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ON A FRACTIONAL SEMILINEAR NEUMANN PROBLEM ARISING IN CHEMOTAXIS

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Abstract. We study a semilinear and nonlocal Neumann problem, which is the fractional analogue of the problem considered by Lin–Ni–Takagi in the '80s. The model under consideration arises in the description of stationary configurations of the Keller–Segel model for chemotaxis, when a nonlocal diffusion for the concentration of the chemical is considered. In particular, we extend to any fractional power $s \in (0,1)$ of the Laplacian (with homogeneous Neumann boundary conditions) the results obtained in [23] for $s = 1/2$. We prove existence and some qualitative properties of non–constant solutions when the diffusion parameter ε is small enough, and on the other hand, we show that for ε large enough any solution must be necessarily constant.

1. Introduction

In this paper we consider the following fractional semilinear Neumann problem on a smooth and bounded domain $\Omega \subset \mathbb{R}^n$,

$$\begin{cases} (-\varepsilon\Delta_N)^s u + u = u^p & \text{on } \Omega, \\ \partial_\nu u = 0 & \text{in } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$, $s \in (0,1)$, $1 < p < \frac{n+2s}{n-2s}$, ∂_ν denote the outward normal derivative to $\partial\Omega$ and $(-\Delta_N)^s$ denotes the Spectral fractional Laplacian.

This problem can be seen as a possible nonlocal analogue of the one studied by Lin, Ni, and Takagi in a series of seminal papers in the eighties ([16, 19]). The case $s = 1/2$ was considered by Stinga and Volzone in [23], here we extend their results to any fractional power $s \in (0,1)$ of the Neumann Laplacian.

There are different possible generalization of the classical local problem to the non local setting, depending on which notion of *fractional Laplacian* and of *Neumann boundary condition* one considers. As already mention, here we focuse on the *spectral Neumann Laplacian*, as done in [23], since this is interesting in applications. At the end of this Introduction, we will comment more in detail on other possible notions, describing their differences and similarities.

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Thanks to the results in [21, 22], in order to study problem (1.1), we will study the associated local extended problem:

$$\begin{cases} \varepsilon\Delta_x U + \frac{1-2s}{y}U_y + U_{yy} = 0 & \text{in } \mathcal{C}, \\ \partial_\nu U = 0 & \text{on } \partial_L\mathcal{C}, \\ U(x,0) > 0 & \text{in } \Omega \\ -\lim_{y\to 0} y^{1-2s}U_y(x,y) + U(x,0) = U(x,0)^p & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $\mathcal{C} := \Omega \times (0,+\infty)$ and $\partial_L\mathcal{C} := \partial\Omega \times (0,+\infty)$. Notice that here, just for convenience, we have preferred to put the parameter ε as a coefficient in the equation, rather than on the weighted normal derivative on $\{y=0\}$, see Remark 2.3.

Let

$$a := 1-2s.$$

Problem (1.1) has a variational structure and the associated energy functional is given by

$$J_{\varepsilon,a}(U) := \frac{1}{2}\iint_{\mathcal{C}} \left(\varepsilon|\nabla_x U|^2 + |U_y|^2\right) y^a dxdy + \int_\Omega |U(x,0)|^2 dx - \frac{1}{p+1}\int_\Omega |U(x,0)|^{p+1}dx.$$

Our main results concern exitence and non–existence of non-constant solutions for problem (1.2), depending on the parameter ε.

More precisely, we have the following:

Theorem 1.1. *There exists $\overline{\varepsilon} > 0$ such that if $\varepsilon < \overline{\varepsilon}$, then problem (1.1) admits at least one non–constant positive weak solution $u^{\varepsilon,a}$. Moreover, if we denote by $U^{\varepsilon,a}$ its extension satisfying* (1.2)*, then we have the following energy estimate:*

$$J_{\varepsilon,a}(U^{\varepsilon,a}) \le C(n,a,p,\Omega)\varepsilon^{\frac{n}{2}}. \tag{1.3}$$

For the notion of weak solution, see Section 4.

In Section 6, we also prove some properties of such solutions, in particular we show that they tend to zero in measure and concentrates around some points as $\varepsilon \to 0$.

On the other hand, we show non–existence of non–constant solutions for sufficiently large ε, according to the following result

Theorem 1.2. *There exists $\varepsilon^* > 0$ such that if $\varepsilon > \varepsilon^*$ then the unique weak solution of problem (1.1) is $u \equiv 1$.*

To prove existence of a weak solution to (1.2) (and thus of (1.1)), we make a standard use of the Mountain Pass Theorem, then for showing that such solution cannot be constant if ε is small enough, the energy estimate (1.3) (proven by means of a suitable competitor) will be crucial.

On the other hand, the proof of Theorem 1.2 crucially relies on some a priori (uniform in ε) L^∞-estimates for weak solutions, which will be obtained by a blow-up technique à la Gidas-Spruck (see Section 5). More precisely, we have the following

Theorem 1.3. *There exists a positive constant $C > 0$, that does not depend on ε, such that, if u is a weak solution of* (1.1)*, then,*

$$\sup_{x\in\Omega} u(x) \le C.$$

With respect to the case $s = 1/2$ studied in [23], we emphasize that here the extended function U is not harmonic but satisfies a weighted equation that, for $\varepsilon = 1$, reads as:

$$\operatorname{div}(y^{1-2s}\nabla U) = 0.$$

Observe that the weight y^{1-2s} could be singular or degenerate (depending on whether s is above or below $1/2$), nevertheless it enjoys some good properties since it belongs to the Muckenhoupt class A_2 (see [10, Page 79]). In particular, this requires some attention when using some results that are classical tools when dealing with harmonic functions, for example Harnack inequality, Hopf's Lemma, regularity results via semigroup theory (see Section 3 for further comments on this last point).

We describe now the Keller-Segel model and how problem (1.1) arises in the study of Chemotaxis. Chemotaxis is the oriented movement of cells towards higher concentrations of some chemical substance in their environment. Keller and Segel proposed the following model to describe the chemotactic aggregation.

