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Robust Control of a Flapping-Wing MAV by Differentiable Wingbeat Modulation

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Abstract: We present in this paper the design of a nonlinear controller for robust control of a flapping-wing Micro Air Vehicle in the longitudinal plane. Control of both pitch and hover dynamics are performed by a single means of actuation, the flapping motion of the wing. This is accomplished through continuous manipulation of two parameters of the wingbeat waveform, namely a frequency shift and a wing bias. Differentiability of the wingbeat function with respect to these parameters allows a theoretically sound application of averaging techniques, an issue that has been frequently overlooked in the literature. Simulation results show the effectiveness of the proposed methodology in dealing with parametric and dynamic model uncertainty.

Keywords: micro air vehicles, averaging, nonlinear systems, nonlinear control, flight control

1. INTRODUCTION

Mimicking insect capabilities is a highly desirable potential means to meeting the goals of enhanced dexterity and maneuverability for small-scale aerial vehicles. Nevertheless, it is now well-understood that the development of a flapping-wing based system represents a significant technical challenge. A major aspect of this challenge is the development of robust motion controllers capable of executing the desired aerial mobility with means for actuation that are constrained in size and power consumption. Recent contributions in the literature (See Doman et al. (2010); Oppenheimer et al. (2011) and references therein) have remarked the possibility of achieving moment and force generation simply by independently shaping the wingbeat waveforms during each cycle, whereas other approaches make use of a larger number of actuated mechanical degrees of freedom, noticeably moving appendages (Bolender (2009); Dyhr et al. (2013)), variable stroke-plane angles (Serrani (2011); Lee et al. (2014)) and variable angle-of-attack of the wing (Humbert and Faruque (2011).) Due to the requirement of periodic and/or time-varying wing motion patterns, the method of averaging has been the approach of choice for stability analysis and control system design for flapping-wing MAVs. Interestingly enough, some of the mentioned references adopt wingbeat functions that do not generally satisfy the requirements for a theoretically sound application of the method of averaging. The aim of this paper is to introduce a novel differentiable wingbeat function that does not require manipulations of the waveform within each period, and is amenable to a rigorous application of averaging theory. The robust control approach of Serrani (2011), which originally made use of two effectors, is adapted to the novel minimal actuation configuration towards the design of a robust controller for stabilization about a desired setpoint.

The paper is organized as follows: In Section 2, the vehicle model and the novel wingbeat function are introduced, and the averaged model developed. The controller design and the stability analysis are presented in Sections 3 and 4, respectively. Simulation results are presented in Section 5, followed by conclusions in Section 6.

2. MAV MODEL AND PROBLEM FORMULATION

We consider a simplified model of the dynamics of a generic insectiform flapping-wing MAV derived in Bolender (2009). In this study, it is assumed that the MAV is comprised of a single rigid body (equivalently, head, torso and abdomen of the insect are set in a fixed configuration) equipped with a pair of moving massless wings. When restricted to the longitudinal plane, the equations of motion of Bolender (2009) read as the non-autonomous system

$$\begin{aligned}
m\ddot{x} &= 2\dot{\mu}^2(t)\,[k_L(\alpha)\sin(\lambda-\theta) \\
&\quad - k_D(\alpha)\,\mathrm{sign}\,\dot{\mu}(t)\,\cos\mu(t)\cos(\lambda-\theta)] \\
m\ddot{z} &= mg - 2\dot{\mu}^2(t)\,[k_L(\alpha)\cos(\lambda-\theta) \\
&\quad + k_D(\alpha)\,\mathrm{sign}\,\dot{\mu}(t)\,\cos\mu(t)\,\sin(\lambda-\theta)] \\
I_{yy}\ddot{\theta} &= 2\dot{\mu}^2(t)\,k_L(\alpha)\,[p_x\cos\lambda - p_z\sin\lambda + y_r\sin\mu(t) \\
&\quad - x_r\cos\mu(t)\sin((\pi/2-\alpha)\,\mathrm{sign}\,\dot{\mu}(t))] \\
&\quad + 2\dot{\mu}^2(t)\,\mathrm{sign}\,\dot{\mu}(t)\,k_D(\alpha)\cos\mu(t)\,[\,p_z\cos\lambda + p_x\sin\lambda \\
&\quad + x_r\cos((\pi/2-\alpha)\,\mathrm{sign}\,\dot{\mu}(t))]
\end{aligned} \tag{1}$$

where $t \in \mathbb{R}_{\geq 0}$ is the time variable, $x \in \mathbb{R}$, $z \in \mathbb{R}$ and $\theta \in [-\pi, \pi)$ are respectively the longitudinal and vertical position of the center of mass and the pitch angle of the vehicle with respect to an inertial coordinate system. The function $t \mapsto \mu(t) \in \mathbb{R}$ is the time-varying wingbeat modulation, assumed to be of class $\mathcal{C}^1$. The vectors $(x_r, y_r, 0)$ and $(p_x, p_y, -p_z)$ are constant model parameters representing respectively the location of the center of lift of the wing in a wing-fixed frame and the location of the hinge root point of the right wing in a body-fixed frame centered at the vehicle center of mass. The parameters g, m and I_{yy} denote respectively acceleration of gravity, mass and moment of inertia of the vehicle about the body-y_b axis. The wingstroke plane angle $\lambda \in [0, \pi/2]$ is assumed fixed, as opposed to other studies where it is regarded as an additional control input Serrani (2011); Lee et al. (2014).The angle-of-attack of the wings, $\alpha \in (0, \pi/4]$, is assumed constant along both upstroke and downstroke, and to instantaneously reverse its sign at the end of each half stroke. For this study, it is assumed that α is a tunable constant parameter. Following Berman and Wang (2007), the lift and drag coefficients are obtained from a slender body approximation as $C_L(\alpha) = C_T \sin 2\alpha$ and $C_D(\alpha) = C_{D\pi/2} \sin^2 \alpha$, respectively, where C_T and $C_{D\pi/2}$ are positive coefficients. Finally, the overall lift and drag force coefficients of the vehicle are defined respectively as $k_L(\alpha) = \frac{\rho}{2} I_w C_L(\alpha)$ and $k_D(\alpha) = \frac{\rho}{2} I_w C_D(\alpha)$, where I_w is the moment of inertia of the wing and ρ is the air density. The reader is referred to Bolender (2009) for further details on the model.

