{s_L} \geq \frac{1}{\alpha^2}, \tag{11}$$

and that, comparing this with constraint (9), $1/\alpha^2 > 1+1/\alpha$ for $\alpha \leq 1/2$, i.e. condition (b). In this sense uniqueness is delivered when the lower bound on the quality differential is higher than that which is sufficient to ensure existence, so that broadly speaking for the price equilibrium to be unique, the quality levels should be sufficiently far apart, by an amount which is determined by the concavity of the distribution.

As a final remark, however, this broad conclusion should be qualified by a discussion of our twin assumptions of zero costs and given quality levels. Generally speaking, the former allows to focus on the firms' strategic choices as driven by demand, and hence to bring out more sharply the role of the distribution of the consumers' willingness to pay (Tirole 1988, Wauthy 1996): the latter is in some sense the key to vertical differentiation – standard results point to the willingness to pay for quality upgradings as supporting equilibrium differentiation, even in the absence of differential quality costs. However, it should also be noticed that in many mixed markets one can arguably look at fixed costs as mainly quality driven, while variable unit costs may be thought of as constant (Lutz & Pezzino 2014); this is a situation where quality can be improved only by capital investments, and the focus on short-run price equilibria allows to normalize constant marginal costs to zero.[22] On the other hand, though the short-run focus is consistent with the assumption of given quality, the latter does beg the question of the likely outcome of a more general framework involving endogenous quality levels: since our model can be looked at as the first stage of a classical two-stage game, one naturally wonders what quality allocations would be supported by a perfect Nash equilibrium in prices and qualities.[23] Though no obvious answer presents itself, under the assumption of zero (low) marginal costs of quality upgrading, the high quality firm is likely to settle at the top quality level, as this allows to raise *ceteris paribus* the consumers' willingness to pay. By contrast, the public firm should tradeoff the welfare gain of higher quality in terms of larger market share, against the welfare loss in terms of lower consumers' surplus at the margin – the net effect of which is very likely to depend on the shape of the distribution of the willingness to pay.[24]

[22]A similar assumption of constant unit costs with respect to quantity can be found in models with covered markets, such as Grilo (1994), though in general the covered-market case allows also to work with a richer setting – like cost functions differing across firms, and unit costs increasing in quantity and quality (Laine & Ma 2015).

[23]In even more general terms, one could also wonder how entry dynamics can be affected by the presence of a welfare maximizing public firm in the framework of uncovered-market vertical differentiation, where 'natural oligopolies' are likely to be found (Shaked & Sutton 1983).

[24]Recall that in the standard profit maximizing framework with zero costs, the high quality firm settles at the top quality (Benassi et al. 2006). As to the public firm, a

4 An example: the Pareto distribution

In this Section we apply our model to the case of a Pareto distribution. While clearly limited to a specific case, we believe that this example can serve as an illustration of the way the distribution of the willingness to pay affects equilibrium outcomes. Suppose then that the consumers' willingness to pay is distributed as a Pareto distribution of the second kind (Johnson et al. 1995), so that the density and the cumulative distributions are respectively $f(\theta,\gamma)=\gamma\,(1+\theta)^{-(1+\gamma)}$ and $F(\theta,\gamma)=1-\frac{1}{(1+\theta)^{\gamma}}$, defined over the support $\Theta=[0,\infty)$, where $\gamma>1$ is a given parameter. It is then easily seen that $\pi(\theta,\gamma)=\frac{1-\gamma\theta}{1+\theta}$ and $\eta(\theta,\gamma)=\frac{\gamma\theta}{1+\theta}$ such that Assumption 1 is satisfied. In particular, Assumption 1(b) holds for any $\alpha=1/\gamma$, so that $1-F(\theta)$ is ρ-concave with $\rho=-1/\gamma$; notice also that in this case we have $\tilde{\theta}=\frac{1}{\gamma-1}>\theta^{\circ}=\frac{1}{\gamma}$ such that $\eta(\tilde{\theta})=1$ and $\pi\left(\theta^{\circ}\right)=0$. In addition, $\eta\left(\theta^{\circ}\right)=\gamma/\left(1+\gamma\right)$, such that the condition $\eta\left(\theta^{\circ}\right)<1/\left(1+k\right)$ set out in Proposition 1 reduces to $k<1/\gamma$. It should be remarked that in this framework γ is an inverse parameter of first order stochastic dominance, so that higher values of γ support lower mean values of the consumers' willingness to pay.[25]

We now perform a numerical simulation with different values of γ, say between $\gamma=2$ and $\gamma=3$, to see the way a shift on the distribution of the willingness to pay affects equilibrium prices. To do so we set $k=1/8<1/\gamma$, which is equivalent to $s_H/s_L=9$. On the basis of Proposition 2 (and condition (11)), this quality ratio delivers a unique equilibrium for $\alpha\geq 1/3$, which is verified as $\alpha=1/\gamma\geq 1/3$, while $\alpha\leq 1/2$ as required by sufficient condition (b) of the same Proposition. Within this framework, we perform

generic consumer's surplus is given by θs_L-p_L: while higher s_L means for given prices lower θ_L (and hence, widening the pool of consumers), it also means that many additional consumers will be "poor" (whose contribution to welfare in terms of surplus will be low).

[25]Mean willingness to pay is $\mu=1/(\gamma-1)$.

three simulations for $\gamma = 2$, $\gamma = 5/2$ and $\gamma = 3$. Table 1 reports equilibrium values of the positions of the marginal consumers, θ_H^* and θ_L^*, obtained for these different values of γ.