Let $w(x,t) > 0$ be the density of a bacteria population and $v(x,t) > 0$ the density of a certain chemoattractant, then we have

$$\begin{cases} \partial_t w = D_1\Delta w - \chi\operatorname{div}(w\nabla(\log v)) & \text{in } \Omega\times(0,+\infty) \\ \partial_t v = D_2\Delta v - aw + bv & \text{in } \Omega\times(0,+\infty) \\ \partial_\nu w = \partial_\nu v = 0 & \text{on } \partial\Omega. \end{cases}$$

In [16, 19], Lin, Ni, and Takagi studied the steady states of the above system, relying on the observation that it could be reduced to the study of a single equation. More precisely, we can rewrite the stationary equation

$$D_1\Delta w - \chi\operatorname{div}(w\nabla(\log v)) = 0$$

as

$$\operatorname{div}(D_1 w\nabla(\log w - \chi D_1^{-1}\log v)) = 0 \tag{1.4}$$

and use the 0–Neumann condition, to see that $w = \lambda v^{\chi/D_1}$ for some positive λ. Indeed, multiplying (1.4) by $\log w - \chi D_1^{-1}\log v$, integrating by parts, and using that $\partial_v w = \partial_\nu v = 0$ on $\partial\Omega$, one gets

$$\int_\Omega D_1 w|\nabla(\log w - \chi D_1^{-1}\log v)|^2\,dx = 0,$$

which implies that $\log w - \chi D_1^{-1}\log v$ must be constant.

Hence the stationary version of the Keller–Segel system reduces to a single equation for v, which can be written in the form

$$\begin{cases} -\varepsilon\Delta u + u = u^p & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $p = \chi/D_1$ and $\varepsilon = D_2/a$.

Possibly different nonlocal approaches to the study of Chemotaxis have been proposed recently in several papers.

In [9], Escudero proposed a model in which the chemical satisfies a classical local diffusion equation while the bacteria satisfy a nonlocal diffusion. In contrast, Stinga and Volzone [23], considered a model in which the diffusion is assumed to be local for the bacteria and nonlocal for the chemical. As already mentioned, in this work we use the same approach, generalizing their results to any fractional power $s \in (0,1)$. Another nonlocal model for chemotaxis was proposed by Huaroto and Neves in [14]. They considered a fractional variant of the Keller-Segel model, in which the velocity (that in the classical model is proportional to the gradient of the concentration of the chemoattractant ∇v), is instead defined through a linear fractional operator. This leads again to an equation for the chemical, in which the diffusion is governed by a fractional Neumann Laplacian.

To conclude this Introduction, we recall other possible notions of nonlocal Neumann Laplacian, describing their differences and similarities, with a particular emphasis on their probabilistic interpretation (for more details, see e.g. [1, Sections 2, 4], [8, Section 7] and references therein).

We start by recalling the probabilistic interpreptation of the *spectral Neumann Laplacian*, considered in this paper. Given a bounded domain $\Omega \subset \mathbb{R}^n$ and a point $x \in \Omega$, let $u(x,t)$ denote the number of particles located at the point x at time t. Let us write u as a superposition of eigenfunctions of the Laplacian with Neumann boundary condition, and let it evolve according to a classical random walk, but in such a way that spectral components relative to high frequencies move slower than the ones relative to low frequencies. More precisely, the time step τ_k is chosen to be larger if the frequency is higher, according to the following relation:

$$\tau_k = \lambda_k^{1-s} h_k^2,$$

where λ_k denotes the Neumann eigenvalue and h_k the space step. The fact of having homogeneous Neumann boundary conditions, means that the process get reflected when it reaches the boundary.

Another possible notion of nonlocal Neumann Laplacian, is the one introduced in [8], where the authors defined the following *nonlocal normal derivative*:

$$\mathcal{N}_s u(x) := c_{n,s} \int_\Omega \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy.$$

With this definition, the problem

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ \mathcal{N}_s u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \end{cases} \tag{1.5}$$

has a variational structure (here $(-\Delta)^s$ denotes the usual fractional Laplacian in $\mathbb{R}^n$). Moreover, as shown in [8, Section 5], $\mathcal{N}_s u$ recovers the classical normal derivative of u in the limit $s \to 1$. Another interesting feature of problem (1.5) is that it can be reformulated through a regional operator with a kernel having logarithmic behaviour at the boundary. Such notion has also a probabilistic interpretation, as described in Section 2 of [8].

More precisely, let us a consider a particle that moves randomly in $\mathbb{R}^n$ according to the following law: a particle situated at a point $x \in \mathbb{R}^n$, can jump at any other point $y \in \mathbb{R}^n$ with a probability that is proportional to $|x-y|^{-n-2s}$. It is well known that the probability density $u(x,t)$ that the particle is located at the point x at time t, solves the fractional heat equation $u_t + (-\Delta)^s u = 0$. If we consider now such a process in a bounded domain Ω, imposing homogeneous Neumann condition $\mathcal{N}_s u = 0$ corresponds to the fact that if a particle reaches a point $x \in \mathbb{R}^n \setminus \overline{\Omega}$, it may jump back at any point $y \in \Omega$ with a probability density proportional to $|x-y|^{-n-2s}$.

Another possibility is to consider the *regional fractional Laplacian*, which corresponds, from a probabilistic point of view, to a censored process: in this case the particle cannot exit the set Ω. As observed in [8], it is possible to define a natural notion of homogeneous Neumann boundary condition for this kind of operator which still gives a variational structure to the problem. However, one cannot consider nonhomogeneous conditions in this setting.

Finally, in [12] a further notion of fractional Neumann Laplacian was considered, in which the classical normal derivative is replaced by a "weighted" classical derivative of the form $\partial_\nu(u/d^{s-1})$, where d denotes the distance from the boundary of Ω.

The paper is organized as follows:

- In Section 2, we introduce the functional setting and the extension problem for the spectral Neumann fractional Laplacian;
- Section 3 deals with regularity results for weak solutions;
- In Section 4, we prove the existence result Theorem 1.1;
- In Section 5, we prove the uniform L^∞ estimate of Theorem 1.3 and, as a consequence, the non–existence result Theorem 1.2;
- Finally, Section 6 contains some properties of the non–constant solutions given by Theorem 1.1. In particular, we show that they concentrate at some points (see Theorem 6.6). A crucial ingredient will be an Harnack inequality contained in [4].

Notations: Let $\mathbb{R}^{n+1} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}\}$ and $\mathbb{R}^{n+1}_+ = \{(x,y) \in \mathbb{R}^{n+1} \,:\, y > 0\}$. We denote by $B(x_0, r)$ the ball of radius r centered at x_0 in $\mathbb{R}^n$, and we set $B_r := B(0,r)$. Moreover, $\widetilde{B}((x_0,y_0),r)$ denotes the ball of radius r centered at $(x_0,y_0) \in \mathbb{R}^{n+1}$ and we set $\widetilde{B}^+((x_0,y_0),r) = \widetilde{B}((x_0,y_0),r) \cap \mathbb{R}^{n+1}_+$. Finally, we will denote with $\widetilde{B}^+_r = \widetilde{B}^+((0,0),r)$.