For the sake of simplicity, and without loss of generality, we let $\lambda = 0$, yielding the simpler model

$$\begin{aligned}
\ddot{x} &= -\frac{2k_L(\alpha)}{m}\dot{\mu}^2(t)\left[\sin\theta + \frac{k_D(\alpha)}{k_L(\alpha)}\operatorname{sign}\dot{\mu}(t)\cos\mu(t)\cos\theta\right]\\
\ddot{z} &= g - \frac{2k_L(\alpha)}{m}\dot{\mu}^2(t)\left[\cos\theta - \frac{k_D(\alpha)}{k_L(\alpha)}\operatorname{sign}\dot{\mu}(t)\cos\mu(t)\sin\theta\right]\\
\ddot{\theta} &= \frac{2k_L(\alpha)}{I_{yy}}\dot{\mu}^2(t)\left[p_x + y_r\sin\mu(t) - x_r\cos\mu(t)\sin\psi(t)\right.\\
&\quad \left. + \frac{k_D(\alpha)}{k_L(\alpha)}\operatorname{sign}\dot{\mu}(t)\cos\mu(t)\,(\,p_z + x_r\cos\psi(t))\right]
\end{aligned} \tag{2}$$

where $\psi(t) := (\pi/2 - \alpha)\operatorname{sign}\dot{\mu}(t)$. For later use, we denote by $\boldsymbol{x} = (x, \dot{x}, z, \dot{z}, \theta, \dot{\theta}) \in \mathcal{X} := \mathbb{R}^4 \times [-\pi, \pi) \times \mathbb{R}$ the state of the system, whereas the output to be controlled is selected as the position of the center of mass, $\boldsymbol{y} = (x, z) \in \mathbb{R}^2$. Let the vector $\boldsymbol{p} \in \mathcal{P} \subset \mathbb{R}^p$ collect all model parameters of (1) subject to uncertainty, where $\mathcal{P}$ is a compact and convex set. System (2) reads as

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{g} + \dot{\mu}^2(t)\boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{p}),\ \boldsymbol{x}(0) = \boldsymbol{x}_0\\
\boldsymbol{y} &= \boldsymbol{h}(\boldsymbol{x})
\end{aligned} \tag{3}$$

with obvious meaning of the vector $\boldsymbol{g} \in \mathbb{R}^6$, the readout map $\boldsymbol{h} : \mathcal{X} \to \mathbb{R}^2$ and the vector field $\boldsymbol{f} : \mathbb{R}_{\geq 0} \times \mathcal{X} \to \mathbb{R}^n$. It is worth noticing that, in spite of the signum function, the right-hand side of the first equation of (3) is a continuously differentiable vector field of $(t, \boldsymbol{x})$ whenever $\mu(\cdot) \in \mathcal{C}^1$.

2.1 Differentiable Wingbeat Modulation

Since no additional mechanical device such as an abdomen (Dyhr et al. (2013)) or a weight-shifting mechanism (Doman et al. (2010)) is assumed to be available for controlling the pitching dynamics, actuation is only exerted through the time-varying wingbeat. Control of the vehicle dynamics (1), in a sense to be specified, will be achieved by instantaneous manipulation of the waveform of the wingbeat via a suitable set of control parameters. In this study, inspired by Hedrick and Daniel (2006), we propose the *novel wingbeat function*

$$\mu(t) = \mu(\tau(t,\delta), b) := \begin{cases} \cos(\tau) + b(t)\tanh(\tau)^3, \\ \qquad \tau \bmod 4\pi \in [0, 2\pi) \\ \cos(\tau) + b(t)\left[1 - \tanh(\tau - 2\pi)^3\right], \\ \qquad \tau \bmod 4\pi \in [2\pi, 4\pi) \end{cases} \tag{4}$$

where

$$\tau(t, \delta) = \omega_0 t + \omega_0 \int_0^t \delta(s)\mathrm{d}s \tag{5}$$

is a rescaled temporal variable, $\omega_0 > 0$ is the wingbeat frequency (regarded as a tunable constant parameter), $\delta(\cdot) \in \mathcal{C}^1$ is the manipulated frequency shift and $b(\cdot) \in \mathcal{C}^1$ is the manipulated wing bias. Letting $|b(t)| < 1$ and $|\delta(t)| < 1$ for all $t \in \mathbb{R}_{\geq 0}$ ensures physical meaning of the wingbeat and that the time scaling $t \mapsto \tau$ is well defined. Notice that $\mu(\cdot)$ is a twice continuously differentiable function of its arguments, and a 4π-periodic function of τ when $b = $ cost. These properties remove altogether the technical difficulties related to averaging discontinuous and/or non-periodic signals that affect most of the work found in the literature (most noticeably, the seminal papers Doman et al. (2010); Oppenheimer et al. (2011)), while yielding a model that is in the appropriate analytic form for averaging. Finally, the control input, henceforth selected as $\boldsymbol{u} = (\delta, b) \in \mathcal{U} := (-1, 1) \times (-1, 1)$, is to be provided by a time-invariant state-feedback controller of the form

$$\begin{aligned}
\dot{\boldsymbol{\xi}} &= \boldsymbol{f}_c(\boldsymbol{\xi}, \boldsymbol{x}),\ \boldsymbol{\xi}(0) = \boldsymbol{\xi}_0\\
\boldsymbol{u} &= \boldsymbol{h}_c(\boldsymbol{\xi}, \boldsymbol{x})
\end{aligned} \tag{6}$$

with state $\boldsymbol{\xi} \in \mathbb{R}^\nu$, where $\boldsymbol{f}_c : \mathbb{R}^\nu \times \mathcal{X} \to \mathbb{R}^\nu$ and $\boldsymbol{h}_c : \mathbb{R}^\nu \times \mathcal{X} \to \mathcal{U}$ are $\mathcal{C}^1$ functions of their arguments. The control objectives will be defined in the next sections, after the averaging procedure has been described.