	$\gamma = 2$	$\gamma = 5/2$	$\gamma = 3$
θ_H^*	0.93629	0.61392	0.45498
θ_L^*	0.25484	0.25318	0.24011

Table 1: Equilibrium marginal values of willingness to pay

From Table 1, we see that in the case of a Pareto distribution, an increase in γ (i.e. lower mean income) leads to a leftward shift of both marginal consumers, together with a decrease in the distance between them. This would point to decreasing income leading to stiffer competition, which is confirmed by Table 2 below, such that relative prices decrease unambiguously with higher values of γ.

	$\gamma = 2$	$\gamma = 5/2$	$\gamma = 3$
p_H^*/p_L^*	30.392	20.399	16.159
$\frac{p_H^*}{s_H^*}$	0.86057	0.57384	0.43111
$\frac{p_L^*}{s_L^*}$	0.25484	0.25318	0.24011

Table 2: Equilibrium relative and hedonic prices

In Table 2 we also report the behaviour of hedonic prices, which would suggest that lower average income – at least in this example – puts a downward pressure on the price per 'unit of quality', which appears to be stronger for the (profit-maximizing) high quality firm. Both relationships (Table 2) are apparently monotone, suggesting that the competition of the public sector (or of the regulated industry) will be more intense in case of lower average income (i.e., the larger the parameter γ).

5 Conclusions

Starting by informal arguments that public firms competing with private firms rely on distributional concerns about inequality, and by formal reasoning that the distribution of the willingness to pay affects the firms' equilibrium choices, in this paper we show that a price equilibrium in a vertically differentiated mixed duopoly with uncovered market exists, if the quality spectrum is wide enough *vis à vis* a measure of the convexity of the distribution of the consumers' willingness to pay, and that such equilibrium is unique if this sufficient condition is tightened. In particular, we show that for a price equilibrium to exist the distribution of the willingness to pay cannot be logconcave, and that sufficient conditions for existence and uniqueness place a lower bound which is higher, the higher the given convexity bound on the income distribution.

By way of example, we apply our basic model to a Pareto distribution, and find that a decrease of average income is (broadly speaking associated) with higher competitive pressure from the public firm, as signaled, e.g., by a decrease of the distance between the marginal willingness to pay for high *vs* low quality goods, and of relative prices – the price of high quality decreases relative to that of low quality; also, the decrease in hedonic prices appears to be stronger for the high quality goods. Though obviously constrained by the specific form of this example, these results confirm that assumptions about the distribution of the consumers' willingness to pay do play a key role in assessing the working of vertically differentiated markets.

Finally, while our results are based on assumptions (notably zero costs and given quality levels) which are obviously restrictive (though arguably reasonable), our framework points to the important general question of comparing the overall performance of 'mixed' *vs* 'pure' oligopoly within vertical differentiated markets. If this sets a general agenda for future research, such a comparison should however take into account that the distributional assumptions required to support the existence of a price equilibrium in the 'mixed' case, are likely to be different from those required to support a perfect equilibrium in prices and qualities in the 'pure' case – as indeed is made clear by the standard textbook assumption of uniform income distribution being inconsistent with having one welfare maximizing firm.

yielding:

$$(1+\alpha)\,\eta\left(\theta_H^*\right)\left(\frac{1}{1-\eta\left(\theta_H^*\right)}+\frac{1}{\alpha^2}\right)<\frac{2\eta\left(\theta_H^*\right)-1}{1-\eta\left(\theta_H^*\right)}+\frac{1}{1-\eta\left(\theta_H^*\right)}+\frac{1}{\alpha^2},$$

from which, after rearrangement,

$$\frac{(1-\alpha)\,\eta\left(\theta_H^*\right)}{1-\eta\left(\theta_H^*\right)}+\frac{1}{\alpha^2}>\frac{(1+\alpha)\,\eta\left(\theta_H^*\right)}{\alpha^2}. \tag{B.3}$$

We now invoke the condition $k \leq \alpha^2/\left(1-\alpha^2\right)$, which is consistent with $k<\alpha$, as $\alpha^2/\left(1-\alpha^2\right)<\alpha$ for $\alpha \leq 1/2$. Under this assumption, $1-\eta\left(\theta_H^*\right)<\alpha^2$: indeed, this is equivalent to $\eta\left(\theta_H^*\right)>1-\alpha^2$, which is true as $\eta\left(\theta_H^*\right)>1/\left(1+k\right)$ and:

$$\frac{1}{1+k}-\left(1-\alpha^2\right)=\frac{-k+\alpha^2+\alpha^2 k}{1+k}>0.$$

There follows that $\frac{(1-\alpha)\eta\left(\theta_H^*\right)}{1-\eta\left(\theta_H^*\right)}>\frac{(1-\alpha)\eta\left(\theta_H^*\right)}{\alpha^2}$, so that (b.3) holds if:

$$\frac{(1-\alpha)\,\eta\left(\theta_H^*\right)}{\alpha^2}+\frac{1}{\alpha^2}>\frac{(1+\alpha)\,\eta\left(\theta_H^*\right)}{\alpha^2},$$

i.e.

$$1>2\eta\left(\theta_H^*\right)\alpha,$$

which is surely true for $\alpha \leq 1/2$, as $2\eta\left(\theta_H^*\right)\alpha<1$.■

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Highlights

- We study the existence of a price equilibrium in a vertically differentiated mixed duopoly with uncovered market exists when a welfare-maximizing public firm producing low quality goods competes against a profit-maximizing private firm producing high quality goods.
- We show that for a price equilibrium to exist the distribution of the willingness to pay cannot be log-concave, and that sufficient conditions for existence and uniqueness place on the quality spectrum a lower bound, which is higher, the higher the given convexity bound on the income distribution.
- We apply our model to a Pareto distribution, and find that a decrease of average income is associated with higher competitive pressure from the public firm, and the decrease in hedonic prices appears to be stronger for the high quality goods.