2. Functional Setting

Let $\{\varphi_k\}_{k\in\mathbb{N}}$ be an Hilbert basis of $L^2(\Omega)$ made by eigenfunctions of $-\Delta$ with Neumann boundary condition and let $\{\lambda_k\}_{k\in\mathbb{N}}$ be the corresponding eigenvalues, i.e.:

$$\begin{cases} -\Delta \varphi_k = \lambda_k \varphi_k & \text{in } \Omega \\ \partial_\nu \varphi_k = 0 & \text{on } \partial\Omega. \end{cases}$$

It is known that $\lambda_0 = 0$, $\int_\Omega \varphi_k = 0$ and $\varphi_k \in C^\infty(\overline{\Omega})$ for any $k \in \mathbb{N}$. It is easy to see that $\|\varphi_k\|^2_{H^1(\Omega)} = 1 + \lambda_k$ and that $\{\varphi_k\}$ are also an orthogonal basis of $H^1(\Omega)$. Therefore we can characterize $H^1(\Omega)$ as

$$H^1(\Omega) = \big\{ u \in L^2(\Omega) \, : \, \|u\|^2_{H^1(\Omega)} = \sum_{k=0}^{+\infty} (1+\lambda_k)|u_k|^2 < +\infty \big\},$$

where $u_k := \langle u; \varphi_k \rangle_{L^2(\Omega)}$.

For any $u \in H^1(\Omega)$, we can define $-\Delta_N u$ (the Neumann Laplacian) as an element of the dual space $H^1(\Omega)'$ in the following way: for any function $v \in H^1(\Omega)$ the action of $-\Delta_N u$ is given by

$$\langle -\Delta_N u; v \rangle = \sum_{k=0}^{+\infty} \lambda_k u_k v_k.$$

In the same way, we can define the natural domain of the ε–fractional Neumann Laplacian $(-\varepsilon\Delta_N)^s$ as the space

$$\mathcal{H}^s_\varepsilon(\Omega) := \big\{ u \in L^2(\Omega) : \sum_{k=1}^{\infty} (\varepsilon\lambda_k)^s \, |u_k|^2 < \infty \big\}.$$

Notice that $\mathcal{H}^s_\varepsilon(\Omega)$ endowed with the scalar product

$$\langle u; v \rangle_{\mathcal{H}^s_\varepsilon} := \langle u; v \rangle_{L^2(\Omega)} + \sum_{k=1}^{\infty} (\varepsilon\lambda_k)^s \, u_k v_k,$$

and the associated norm

$$\|u\|_{\mathcal{H}^s_\varepsilon} := \|u\|_{L^2(\Omega)} + \left(\sum_{k=1}^{\infty} (\varepsilon\lambda_k)^s \, |u_k|^2 \right)^{\frac{1}{2}}$$

is a Hilbert space.

As before, we see $(-\varepsilon\Delta_N)^s$ as an element of $\mathcal{H}^s_\varepsilon(\Omega)'$ whose action is given by

$$\langle (-\varepsilon\Delta_N)^s u; v \rangle = \sum_{k=0}^{+\infty} (\varepsilon\lambda_k)^s u_k v_k \quad \text{for any } v \in \mathcal{H}^s_\varepsilon(\Omega). \tag{2.1}$$

Remark 2.1. *Let $H^s(\Omega)$ be the usual fractional Sobolev space, defined as the closure of $C^\infty(\Omega)$ with respect to the norm*

$$\|u\|_{H^s(\Omega)} := \|u\|_{L^2(\Omega)} + [u]_{H^s(\Omega)} = \|u\|_{L^2(\Omega)} + \left(\iint_{\Omega\times\Omega} \frac{|u(x)-u(\bar{x})|^2}{|x-\bar{x}|^{n+2s}} \, dx \, d\bar{x} \right)^{\frac{1}{2}}.$$

It was shown in [5, Lemma 7.1] *that*[1]

$$\mathcal{H}^{\varepsilon,s}(\Omega) = H^s(\Omega) \tag{2.2}$$

and

$$\| \, (-\varepsilon\Delta_N)^{\frac{s}{2}} \, u\|^2_{L^2(\Omega)} \sim \varepsilon^s [u]^2_{H^s(\Omega)}. \tag{2.3}$$

[1] In [5, Lemma 7.1] they require the functions to have 0 average, however this is not needed in the proof.

As explained in the Introduction, we will consider an extension problem in the half–space $\mathbb{R}^{n+1}_+ := \{(x,y) \in \mathbb{R}^{n+1} \,|\, x \in \mathbb{R}^n,\ y > 0\}$, associated to (1.1). To this aim, let us introduce a suitable weighted Sobolev space. Let $\mathcal{C} := \Omega \times \mathbb{R}_+$ and let $a = 1-2s$; we define

$$H^1(\mathcal{C}, y^a) := \left\{ V \in L^1_{\mathrm{loc}}(\mathcal{C}) \,|\, y^a(|V|^2 + |\nabla V|^2) \in L^1(\mathcal{C}) \right\}.$$

In [21, 22] it was shown that is possible to realize the spectral fractional Neumann Laplacian as a Dirichlet to Neumann map, as the following Theorem states:

Theorem 2.2 ([21, 22]). *For every $u \in \mathcal{H}^s_\varepsilon(\Omega)$ with $\int_\Omega u(x) = 0$ there exists a unique $U \in H^1(\mathcal{C}, y^a)$ such that $\int_\Omega U(x,y)dx = 0$ for all $y > 0$ and*

$$\begin{cases} \varepsilon \Delta_x U + \dfrac{a}{y} U_y + U_{yy} = 0 & \text{in } \mathcal{C} \\ U(x,0) = u(x) & \text{in } \Omega \\ \partial_\nu U = 0 & \text{on } \partial_L \mathcal{C}. \end{cases} \tag{2.4}$$

The function U is given by

$$U(x,y) = \sum_{k=1}^{\infty} \rho(\varepsilon^{1/2} \lambda_k^{1/2} y) u_k \varphi_k(x) \tag{2.5}$$

where $\rho(t)$ is the unique solution of

$$\begin{cases} \rho''(t) + \frac{1-2s}{t} \rho'(t) = \rho(t) & t > 0 \\ -\lim\limits_{t \to 0^+} t^{1-2s} \rho'(t) = c_s \\ \rho(0) = 1, \end{cases}$$

Here c_s is a positive constant depending on s.

Moreover, we have

$$c_s(-\varepsilon \Delta_N)^s u(x) = -\lim_{y \to 0+} y^a U_y(x,y). \tag{2.6}$$

Finally, U is the unique minimizer of the functional

$$\mathcal{F}(U) = \frac{1}{2} \iint_{\mathcal{C}} \left(\varepsilon \left| \nabla_x U \right|^2 + |U_y|^2 \right) y^a dxdy \tag{2.7}$$

among all functions of $H^1(\mathcal{C}, y^a)$ whose trace over Ω is u.