2.2 Averaging

We start by defining the relation between differentiation with respect to t and τ. For a given $\mathcal{C}^1$ function $\chi : t \mapsto \chi(t)$, we let $\dot{\chi}(t) := \mathrm{d}\chi(t)/\mathrm{d}t$ and $\chi'(t) := \mathrm{d}\chi(t)/\mathrm{d}\tau$, where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \omega_0(1+\delta)\frac{\mathrm{d}}{\mathrm{d}\tau}, \quad \frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{1}{\omega_0(1+\delta)}\frac{\mathrm{d}}{\mathrm{d}t} \tag{7}$$

Accordingly, equation (3) can be rewritten as

$$\begin{aligned}
\boldsymbol{x}' &= \frac{1}{\omega_0}\left[\boldsymbol{\varphi}_0(\boldsymbol{u}) + \omega_0^2\,\boldsymbol{\varphi}_1(\tau, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) + \omega_0\,\boldsymbol{\varphi}_2(\tau, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p})\dot{b}\right.\\
&\quad \left. + \boldsymbol{\varphi}_3(\tau, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p})\dot{b}^2\right]
\end{aligned} \tag{8}$$

Under the given assumptions, it can be verified that all $\boldsymbol{\varphi}_i(\cdot)$, $i \in \{0, \ldots, 3\}$, are continuously differentiable and *bounded* functions of their arguments. Letting, by an appropriate choice of α,

$$\omega_0 = \frac{1}{\varepsilon}, \quad \frac{k_L(\alpha)}{m} = \mathcal{O}(\varepsilon^2), \quad \frac{k_L(\alpha)}{I_{yy}} = \mathcal{O}(\varepsilon^2) \tag{9}$$

where $\varepsilon \in (0, 1)$ is a tunable parameters, Eq. (8) reads as

$$\begin{aligned}
\boldsymbol{x}' &= \varepsilon\left[\boldsymbol{\varphi}_0(\boldsymbol{u}) + \boldsymbol{\phi}_1(\tau, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) + \varepsilon\,\boldsymbol{\phi}_2(\tau, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p})\dot{b}\right.\\
&\quad \left. + \varepsilon^2\boldsymbol{\phi}_3(\tau, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p})\dot{b}^2\right]
\end{aligned} \tag{10}$$

where $\boldsymbol{\phi}_i(\cdot)$, $i \in \{1, \ldots, 3\}$, are continuously differentiable and bounded functions. System (10) can now be averaged with respect to $\tau \in [0, 4\pi]$, using ε as the averaging parameter. Application of the averaging operator

$$\operatorname{avg}(\cdot) : u(\cdot) \mapsto \frac{1}{4\pi}\int_0^{4\pi} u\,(\tau)\,\mathrm{d}\tau$$

to the right-hand side of (10) and reverting to the original coordinates yields the *perturbed averaged model*[1]

[1] Note that the controller dynamics (6) are assumed time-invariant, hence they are left unchanged by averaging.

$$\dot{\boldsymbol{x}} = \boldsymbol{f}_{\mathrm{avg}}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{p}) + \varepsilon\boldsymbol{\Delta}_{\mathrm{avg}}(\boldsymbol{x},\boldsymbol{u},\dot{\boldsymbol{u}},\boldsymbol{p},\varepsilon) \tag{11}$$

where the perturbation $\boldsymbol{\Delta}_{\mathrm{avg}}(\cdot)$ accounts for the last two terms in the right-hand side of (10), hence vanishes at $\dot{b}=0$. The *unperturbed averaged model*, obtained from (11) by setting $\boldsymbol{\Delta}_{\mathrm{avg}}=0$, has the following expression:

$$\begin{aligned}
\ddot{x} &= -\frac{\omega_0^2}{m}(1+\delta)^2\left[k_L\beta_1(b)\sin\theta + k_D\beta_2(b)\cos\theta\right]\\
\ddot{z} &= g - \frac{\omega_0^2}{m}(1+\delta)^2\left[k_L\beta_1(b)\cos\theta - k_D\beta_2(b)\sin\theta\right]\\
\ddot{\theta} &= \frac{\omega_0^2}{I_{yy}}(1+\delta)^2 k_L\big[\beta_1(b)p_x + \beta_2(b)x_r\cos\alpha + \beta_3(b)y_r\big]\\
&\quad + \frac{\omega_0^2}{I_{yy}}(1+\delta)^2 k_D\big[\beta_2(b)p_x + \beta_2(b)x_r\sin\alpha\big]
\end{aligned} \tag{12}$$

where $\beta_i(\cdot)$ are continuous functions satisfying

$$\beta_1(0)=1\,,\quad \beta_2(0)=0\,,\quad \beta_3(0)=0$$

A polynomial approximation over the interval $b\in[-1,1]$ yields the following expressions for $\beta_i(\cdot)$:

$$\begin{aligned}
&\beta_1(b) = 1 + 0.1639\,b^2\,,\quad \beta_2(b) = -0.1289\,b^2 + 0.024\,b^3\\
&\beta_3(b) = 0.4409\,b + 0.05\,b^2 - 0.00078\,b^3
\end{aligned} \tag{13}$$

The above relationships reveal the pitch dynamics can be controlled by the wing bias parameter through the term $\beta_3(b)y_r$, due to the fact that – necessarily – $y_r\neq 0$. On the other hand, when $p_x\neq 0$, a nonzero pitching moment exists, which must be compensated by a nonzero wing bias at trim. This suggests that the choice $p_x=0$ is a desirable trait for the vehicle design, which shall be hereby adopted in the unperturbed model. Finally, letting

$$\rho(b) := \sqrt{k_L^2\beta_1^2(b) + k_D^2\beta_2^2(b)}\,,\ \ \gamma(b) := \arctan\frac{k_D^2\beta_2(b)}{k_L^2\beta_1(b)}$$

and

$$\begin{aligned}
\mu_1(b) &:= \frac{\omega_0^2}{m}\rho(b)\,,\quad \mu_2 := \frac{\omega_0^2}{m}k_L y_r\\
d_\theta(b) &:= \frac{1}{y_r}\left[x_r\cos\alpha + \frac{k_D}{k_L}\left(p_z + x_r\sin\alpha\right)\right]\beta_2(b)
\end{aligned}$$

yields the final form of the unperturbed averaged model:

$$\begin{aligned}
\ddot{x} &= -\mu_1(b)(1+\delta)^2\sin\left(\theta+\gamma(b)\right)\\
\ddot{z} &= g - \mu_1(b)(1+\delta)^2\cos\left(\theta+\gamma(b)\right)\\
\ddot{\theta} &= \mu_2(1+\delta)^2\left[\beta_3(b) + d_\theta(b)\right]
\end{aligned} \tag{14}$$