Remark 2.3.
- *The explicit expression of the solution U_1 to problem (2.4) for $\varepsilon = 1$, can be found, for example, in* [3, Lemma 3.4]. *By a simple scaling argument, one can see that $U(x,y) := U_1(x, \varepsilon^{1/2} y)$ solves (2.4).*
- *The previous result states that it is possible to realize the spectral fractional Laplacian as a Dirichlet to Neumann operator provided $u_\Omega := \frac{1}{|\Omega|} \int_\Omega u = 0$. To give a suitable definition of extension also for functions that do not have zero average, it is enough to consider the extension $\widetilde{U}$ of $u - u_\Omega$ and define $U :\, = \widetilde{U} + u_\Omega$. Clearly both U and $\widetilde{U}$ solve the same equation with Neumann boundary condition on $\partial_L \mathcal{C}$, and we have*

$$c_s(-\varepsilon \Delta_N)^s u = c_s(-\varepsilon \Delta_N)^s \widetilde{u} = -\lim_{y \to 0+} y^a \widetilde{U}_y(x,y) = -\lim_{y \to 0+} y^a U_y(x,y).$$

- *Observe that, for the sake of readibility, in our problem* (1.1)*, we are considering* $c_s = 1$*, which is not restrictive up to renaming* ε*.*

Let us define now the space $\mathrm{H}^\varepsilon(\mathcal{C}, y^a)$ as the completion of $H^1(\mathcal{C}, y^a)$ with respect to the norm

$$\|U\|^2_{\varepsilon,a} := \iint_{\mathcal{C}} \left(\varepsilon|\nabla_x U|^2 + |U_y|^2\right) y^a dxdy + \int_\Omega |U(x,0)|^2 dx.$$

Notice that $\mathrm{H}^\varepsilon(\mathcal{C}, y^a)$, endowed with the scalar product

$$(v,w)_{\varepsilon,a} = \iint_{\mathcal{C}} \left(\varepsilon \nabla_x U \cdot \nabla_x V + U_y V_y\right) y^a dxdy + \int_\Omega U(x,0)V(x,0)dx,$$

is a Hilbert space. Clearly, the extension U defined in the previous observation belongs to this space.

We observe that constant functions belong to $\mathrm{H}^\varepsilon(\mathcal{C}, y^a)$ but not to $H^1(\mathcal{C}, y^a)$, hence the inclusion

$$H^1(\mathcal{C}, y^a) \subset \mathrm{H}^\varepsilon(\mathcal{C}, y^a)$$

is strict. This can be seen easily in the following way: without loss of generality, let $V \equiv 1$, and let, for every $n \in \mathbb{N}$,

$$V_n(x,y) = \max\left\{0, 1 - \frac{y}{n}\right\} \quad \text{for } (x,y) \in \mathcal{C}.$$

Then, obviously $V_n \in H^1(\mathcal{C}, y^a)$, $V_n(\cdot,0) = V(\cdot,0)$ on Ω, moreover

$$\iint_{\mathcal{C}} \left(\varepsilon|\nabla_x V_n|^2 + |\partial_y V_n|^2\right) y^a \, dx\, dy = \frac{|\Omega|}{n^2} \int_0^n y^{1-2s}\, dy = \frac{|\Omega|}{2-2s} \frac{1}{n^{2s}}.$$

Hence,

$$\|V_n - V\|_{\varepsilon,a} \longrightarrow 0 \quad \text{as } n \to +\infty.$$

We recall that, by a result of Nekvinda [18] (see also [17]), we know that there exists a linear and bounded trace operator

$$T^a : H^1(\mathcal{C}, y^a) \to H^s(\Omega).$$

By Remark 2.1, we have that

$$T^a : H^1(\mathcal{C}, y^a) \to \mathcal{H}^{\varepsilon,s}(\Omega). \tag{2.8}$$

For simplicity of notation, in the following we will denote by $V(\cdot,0)$, the trace over Ω of a function $V \in H^1(\mathcal{C}, y^a)$.

In the following Proposition, we extend the trace operator defined on $H^1(\mathcal{C}, y^a)$ to the larger space $\mathrm{H}^\varepsilon(\mathcal{C}, y^a)$.

Proposition 2.4. *For all* $V \in H^1(\mathcal{C}, y^a)$*, there exists a constant* C_s *independent on* ε *such that*

$$\begin{aligned}\left\|(-\varepsilon\Delta_N)^{\frac{s}{2}}\, V(\cdot,0)\right\|^2_{L^2(\Omega)} &= \sum_{k=1}^{\infty} (\varepsilon\lambda_k)^s \left|\langle V(\cdot,0); \varphi_k\rangle_{L^2(\Omega)}\right|^2 \\ &\le C_s \iint_{\mathcal{C}} \left(\varepsilon\, |\nabla_x V|^2 + |V_y|^2\right) y^a dxdy,\end{aligned}$$

where the equality holds if and only if V solves problem (2.4).

Moreover, there exists a unique bounded linear operator $T^{\varepsilon,a} : \mathrm{H}^\varepsilon(\mathcal{C}, y^a) \to \mathcal{H}^{\varepsilon,s}(\Omega)$, whose operator norm is proportional to $\varepsilon^{\frac{s}{2}}$, such that $T^{\varepsilon,a}V(\cdot) = V(\cdot,0)$ for any $V \in H^1(\mathcal{C}, y^a)$.

Proof. The proof is an adaptation of [23, Lemma 2.4]. Let $u(\cdot) = V(\cdot,0)$ be the trace of V over Ω. Without loss of generality, we may assume $\int_\Omega u = 0$. Let U be the solution of problem (2.4) with $U(\cdot,0) = u(\cdot)$ over Ω. Since U is a minimizer of the functional (2.7) among all functions having the same trace over Ω, we have

$$\iint_{\mathcal{C}} \left(\varepsilon|\nabla_x U|^2 + |U_y|^2\right) y^a dxdy \leq \iint_{\mathcal{C}} \left(\varepsilon|\nabla_x V|^2 + |V_y|^2\right) y^a dxdy.$$

Using the explicit formula (2.5) for U and recalling that each φ_k has zero average we find

$$\begin{aligned}&\iint_{\mathcal{C}} \left(\varepsilon|\nabla_x U|^2 + |U_y|^2\right) y^a dxdy \\ &\quad = \int_0^{+\infty} \left(\sum_{k=1}^{+\infty} \rho^2\left((\varepsilon\lambda_k)^{1/2} y\right) u_k^2 \varepsilon\lambda_k + \sum_{k=1}^{+\infty} \dot{\rho}^2\left((\varepsilon\lambda_k)^{1/2} y\right) u_k^2 \varepsilon\lambda_k \right) y^a dy.\end{aligned}$$

Performing that change of variable $(\varepsilon\lambda_k)^{1/2} y = t$ and recalling that $a = 1-2s$ we find

$$\begin{aligned}\iint_{\mathcal{C}} \left(\varepsilon|\nabla_x U|^2 + |U_y|^2\right) y^a dxdy &= \left(\int_0^{+\infty} \left(\rho^2(t) + \dot{\rho}^2(t)\right) t^{1-2s} dt\right) \cdot \left(\sum_{k=1}^{+\infty} (\varepsilon\lambda_k)^s u_k^2\right) \\ &= C_s \left\|(-\varepsilon\Delta)^{\frac{s}{2}} u\right\|^2_{L^2(\Omega)},\end{aligned}$$

where

$$C_s = \int_0^{+\infty} \left(\rho^2(t) + \dot{\rho}^2(t)\right) t^{1-2s} dt.$$

Hence the inequality is established.

Finally, the trace operator $T^{\varepsilon,a}$ in (2.8) can be extended by density from $H^1(\mathcal{C}, y^a)$ to $\mathrm{H}^\varepsilon(\mathcal{C}, y^a)$. ∎

Remark 2.5. *Using Proposition 2.4, the fact that $\mathcal{H}^{\varepsilon,s}(\Omega) = H^s(\Omega)$ (see (2.2)-(2.3)) and the fractional Sobolev embedding, we have for any $V \in \mathcal{H}^{\varepsilon,s}(\Omega)$*

$$\|V(\cdot,0)\|_{L^{\frac{2n}{n-2s}}(\Omega)} \leq C \|V(\cdot,0)\|_{H^s(\Omega)} \leq \frac{C}{\varepsilon^{s/2}} \|V\|_{\varepsilon,a}. \tag{2.9}$$

As a consequence, the operator $T^{\varepsilon,a}$ is compact from $\mathrm{H}^\varepsilon(\mathcal{C}, y^a)$ to $L^q(\Omega)$ for every $1 \leq q < \frac{2n}{n-2s}$.

3. Regularity of solutions

In this section we want to study the regularity of solutions to

$$(-\varepsilon\Delta_N)^s u + u = f \quad \text{ in } H^{-s}(\Omega). \tag{3.1}$$

First of all, we notice that the unique solution of (3.1) is given by

$$u(x) = \sum_{k=1}^{+\infty} \frac{1}{1+(\varepsilon\lambda_k)^s} f_k \varphi_k(x) \tag{3.2}$$

where $f_k = \langle f; \varphi_k \rangle$. Now for every $u \in L^2(\Omega)$ we define the fractional heat semigroup as

$$e^{-t(-\varepsilon\Delta_N)^s} u(x) = \sum_{k=1}^{\infty} e^{-t(\varepsilon\lambda_k)^s t} u_k \varphi_k(x). \tag{3.3}$$

Thanks to this definition, we can rewrite (3.2) as

$$u(x) = \int_0^{+\infty} e^{-t-t(-\varepsilon\Delta_N)^s} f(x) dt. \tag{3.4}$$

Let $s \in (0,1)$ and $u \in \mathcal{H}^{s,\varepsilon}(\Omega)$ be a function with zero average, we define the function v_s as

$$v_s(x,t) := e^{-t(-\varepsilon\Delta_N)^s} u(x).$$

If $s = 1/2$, we can observe that $v := v_{1/2}$ is the unique solution of

$$\begin{cases} \varepsilon\Delta_x v + v_{tt} = 0, & \text{in } \mathcal{C} \\ \partial_\nu v = 0, & \text{on } \partial_L \mathcal{C} \\ v(x,0) = u(x), & \text{on } \Omega, \end{cases}$$

see for instance [23, Theorem 2.1]. Hence, one can write $v_{1/2}$ as a convolution between u and a kernel $P_t(x)$ that behaves like

$$P_t(x) \sim \frac{t}{(t^2+|x|^2)^{\frac{n+1}{2}}} \quad \text{if } x \in \Omega \text{ and } 0 < t < 1,$$

and is uniformly bounded in x if $t > 1$. Thanks to this information and (3.4) it is possible to obtain an integral representation of u in terms of f and $P_{\sqrt{\varepsilon}t}(x)$. Such representation was used in [23, Theorem 3.5] to obtain regularity results for u. If $s \neq \frac{1}{2}$ it is not clear if v_s solves an elliptic equation in $\mathcal{C}$ and therefore if an integral representation of u in terms of f and a suitable kernel even exists. Hence, in order to obtain some regularity results, we decided to follow an approach based only on semigroup theory.

It is well known (see for example [7, Theorem 1.3.2. and Theorem 1.3.3.]), that the fractional Neumann heat semigroup is contractive from $L^p(\Omega)$ to $L^p(\Omega)$ for every $p \in [1,\infty]$, namely

$$\left\| e^{-t(-\varepsilon\Delta_N)^s} u \right\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)}. \tag{3.5}$$

Moreover, since the first eigenvalue of the Neumann Laplacian vanishes, we have that the following $L^\infty - L^1$ estimate holds true (see e.g. [6, Theorem 2.10 (b), Remark 2.11] and references therein):

$$\left\| e^{-t(-\varepsilon\Delta_N)^s} u \right\|_{L^\infty(\Omega)} \leq C_\varepsilon e^t t^{-\frac{n}{2s}} \|u\|_{L^1(\Omega)} \quad \text{for all } t > 0, \tag{3.6}$$

here C_ε is a constant that depends on s, n, Ω and ε. In order to obtain an $L^q - L^p$ estimate we need to recall the statement of the Riesz–Thorin interpolation Theorem, for a proof see for instance [20, Theorem 2.1].

Theorem 3.1. *Let (X, μ) and (Y, ν) be a pair of $\sigma-$finite measure space, and let $1 \leq p_0, q_0, p_1, q_1 \leq +\infty$. If*

$$T : L^{p_0}(X, \mu) + L^{p_1}(X, \mu) \mapsto L^{q_0}(Y, \nu) + L^{q_1}(Y, \nu)$$

is a bounded and linear operator, i.e. there exist constants M_0 and M_1 such that

$$\begin{cases} \|T(f)\|_{L^{q_0}(Y)} \leq M_0 \|f\|_{L^{p_0}(X)} \\ \|T(f)\|_{L^{q_1}(Y)} \leq M_1 \|f\|_{L^{p_1}(X)}, \end{cases}$$

then for every p, q, and $t \in [0, 1]$, that fulfill the relations

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1} \qquad \text{and} \qquad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1},$$

there exists a constant M such that

$$\|T(f)\|_{L^q(Y)} \leq M \|f\|_{L^p(X)}.$$

Moreover the constant M satisfies $M \leq M_0^{1-t} M_1^t$.