System (14) has a manifold of equilibrium points of the form $\boldsymbol{x}=(x_d,0,z_d,0,0,0)\in\mathcal{X}$, when the input is selected as $\boldsymbol{u}=(\delta_*,0)$, where

$$\delta_* := \frac{\omega_*^2(\alpha)}{\omega_0^2} - 1\,,\quad \omega_*(\alpha) := \sqrt{\frac{mg}{k_L(\alpha)}}$$

and $\omega_*(\alpha)$ is the *hovering flapping frequency* (Doman et al. (2010).) Consistently with (9), the fundamental wingbeat frequency, ω_0, is then chosen on the basis of the *nominal values* of the model parameters as $\omega_0=\omega_{*,\mathrm{nom}}(\alpha)$. Finally, known bounds of the form

$$\begin{aligned}
&0<\underline{\mu}_1\le\mu_1(b)\le\bar{\mu}_1\,,\quad 0<\underline{\mu}_2\le\mu_2\le\bar{\mu}_2\,,\quad \underline{\delta}_*\le\delta_*\le\bar{\delta}_*\\
&|d_\theta(b)|\le\bar{d}_\theta|b|^2\,,\quad |\gamma(b)|\le\bar{\gamma}_1|b|\,,\quad |\gamma(b)|\le\bar{\gamma}_2|b|^2
\end{aligned} \tag{15}$$

can be established for all $\boldsymbol{p}\in\mathcal{P}$ and all $\boldsymbol{u}\in\mathcal{U}$.

The control objective is stated as follows: *Given the averaged model* (11)*, find a dynamic controller* (6) *such that the resulting closed-loop system has a GAS and LES equilibrium at the origin, robustly with respect to all values of the model parameters satisfying* (15).

The focus of the paper is on the design of a controller that satisfies the objectives for the unperturbed averaged model (14). Extension of these results to the perturbed averaged model can be accomplished by resorting to an appropriate *small-gain* argument (in the parameter ε) to accommodate the perturbation term $\boldsymbol{\Delta}_{\mathrm{avg}}(\cdot)$.

3. DESIGN FOR THE UNPERTURBED MODEL

The proposed controller is comprised of two separate control loops: In the first-level loop, the wingbeat frequency shift and the pitch angle act as control inputs for the forward and vertical dynamics. The second-level loop regulates the pitch dynamics via the wing bias.

3.1 Control of the Forward and Vertical Dynamics

Using the transformations $\theta\mapsto\eta:=\theta+\gamma(b)$ and $(\delta,\eta)\mapsto(v_x,v_z)$ defined as

$$v_x := (1+\delta)^2\sin(\eta)\,,\quad v_z := 1-(1+\delta)^2\cos(\eta) \tag{16}$$

one obtains for the forward and vertical dynamics of (14)

$$\begin{aligned}
\ddot{x} &= -\mu_1(b)v_x\\
\ddot{z} &= \mu_1(b)\left[v_z + \delta_* + \nu(b)\right]
\end{aligned} \tag{17}$$

where

$$\nu(b) := \frac{\omega_*^2}{\omega_0^2}\frac{k_L(\alpha)-\rho(b)}{\rho(b)} \tag{18}$$

is a disturbance vanishing at $b=0$ and satisfying

$$|\nu(b)|\le\bar{\nu}_1\,|b|\,,\quad |\nu(b)|\le\bar{\nu}_2\,|b|^2$$

for some $\bar{\nu}_1,\bar{\nu}_2>0$ and all $b\in(-1,1)$. The map (16) is a diffeomorphism in the domain $\mathcal{D}=(-1,1)\times(-\pi/2,\pi/2)$ onto its range, $\mathcal{R}(\mathcal{D})$, where v_z satisfis $|v_z|<1$, since $\eta=\arctan\left(v_x/(1-v_z)\right)$. Next, bounded virtual controls

$$|v_{\mathrm{cmd}}^x|\le\bar{v}_x\,,\qquad |v_{\mathrm{cmd}}^z|\le\bar{v}_z\,,\qquad \bar{v}_x,\bar{v}_z>0 \tag{19}$$

are determined to robustly stabilize the set-point $(x_d,z_d)\in\mathbb{R}^2$ when $(v_x,v_z)=(v_{\mathrm{cmd}}^x,v_{\mathrm{cmd}}^z)$. The bounds $\bar{v}_x$ and $\bar{v}_z$ shall be chosen so that, using the inverse of the map (16),

$$|v_x|\le\bar{v}_x \ \text{ and } \ |v_z|\le\bar{v}_z \implies |\delta|\le\bar{\delta} \ \text{ and } \ |\eta|\le\bar{\eta}$$

for some prescribed $\bar{\delta}\in(0,1)$ and $\bar{\eta}\in(0,\pi/2)$. Due to the uncertainty on $\mu_1(b)$, the design of the virtual controls v_{cmd}^x, v_{cmd}^z can not be accomplished by the methodologies originally proposed in Teel (1992), and successive refinements in Marchand and Hably (2005). Conversely, the approach of Marconi and Isidori (2000) applies. Following this latter work, let sat : $\mathbb{R}\to\mathbb{R}$, denote a *differentiable unit saturation function*. The x-dynamics are first augmented with the integral error $\dot{x}_1=x-x_d$, followed by the change of coordinates

$$x_2 := x - x_d + \ell_1^x\,\mathrm{sat}\left(\frac{\kappa_1^x}{\ell_1^x}x_1\right)\,,\quad x_3 := \dot{x} + \ell_2^x\,\mathrm{sat}\left(\frac{\kappa_2^x}{\ell_2^x}x_2\right)$$

whereas the virtual control is selected as

$$v_{\mathrm{cmd}}^x := \ell_3^x\,\mathrm{sat}\left(\frac{\kappa_3^x}{\ell_3^x}x_3\right) \tag{20}$$

where $\ell_i^x>0$, $\kappa_i^x>0$ are suitable gains. Note that $\ell_3^x\le\bar{v}_x$. In the new coordinates, the augmented x-dynamics read as

$$\dot{x}_1 = -\ell_1^x \operatorname{sat}\left(\frac{\kappa_1^x}{\ell_1^x} x_1\right) + x_2$$
$$\dot{x}_2 = -\ell_2^x \operatorname{sat}\left(\frac{\kappa_2^x}{\ell_2^x} x_2\right) + x_3 + \kappa_1^x \operatorname{sat}'\left(\frac{\kappa_1^x}{\ell_1^x} x_1\right)\dot{x}_1 \tag{21}$$
$$\dot{x}_3 = -\mu_1(b)\left[\ell_3^x \operatorname{sat}\left(\frac{\kappa_3^x}{\ell_3^x} x_3\right) + d_x\right] + \kappa_2^x \operatorname{sat}'\left(\frac{\kappa_2^x}{\ell_2^x} x_2\right)\dot{x}_2$$

where $d_x := v_x - v^x_{\text{cmd}}$. Similarly as in Serrani (2011), the following result is established from application of the methodology of Marconi and Isidori (2000):