We can now show the following

Proposition 3.2. *For every $p \in [1, +\infty]$ and $q \in [p, +\infty]$, it holds*

$$\left\| e^{-t(-\varepsilon\Delta_N)^s} u \right\|_{L^q(\Omega)} \leq C_\varepsilon^{\left(\frac{1}{p}-\frac{1}{q}\right)} e^{t\left(\frac{1}{p}-\frac{1}{q}\right)} t^{-\frac{n}{2s}\left(\frac{1}{p}-\frac{1}{q}\right)} \|u\|_{L^p(\Omega)} \tag{3.7}$$

where C_ε is the same constant of (3.6) and we adopt the convention that $0 = \frac{1}{\infty}$.

Proof. Let us denote by

$$\left\| e^{-t(-\varepsilon\Delta_N)^s} \right\|_{\mathcal{L}(L^p, L^q)}$$

the operator norm of $e^{-t(-\varepsilon\Delta_N)^s}$ as an operator from $L^p(\Omega)$ to $L^q(\Omega)$. From Theorem 3.1 and estimates (3.5) and (3.6) we have (writing $\frac{1}{p} = \frac{t}{1} + \frac{1-t}{\infty}$ with $t = \frac{1}{p}$)

$$\left\| e^{-t(-\varepsilon\Delta_N)^s} \right\|_{\mathcal{L}(L^p, L^\infty)} \leq \left\| e^{-t(-\varepsilon\Delta_N)^s} \right\|_{\mathcal{L}(L^1, L^\infty)}^{\frac{1}{p}} \left\| e^{-t(-\varepsilon\Delta_N)^s} \right\|_{\mathcal{L}(L^\infty, L^\infty)}^{1-\frac{1}{p}} \leq C_\varepsilon^{\frac{1}{p}} e^{\frac{t}{p}} t^{-\frac{n}{2sp}}.$$

Now, for any $q \geq p$ we have, again by Theorem 3.1 and estimates (3.5) and (3.6) (this time we write $\frac{1}{q} = \frac{t}{p} + \frac{1-t}{\infty}$ with $t = \frac{p}{q}$)

$$\begin{aligned}\left\| e^{-t(-\varepsilon\Delta_N)^s} \right\|_{\mathcal{L}(L^p,L^q)} &\leq \left\| e^{-t(-\varepsilon\Delta_N)^s} \right\|_{\mathcal{L}(L^p,L^p)}^{\frac{p}{q}} \left\| e^{-t(-\varepsilon\Delta_N)^s} \right\|_{\mathcal{L}(L^p,L^\infty)}^{1-\frac{p}{q}} \\ &\leq \left(C_\varepsilon^{\frac{1}{p}} e^{\frac{t}{p}} t^{-\frac{n}{2sp}} \right)^{1-\frac{p}{q}} \\ &= C_\varepsilon^{\left(\frac{1}{p}-\frac{1}{q}\right)} e^{t\left(\frac{1}{p}-\frac{1}{q}\right)} t^{-\frac{n}{2s}\left(\frac{1}{p}-\frac{1}{q}\right)},\end{aligned}$$

that is equivalent to the desidered estimate. □

Interior and boundary regularity for solutions of the equation

$$(-\Delta)^s u = f,$$

have been established in [5] for both the cases of 0–Dirichlet and 0–Neumann boundary conditions. In particular, regularity in Hölder spaces has been considered.

Here, we start by considering regularity in L^q spaces for our equation (3.1), extending the results in [23, Theorem 3.5] to any fractional power $s \in (0, 1)$.

Theorem 3.3. *Let u be a solution of (3.1). Then the following facts hold:*

(i) If $f \in L^p$ with $1 \leq 2sp < n$ then $u \in L^q(\Omega)$ for every $p \leq q < p^*$ where $p^* = \frac{np}{n-2sp}$ and

$$\|u\|_{L^q(\Omega)} \leq C_{\varepsilon,n,p,q,s,\Omega} \|f\|_{L^p(\Omega)}.$$

(ii) If $f \in L^p(\Omega)$ with $2sp > n$ then $u \in L^\infty(\Omega)$ and

$$\|u\|_{L^\infty(\Omega)} \leq C_{\varepsilon,n,p,s,\Omega} \|f\|_{L^p(\Omega)}.$$

Proof. If $f \in L^p(\Omega)$ then we can write

$$u(x) = (I + (-\varepsilon\Delta_N)^s)^{-1} f(x) = \int_0^{+\infty} e^{-t-(t\varepsilon^s)(-\Delta_N)^s} f(x) dt.$$

If $2sp < n$ and $q \in [p, \frac{np}{n-2sp})$ we can use (3.7) to infer that

$$\begin{aligned}\|u\|_{L^q(\Omega)} &\leq \int_0^{+\infty} e^{-t} \left\| e^{-t(-\varepsilon\Delta)^s} f \right\|_{L^q(\Omega)} dt \\ &\leq C_\varepsilon^{\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p(\Omega)} \int_0^{+\infty} e^{-t\left(1-\left(\frac{1}{p}-\frac{1}{q}\right)\right)} t^{-\frac{n}{2s}\left(\frac{1}{p}-\frac{1}{q}\right)} dt\end{aligned} \tag{3.8}$$

where C_ε is a constant that depends on s, n, Ω, and ε. The conclusion follows observing that the last integral in (3.8) converges if and only if $q < \frac{np}{n-2sp}$. Assume now $2sp > n$, thanks again to (3.7) we have

$$\|u\|_{L^\infty(\Omega)} \leq C_\varepsilon^{\frac{1}{p}} \|f\|_{L^p(\Omega)} \int_0^{+\infty} e^{-t\left(1-\frac{1}{p}\right)} t^{-\frac{n}{2sp}} dt, \tag{3.9}$$

where C_ε is again a constant that depends on s, n, Ω, and ε. Again the conclusion follows observing that the integral in (3.9) converges if and only if $2sp > n$. $\square$

Theorem 3.4. *Let u be a solution of (3.1) with $f \in L^p(\Omega)$, we have:*

(i) If $2sp > n$ let $\alpha = 2s - \frac{n}{p}$.
- *If $\alpha < 1$ then $u \in C^{0,\alpha}(\overline{\Omega})$;*
- *If $\alpha = 1$ then $u \in C^1(\overline{\Omega})$;*
- *If $1 < \alpha < 2$ then $u \in C^{1,\alpha-1}(\overline{\Omega})$.*

In all three cases the corresponding seminorm is bounded by a constant (that depends on s, ε, n, α and Ω) times the norm of f in $L^p(\Omega)$.

(ii) If $f \in L^\infty(\Omega)$, we have:
- *if $s < \frac{1}{2}$ then $u \in C^{0,2s}(\overline{\Omega})$;*
- *if $s = \frac{1}{2}$ then $u \in C^1(\overline{\Omega})$;*
- *if $s > \frac{1}{2}$ then $u \in C^{1,2s-1}(\overline{\Omega})$.*

In all three cases the corresponding seminorm is bounded by a constant (that depends on s, ε, n and Ω) times the norm of f in $L^\infty(\Omega)$.