Proposition 1. Let the gains κ_i^x, ℓ_i^x, $i = 1, 2$, be parameterixed as follows

$$\kappa_1^x = \varepsilon_1^x \kappa_2^x, \qquad \ell_1^x = (1 - a_1^x)\frac{\ell_2^x}{\kappa_2^x} + a_1^x \frac{\ell_2^x}{8\,\varepsilon_1^x \kappa_2^x}$$
$$\kappa_2^x = \varepsilon_2^x \kappa_3^x, \qquad \ell_2^x = (1 - a_2^x)\frac{2\ell_3^x}{\kappa_3^x} + a_2^x \frac{\underline{\beta}_1 \ell_3^x}{8\,\varepsilon_2^x \kappa_3^x}$$

where

$$\varepsilon_1^x \in \left(0, \frac{1}{8}\right), \quad \varepsilon_2^x \in \left(0, \frac{\underline{\beta}_1}{16}\right), \quad a_1^x \in (0,1), \quad a_2^x \in (0,1)$$

Then, for any $\kappa_3^x > 0$ and any $\ell_3^x \in (0, \bar{v}_x]$ the state of system (21) satisfies an a-$\mathcal{L}_\infty$ bound from the set $\mathcal{X}_x = \mathbb{R}^3$ with restriction $\Delta_x = \ell_3^x/2$ on the disturbance d_x. Furthermore, for any signal d_x satisfying $\|d_x\|_{\text{a}} < \Delta_x$, the state of (21) satisfies the asymptotic bounds

$$\|x_1\|_{\text{a}} \le \frac{4}{\kappa_1^x \kappa_2^x \kappa_3^x}\|d_x\|_{\text{a}}, \qquad \|x_2\|_{\text{a}} \le \frac{4}{\kappa_2^x \kappa_3^x}\|d_x\|_{\text{a}}$$
$$\|x_3\|_{\text{a}} \le \frac{2}{\kappa_3^x}\|d_x\|_{\text{a}} \tag{22}$$

Proof. See Marconi and Isidori (2000). □

A similar design is accomplished for the z-dynamics. Augmentation with the integral error $\dot{z}_1 = z - z_d$, followed by the coordinate change

$$z_2 := z - z_d + \ell_1^z \operatorname{sat}\left(\frac{\kappa_1^z}{\ell_1^z} z_1\right), \quad z_3 := \dot{z} + \ell_2^z \operatorname{sat}\left(\frac{\kappa_2^z}{\ell_2^z} z_2\right)$$

and the selection of the virtual control

$$v^z_{\text{cmd}} := -\ell_3^z \operatorname{sat}\left(\frac{\kappa_3^z}{\ell_3^z} z_3\right)$$

where $\ell_i^z > 0$, $\kappa_i^z > 0$ are suitable gains, yield the system

$$\dot{z}_1 = -\ell_1^z \operatorname{sat}\left(\frac{\kappa_1^z}{\ell_1^z} z_1\right) + z_2$$
$$\dot{z}_2 = -\ell_2^z \operatorname{sat}\left(\frac{\kappa_2^z}{\ell_2^z} z_2\right) + z_3 + \kappa_1^z \operatorname{sat}'\left(\frac{\kappa_1^z}{\ell_1^z} z_1\right)\dot{z}_1 \tag{23}$$
$$\dot{z}_3 = \beta_1\left[d_z - \ell_3^z \operatorname{sat}\left(\frac{\kappa_3^z}{\ell_3^z} z_3\right) + \delta_*\right] + \kappa_2^z \operatorname{sat}'\left(\frac{\kappa_2^z}{\ell_2^z} z_2\right)\dot{z}_2$$

where $d_z := v_z - v^z_{\text{cmd}} + \nu(b)$.

Proposition 2. Let the gains κ_i^z, ℓ_i^z, $i = 1, 2$, be parameterized as follows

$$\kappa_1^z = \varepsilon_1^z \kappa_2^z, \qquad \ell_1^z = (1 - a_1^z)\frac{\ell_2^z}{\kappa_2^z} + a_1^z \frac{\ell_2^z}{8\,\varepsilon_1^z \kappa_2^z}$$
$$\kappa_2^z = \varepsilon_2^z \kappa_3^z, \qquad \ell_2^z = (1 - a_2^z)\frac{2\ell_3^z}{\kappa_3^z} + a_2^z \frac{\underline{\beta}_1 \ell_3^z}{8\,\varepsilon_2^z \kappa_3^z}$$

where

$$\varepsilon_1^z \in \left(0, \frac{1}{8}\right), \quad \varepsilon_2^z \in \left(0, \frac{\underline{\beta}_1}{24}\right), \quad a_1^z \in (0,1), \quad a_2^z \in (0,1)$$

Then, for any $\kappa_3^z > 0$ and any $\ell_3^z \in (0, \bar{v}_z]$ the state of system (23) satisfies an a-$\mathcal{L}_\infty$ bound from the set $\mathcal{X}_z = \mathbb{R}^3$ with restrictions $\Delta_z = \ell_3^z/2$, $\Delta_\delta = \ell_2^z/3$ on the disturbance inputs d_z and δ_*, respectively. Furthermore, for any *constant* δ_* satisfying $|\delta_*| < \Delta_\delta$, system (23) has a GAS and locally LES equilibrium at

$$z_1^* = \frac{\delta_*}{\kappa_1^z \kappa_2^z \kappa_3^z}, \quad z_2^* = \frac{\delta_*}{\kappa_2^z \kappa_3^z}, \quad z_3^* = \frac{\delta_*}{\kappa_3^z}$$

whenever $d_z = 0$. For arbitrary locally essentially bounded signals d_z satisfying $\|d_z\|_{\text{a}} < \Delta_z$, the state of (23) in the *error coordinates*

$$\tilde{z}_1 := z_1 - z_1^*, \quad \tilde{z}_2 := z_2 - z_2^*, \quad \tilde{z}_3 := z_3 - z_3^*$$

satisfies the asymptotic bounds

$$\|\tilde{z}_1\|_{\text{a}} \le \frac{4}{\kappa_1^z \kappa_2^z \kappa_3^z}\|d_z\|_{\text{a}}, \qquad \|\tilde{z}_2\|_{\text{a}} \le \frac{4}{\kappa_2^z \kappa_3^z}\|d_z\|_{\text{a}}$$
$$\|\tilde{z}_3\|_{\text{a}} \le \frac{2}{\kappa_3^z}\|d_z\|_{\text{a}} \tag{24}$$

Proof. See Marconi and Isidori (2000); Serrani (2011). □

3.2 Control of the Pitch Dynamics

Regulation of (v_x, v_z) to the virtual control $(v^x_{\text{cmd}}, v^z_{\text{cmd}})$ is accomplished next by controlling the pitch dynamics. Select

$$\delta = \left((v^x_{\text{cmd}})^2 + (1 - v^y_{\text{cmd}})^2\right)^{\frac{1}{4}} - 1 \tag{25}$$

and define

$$\theta_{\text{cmd}} := \arctan\left(\frac{v^x_{\text{cmd}}}{1 - v^z_{\text{cmd}}}\right) \tag{26}$$

to obtain

$$v^x_{\text{cmd}} = (1+\delta)^2 \sin(\theta_{\text{cmd}})$$
$$v^z_{\text{cmd}} = 1 - (1+\delta)^2 \cos(\theta_{\text{cmd}}) \tag{27}$$

Apply the change of variables $(\theta, \dot{\theta}) \mapsto (\theta_1, \theta_2)$ defined as

$$\theta_1 := \theta - \theta_{\text{cmd}}\,, \quad \theta_2 := \dot{\theta} + \ell_1^\theta \operatorname{sat}\left(\frac{\kappa_1^\theta}{\ell_1^\theta}\theta_1\right)$$

where $\ell_1^\theta > 0$ and $\kappa_1^\theta > 0$ are design parameters, to write the pitch dynamics in the form

$$\dot{\theta}_1 = -\ell_1^\theta \operatorname{sat}\left(\frac{\kappa_1^\theta}{\ell_1^\theta}\theta_1\right) + \theta_2 + \dot{\theta}_{\text{cmd}}$$
$$\dot{\theta}_2 = \mu_2(1+\delta)^2 \tau(b) + \kappa_1^\theta \operatorname{sat}'\left(\frac{\kappa_1^\theta}{\ell_1^\theta}\theta_1\right)\dot{\theta}_1 \tag{28}$$

where $\tau(b) := \beta_3(b) + d_\theta(b)$. This last term is viewed as a nonlinear uncertain control function. Owing to the fact that $d_\theta(b) = \mathcal{O}(|b|^2)$, whereas $\beta_3(b) = \mathcal{O}(|b|)$, the following property is established by a suitable upper bound for b:

Property 3. There exist $\bar{b} \in (0,1)$ and $\underline{\tau} > 0$ such that

$$\tau\,(b\,\text{sign}(b)) = \tau\,(|b|)\,\text{sign}(b)\,, \quad |\tau(b)| \ge \underline{\tau}\,|b| \tag{29}$$

for all $|b| \le \bar{b}$ and all $\boldsymbol{p} \in \mathcal{P}$.

The above property allows the use of the control

$$b = -\ell_2^\theta \operatorname{sat}\left(\frac{\kappa_2^\theta}{\ell_2^\theta}\theta_2\right) \tag{30}$$

to achieve robust stabilization of system (28).

Proposition 4. Let the gain κ_1^θ, ℓ_1^θ be parameterized as

$$\kappa_1^\theta = \varepsilon_1^\theta \kappa_2^\theta\,, \qquad \ell_1^\theta = (1-a_1^\theta)\frac{2\ell_2^\theta}{\kappa_2^\theta} + a_1^\theta\,\frac{\underline{\mu}_2(1-\bar{\delta})^2\underline{\tau}\,\ell_2^\theta}{4\,\varepsilon_1^\theta\kappa_2^\theta}$$

where $\varepsilon_1^\theta \in \left(0, \frac{\underline{\mu}_2(1-\bar{\delta})^2\underline{\tau}}{8}\right)$, $a_1^\theta \in (0,1)$ Then, for any $\kappa_2^\theta > 0$ and any $\ell_2^\theta \in (0, \bar{b}]$, system (28) is input-to-state stable with restriction $\Delta_{\dot{\theta}_{\rm cmd}} = \ell_1^\theta/2$ on $\dot{\theta}_{\rm cmd}$, and its state satisfies the asymptotic bounds

$$\|\theta_1\|_{\rm a} \leq \frac{2}{\kappa_1^\theta}\|\dot{\theta}_{\rm cmd}\|_{\rm a},\ \|\theta_2\|_{\rm a} \leq \frac{8\kappa_1^\theta}{\underline{\mu}_2(1-\bar{\delta})^2\underline{\tau}\kappa_2^\theta}\|\dot{\theta}_{\rm cmd}\|_{\rm a} \quad (31)$$

Proof. The proof follows from Property 3 and the arguments in Marconi and Isidori (2000); Serrani (2011). □

4. STABILITY ANALYSIS

The stability analysis of the overall interconnected system will be carried out using the small-gain theorem, and in particular the asymptotic formulation for systems with saturated feedback due to Teel (1996). The closed-loop system is comprised of interconnected locally-input-to-state stable subsystems, where the relevant coupling inputs are highlighted. Since the bounds (22), (24) and (31) are only valid when the corresponding inputs satisfy appropriate restrictions, the first goal of the stability analysis is to show that such restrictions are met in finite time for the disturbance signals $(d_x, d_z, \dot{\theta}_{\rm cmd})$ via an appropriate selection of the tunable gains. Then, the analysis shall proceed by showing that a small-gain interconnection involving the asymptotic bounds can be attained, which leads to global asymptotic stability by virtue of the small-gain theorem of Teel (1996). Local exponential stability follows from the properties of the linearized system.