Proof. In booth cases (i) and (ii) we can use Theorem 3.3 to infer that $u \in L^\infty(\Omega)$, hence $(-\varepsilon\Delta_N)^s u$ belongs to the same L^p space as f, now the conclusion follows by [13, Corollary 4.3][2]. $\square$

The following Theorem contains regularity results in Hölder spaces, which follows easily from the ones in [5].

Theorem 3.5. *Let u be a solution of (3.1) with $f \in C^{0,\alpha}(\overline{\Omega})$, we have :*

(i) If $0 < \alpha + 2s < 1$, then $u \in C^{0,\alpha+2s}(\overline{\Omega})$ and

$$[u]_{C^{0,\alpha+2s}(\overline{\Omega})} \leq C \left(\|u\|_{H^s(\Omega)} + \|f\|_{C^{0,\alpha}(\overline{\Omega})} \right).$$

(ii) If $1 < \alpha + 2s < 2$, then $u \in C^{1,\alpha+2s-1}(\overline{\Omega})$

$$[u]_{C^{1,\alpha+2s-1}(\overline{\Omega})} \leq C \left(\|u\|_{H^s(\Omega)} + \|f\|_{C^{0,\alpha}(\overline{\Omega})} \right).$$

Proof. We will only prove (i), since (ii) follows in the same way. Since $f \in C^{0,\alpha}(\overline{\Omega})$ we can use (ii) of Theorem 3.4 to infer that $u \in C^{0,2s}(\overline{\Omega})$.

Therefore, if we define $h = f - u$ we have that $h \in C^{0,\min\{2s,\alpha\}}(\overline{\Omega})$ and u solves

$$\begin{cases} (-\Delta_N)^s u = h & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases}$$

By the regularity result of [5, Theorem 1.4] and the triangle inequality we have

$$\begin{aligned} [u]_{C^{0,\min\{2s,\alpha\}+2s}(\overline{\Omega})} &\leq C \left(\|u\|_{H^s(\Omega)} + \|h\|_{C^{0,\min\{2s,\alpha\}}(\overline{\Omega})} \right) \\ &\leq C \left(\|u\|_{H^s(\Omega)} + \|u\|_{C^{0,2s}(\overline{\Omega})} + \|f\|_{C^{0,\alpha}(\overline{\Omega})} \right). \end{aligned} \tag{3.10}$$

[2]Even if the controls on the seminorms of u in term of the L^p norms of f are not explicitely stated in [13, Corollary 4.3], a careful reading of the proof shows that they hold true.

Again by (ii) of Theorem 3.4 we have

$$\|u\|_{C^{0,2s}(\overline{\Omega})} \leq C\|f\|_{L^\infty(\Omega)}.$$

We iterate this procedure a finite number of times N (with N such that $2sN \geq \alpha$) to get the conclusion. □

Thanks to the above results, we can prove that any solution of our nonlinear problem (1.1) is in $C^{1,\alpha}(\overline{\Omega})$.

Theorem 3.6. *Let u be any weak solution of (1.1), then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$.*

Proof. Thanks to (ii) of Theorem 3.4, Theorem 3.5 and a standard iterative argument, it is enough to show that $u \in L^\infty(\Omega)$. By the Sobolev embedding we have that $u \in L^q(\Omega)$ where $q = \frac{2n}{n-2s}$, hence $u^p \in L^{\frac{q}{p}}(\Omega)$. Assume now that $u \in L^r(\Omega)$ for some $r \geq q$. Then $u^p \in L^{\frac{r}{p}}(\Omega)$ and we have three cases. If $\frac{2sr}{p} > n$ we can use (ii) of Theorem 3.3 to obtain that $u \in L^\infty(\Omega)$. If $\frac{2sr}{p} = n$, since Ω is bounded we can find $\eta > 0$ such that $u^p \in L^{\frac{r}{p}-\eta}(\Omega)$. Using (i) of Theorem 3.3 we find that $u^p \in L^{\widetilde{q}}(\Omega)$ for every $\widetilde{q} < \dfrac{\left(\dfrac{r}{p}-\eta\right)^*}{p}$, where

$$\left(\frac{r}{p}-\eta\right)^* = \frac{n\left(\dfrac{r}{p}-\eta\right)}{n-2s\left(\dfrac{r}{p}-\eta\right)}.$$

Now we want to choose an $\eta > 0$ such that $\frac{\left(\frac{r}{p}-\eta\right)^*}{p} > \frac{n}{2s}$, a straightforward computation shows that is enough to choose $\eta < \frac{n}{2s(p+1)}$.

Hence we can use again (ii) of Theorem 3.3 to obtain that $u \in L^\infty(\Omega)$. If $\frac{2sr}{p} < n$ then $u \in L^\sigma(\Omega)$ for every $\sigma < \left(\frac{r}{p}\right)^*$. Observe that, since $r \geq q$ we have

$$\left(\frac{r}{p}\right)^* = \frac{\dfrac{nr}{p}}{n-\dfrac{2sr}{p}} = \frac{nr}{np-2sr} \geq r\left(\frac{n}{np-2sq}\right) := r\lambda$$

where $\lambda > 1$ (since $q = 2n/(n-2s)$ and $p < (n+2s)/(n-2s)$). Hence $u \in L^{r\lambda}(\Omega)$. If we iterate this argument k times, we find $u \in L^{r\lambda^k}(\Omega)$, therefore, choosing k as the smallest integer such that $\frac{2sr\lambda^k}{p} \geq n$, we reduce to one of the two previous cases. □

The previous regularity result together with [4, Lemmma 4.5], implies the following

Corollary 3.7. *Let u be any weak solution of (1.1) and let U be its extension satisfying* (2.4).

Then, for any half–ball $\widetilde{B}^+((x_0,0),r) \subset \mathcal{C}$, we have that $y^a U_y \in C^{0,\beta}(\overline{\widetilde{B}^+((x_0,0),r)})$, for some $\beta > 0$.

4. Existence of non–constant solutions for small ε

In this Section we establish existence of positive non–constant solutions to problem (1.1) for ε small enough.

In the sequel, we will adopt the following notation for the positive part of u: $u_+ = \max\{0, u\}$.