Before proceeding with the stability analysis, we establish important results that will be instrumental in the sequel. The proofs are omitted due to space limitation.

Proposition 5. The time derivatives of the wing bias, $\dot{b}$, and the pitch angle command, $\dot{\theta}_{\rm cmd}$, are globally bounded, with bounds that scale with the gains κ_3^x, κ_3^z and κ_2^θ.

From the bound

$$|\dot{b}| \leq \kappa_2^\theta M_\theta\,, \qquad |\dot{\theta}_{\rm cmd}| \leq \kappa_3^x M_x + \kappa_3^z M_z \qquad (32)$$

it follows that the restriction $\|\dot{\theta}_{\rm cmd}\|_a < \ell_1^\theta/2$ can be enforced by a suitable choice of κ_3^x and κ_3^z, once ℓ_1^θ has been selected. To ensure that ℓ_1^θ does not depend on the gain κ_2^θ, which is to be selected next, we scale the saturation level ℓ_2^θ as $\ell_2^\theta := \sigma_2^\theta\kappa_2^\theta$ where $\sigma_2^\theta > 0$ is a *fixed number*. Using

$$|d_x| \leq (1+\bar{\delta})^2|\theta_1| + (1+\bar{\delta})^2\bar{\gamma}_2\,|b|^2$$
$$|d_z| \leq (1+\bar{\delta})^2|\theta_1| + (1+\bar{\delta})^2\bar{\gamma}_2\,|b|^2 + \bar{\nu}_2\,|b|^2$$

and the (now valid) asymptotic bound

$$\|\theta_1\|_a \leq \frac{2}{\kappa_1^\theta}\|\dot{\theta}_{\rm cmd}\|_a \leq \frac{2}{\epsilon_1^\theta\kappa_2^\theta}(M_x+M_z)\max\{\kappa_3^x,\kappa_3^z\}$$

it is possible to prove that the restrictions $\|d_x\|_a < \ell_3^x/2$ and $\|d_z\|_a < \ell_3^z/3$ are attained by selecting k_2^θ sufficiently small first, and then adjusting the values of κ_3^x and κ_3^z accordingly. Next, it will be shown that a small-gain interconnection can be established using the asymptotic bounds. First, notice that

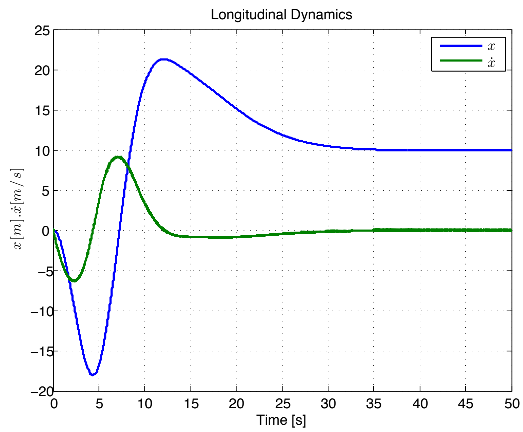

Fig. 1. Motion in the forward direction, $(x(t), \dot{x}(t))$.

$$\begin{aligned}\|\dot{\theta}_{\rm cmd}\|_a &\leq B_x\|\dot{v}^x_{\rm cmd}\|_a + B_z\|\dot{v}^z_{\rm cmd}\|_a \\ &\leq \kappa_3^x B_x Q_x\|d_x\|_a + \kappa_3^z B_z Q_z\|d_z\|_a \end{aligned} \qquad (33)$$

where $Q_x > 0$ and $Q_z > 0$ are constants that depend only on $\bar{\mu}_1$ and the scaling parameters ε_i^x and ε_i^z, $i = 1, 2$. Since

$$\|d_x\|_a \leq (1+\bar{\delta})^2\|\theta_1\|_a + (1+\bar{\delta})^2\bar{\gamma}_1\|b\|_a$$
$$\|d_z\|_a \leq (1+\bar{\delta})^2\|\theta_1\|_a + \left[(1+\bar{\delta})^2\bar{\gamma}_1 + \bar{\nu}_1\right]\|b\|_a \qquad (34)$$

it follows that

$$\|\dot{\theta}_{\rm cmd}\|_a \leq 2\max\{\kappa_3^x,\kappa_3^z\}\max\{Q_\theta\|\theta_1\|_a, Q_b\|b\|_a\} \qquad (35)$$

for some positive constants Q_θ and Q_b. Since

$$\|\theta_1\|_{\rm a} \leq \frac{2}{\kappa_1^\theta}\|\dot{\theta}_{\rm cmd}\|_{\rm a}$$

$$\|b\|_a \leq \kappa_2^\theta\|\theta_2\|_a \leq \frac{8\kappa_1^\theta}{\underline{\mu}_2(1-\bar{\delta})^2\underline{\tau}}\|\dot{\theta}_{\rm cmd}\|_{\rm a} =: 2\kappa_1^\theta\bar{\mu}_b\|\dot{\theta}_{\rm cmd}\|_{\rm a}$$

From the above bounds, it is readily seen that, once κ_2^θ has been determined, hence $\kappa_1^\theta = \varepsilon_1^\theta\kappa_2^\theta$ has been fixed, the composition of the gains forms a simple contraction if κ_3^x and κ_3^z satisfy the additional requirement

$$\max\{\kappa_3^x,\kappa_3^z\} < \frac{1}{4}\min\left\{\frac{\kappa_1^\theta}{Q_\theta}, \frac{1}{\kappa_1^\theta\bar{\mu}_b Q_b}\right\}$$

Finally, using the fact that $\dot{b}$ is globally bounded, as established in Proposition 5, it is possible to enforce a small-gain interconnection for the perturbed averaged systems by selecting $\varepsilon = 1/\omega_0$ sufficiently small through an appropriate choice of the angle-of-attack, α. Details are omitted for reasons of space limitations.

5. SIMULATION RESULTS

A simulation study has been conducted using a vehicle model adapted from Serrani (2011). Closed-loop simulations have been performed using the proposed controller applied on the time-accurate model (2). A uniformly-distributed 15% uncertainty with respect to the nominal values has been adopted in the coefficient the simulation model, yielding a mismatch between the carrier and the hovering frequencies for the wingbeat. Furthermore, the value of p_x has been set to 15% of the corresponding value for p_z, hence departing from the assumption $p_x = 0$. To facilitate comparison with previous results reported in Serrani (2011), the vehicle is commanded to perform the same maneuver, consisting of a simultaneous step change

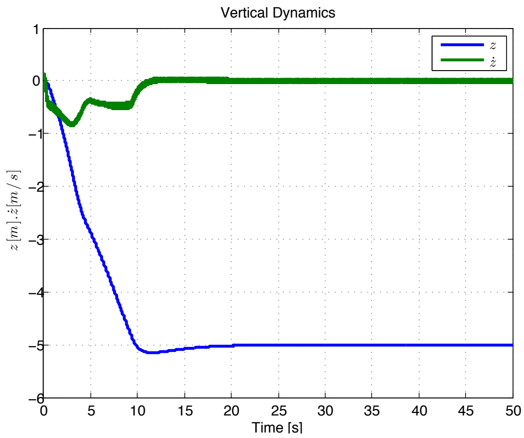

Fig. 2. Motion in the vertical direction, $(z(t), \dot{z}(t))$.

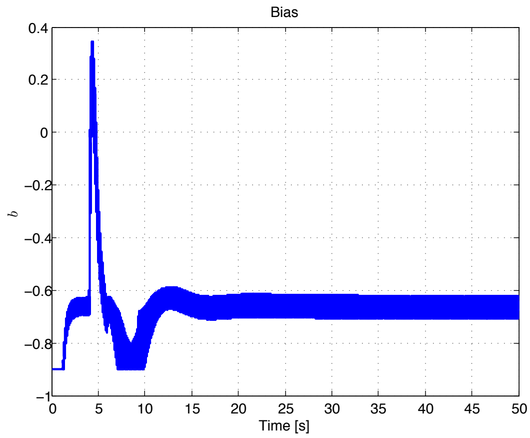

Fig. 3. Wing-bias control input, $b(t)$

in both the x and z coordinates from an initial configuration $(x_0, \dot{x}_0) = (0, 0)$, $(z_0, \dot{z}_0) = (0, 0)$, $(\theta_0, \dot{\theta}_0) = (\pi/6, 0)$ to the desired set point $x_d = 10$ [m], $z_d = -5$ [m]. The setpoint tracking performance for the forward and vertical motion is shown in Figure 1 and Figure 2, respectively. The effect of the bounded control input is clearly visible in the transient behavior for the vertical dynamics, which converges to the setpoint in less than 20 [s]. Since, as seen clearly in Figure (3), the averaged wing-bias input $b(t)$ does not vanish, the angle $\gamma(b(t))$ in the averaged model (14) converges to a nonzero value. It is worth noting that the uncertainty on the parameter p_x is effectively compensated by the controller. The simulation results confirm that the proposed methodology is capable of handling a fair amount of model uncertainty, including parametric uncertainty, uncertainty arising from the mismatch between the averaged and the time-accurate model, as well as dynamic uncertainty on the averaged model used for control design (that is, the mismatch between the *unperturbed* and the *perturbed* averaged models.

6. CONCLUSIONS

We have proposed in this paper a methodology for the design of a robust controller to globally stabilize the averaged longitudinal model of a flapping-wing MAV to a constant desired configuration. The novelty of the method, which constitutes a sizable improvement over our previous results, is that – in principle – a single actuator is needed to control both hovering and pitching motion, as the control action involves shaping the parameters of the wingbeat, without resorting to additional control effectors (such as moving appendages, independently controlled angle-of-attack of the wing or stroke-plane angle). In addition, the proposed wingbeat function, being continuously differentiable with respect to the control parameters, remove the mathematical difficulties that hinder a theoretically sound application of averaging methods to most – if not all – the approaches proposed in the literature for control of flapping-wing MAVs. Current work is addressing the extension of the proposed methodology to a 6-DOF model.

REFERENCES

Berman, G.J. and Wang, Z.J. (2007). Energy-minimizing kinematics in hovering insect flight. *Journal of Fluid Mechanics*, 582(2007), 153.

Bolender, M. (2009). Rigid multi-body equations-of-motion for flapping wing MAVs using Kane's equations. In *Proceedings of the AIAA Guidance, Navigation and Control Conference*. Chicago, IL.

Doman, D., Oppenheimer, M., and Sigthorsson, D. (2010). Wingbeat shape modulation for flapping-wing micro-air-vehicle control during hover. *Journal of Guidance, Control, and Dynamics*, 33(3), 724–739.

Dyhr, J.P., Morgansen, K.A., Daniel, T.L., and Cowan, N.J. (2013). Flexible strategies for flight control: An active role for the abdomen. *Journal of Experimental Biology*, 216(9), 1523–1536.

Hedrick, T.L. and Daniel, T.L. (2006). Flight control in the hawkmoth manduca sexta: The inverse problem of hovering. *The Journal of Experimental Biology*, 209, 3114–3130.

Humbert, J. and Faruque, I. (2011). Analysis of insect-inspired wingstroke kinematic perturbations for longitudinal control. *Journal of Guidance, Control, and Dynamics*, 34(2), 618–623.

Lee, J.S., Kim, J.K., and Han, J.H. (2014). Stroke plane control for longitudinal stabilization of hovering flapping wing air vehicles. *Journal of Guidance, Control, and Dynamics*, 38(4), 800–806.

Marchand, N. and Hably, A. (2005). Global stabilization of multiple integrators with bounded controls. *Automatica*, 41(12), 2147–2152.

Marconi, L. and Isidori, A. (2000). Robust global stabilization of a class of uncertain feedforward nonlinear systems. *Systems & Control Letters*, 41, 281–290.

Oppenheimer, M.W., Doman, D.B., and Sigthorsson, D.O. (2011). Dynamics and control of a biomimetic vehicle using biased wingbeat forcing functions. *Journal of Guidance, Control, and Dynamics*, 34(1), 204–217.

Serrani, A. (2011). Robust nonlinear control design for a minimally-actuated flapping-wing MAV in the longitudinal plane. In *Proceedings of the 50th IEEE Conference on Decision and Control*. Orlando, FL.

Teel, A. (1992). Global stabilization and restricted tracking for multiple integrators with bounded controls. *Systems & Control Letters*, 18(9), 165–171.

Teel, A. (1996). A nonlinear small gain theorem for the analysis of control systems with saturations. *IEEE Transactions on Automatic Control*, 41(9), 1256–1270.