In order to define a notion of solution for system (1.1), we consider the following extended problem

$$\begin{cases} \varepsilon \Delta_x U + \frac{a}{y} U_y + U_{yy} = 0, & \text{in } \mathcal{C}, \\ \partial_\nu U = 0, & \text{on } \partial_L \mathcal{C}, \\ -\lim_{y\to 0} y^a U_y(x,y) + U(x,0) = U(x,0)_+^p, & \text{in } \Omega. \end{cases} \tag{4.1}$$

Definition 4.1.
- *We say that a function $U \in \mathrm{H}^\varepsilon(\mathcal{C}, y^a)$ is a weak solution of (4.1) if for every $\Phi \in \mathrm{H}^\varepsilon(\mathcal{C}, y^a)$ we have*

$$\iint_{\mathcal{C}} (\varepsilon \nabla_x U \cdot \nabla_x \Phi + U_y \Phi_y)\, y^a dxdy + \int_\Omega U(x,0)\Phi(x,0)dx = \int_\Omega U(x,0)_+^p \Phi(x,0)dx. \tag{4.2}$$

- *We say that a function $u \in H^s(\Omega)$ is a weak solution of (1.1) if its extension in $\mathcal{C}$ satisfying problem* (2.4) *is a weak solution of (4.1).*

We remark that working in the space $\mathrm{H}^\varepsilon(\mathcal{C}, y^a)$ allows us to choose constant functions as test functions in the weak formulation (4.2).

It is easy to see that critical points of the functional

$$J_{\varepsilon,a}(U) := \frac{\|U\|_{\varepsilon,a}^2}{2} - \frac{1}{p+1}\int_\Omega |U(x,0)|^{p+1}dx =$$
$$\frac{1}{2}\iint_{\mathcal{C}} \left(\varepsilon|\nabla_x U|^2 + |U_y|^2\right) y^a dxdy + \frac{1}{2}\int_\Omega |U(x,0)|^2 dx - \frac{1}{p+1}\int_\Omega |U(x,0)_+|^{p+1}dx$$

are weak solutions of (4.1).

The following result establishes the existence of non–constant solutions to (1.1) provided the parameter ε is sufficiently small.

Theorem 4.2. *If ε is small enough, there exists at least one non–constant positive weak solution $U^{\varepsilon,a}$ of (4.1) In particular, the function $u^{\varepsilon,a} = U^{\varepsilon,a}(\cdot, 0)$ is a weak solution to (1.1).*

Moreover, there exists a positive constant $C = C(n, a, p, \Omega)$, which does not depend on ε, such that

$$J_{\varepsilon,a}(U^{\varepsilon,a}) \le C\varepsilon^{\frac{n}{2}}.$$

Proof. The proof is an application of the Mountain Pass Theorem by Ambrosetti and Rabinowitz [2]. Similarly to [23], one can show that the functional $J_{\varepsilon,a}$ is $C^{1,0}_{\mathrm{loc}}(\mathrm{H}^\varepsilon(\mathcal{C}, y^a), \mathbb{R})$ and its derivative $J'_{\varepsilon,a}$ is locally Lipschitz continuous. We proceed now as follows.

Step 1: The functional $J_{\varepsilon,a}$ satisfies the Palais–Smale condition. Let $V_k \in \mathrm{H}^\varepsilon(\mathcal{C}, y^a)$ be a sequence such that $\{J_{\varepsilon,a}(V_k)\}_k$ is bounded and $J'_{\varepsilon,a}(V_k)$ tends to 0 as $k \to \infty$. We want to show that, up to subsequences, $V_k \to V$ in $\mathrm{H}^\varepsilon(\mathcal{C}, y^a)$, for some V. Since $J'_{\varepsilon,a}(V_k)$ goes to zero we have

$$\left|\langle J'_{\varepsilon,a}(V_k); V_k\rangle\right| \leq \|V_k\|_{\varepsilon,a}$$

if k is large enough. Hence, we have

$$\left|\|V_k\|^2_{\varepsilon,a} - \int_\Omega (V_k)_+^{p+1}(x,0)\,dx\right| \leq \|V_k\|_{\varepsilon,a},$$

and by this expression and the fact that $J_{\varepsilon,a}(V_k)$ is bounded we deduce that

$$\|V_k\|^2_{\varepsilon,a} \leq C + \frac{2}{p+1}(\|V_k\|^2_{\varepsilon,a} + \|V_k\|_{\varepsilon,a}).$$

Since $p > 1$ we have that the sequence $\{V_k\}_k$ is bounded in $\mathrm{H}^\varepsilon(\mathcal{C}, y^a)$ and therefore, up to subsequences, it weakly converges to some $V \in \mathrm{H}^\varepsilon(\mathcal{C}, y^a)$. By the compactness of the trace operator (see Remark 2.5), we also have $V_k(\cdot, 0) \to V(\cdot, 0)$ strongly in $L^q(\Omega)$ for every $1 \leq q < \frac{2n}{n-2s}$ (and thus, in particular, in $L^{p+1}(\Omega)$).

In order to conclude that such convergence is strong, it is enough to show that

$$\|V_k\|_{\varepsilon,a} - \|V\|_{\varepsilon,a} \to 0. \tag{4.3}$$

Again, since $J'_{\varepsilon,a}(V_k) \to 0$, we also have

$$\left|\langle J'_{\varepsilon,a}(V_k); V_k - V\rangle\right| \to 0. \tag{4.4}$$

Moroever, using the Hölder inequality, we have that

$$\int_\Omega V_k^p(x,0)(V_k(x,0) - V(x,0))\,dx \leq \|V_k(x,0)\|^p_{L^{p+1}}\|V_k(x,0) - V(x,0)\|_{L^p} \to 0, \tag{4.5}$$

and

$$\begin{aligned}&\int_\Omega V_k(x,0)(V_k(x,0) - V(x,0))\,dx \\ &\qquad = \int_\Omega (V_k(x,0) - V(x,0))^2\,dx + \int_\Omega V(x,0)(V_k(x,0) - V(x,0))\,dx \to 0,\end{aligned} \tag{4.6}$$

Recalling that

$$\begin{aligned}\langle J'_{\varepsilon,a}(V_k); V_k - V\rangle = \frac{1}{2}&\iint_{\mathcal{C}} y^a(\varepsilon\nabla_x V_k\,,\,\partial_y V_k)\cdot(\nabla(V_k - V))\,dx\,dy \\ &+ \int_\Omega V_k(x,0)(V_k(x,0) - V(x,0))\,dx \\ &- \int_\Omega V_k^p(x,0)(V_k(x,0) - V(x,0))\,dx,\end{aligned} \tag{4.7}$$

and combining (4.4)-(4.5)-(4.6), we get

$$\iint_{\mathcal{C}} y^a(\varepsilon\nabla_x V_k\,,\,\partial_y V_k)\cdot(\nabla(V_k - V))\,dx\,dy \to 0. \tag{4.8}$$

Observing that

$$
\begin{aligned}
2\iint_{\mathcal{C}} & y^a(\varepsilon\nabla_x V_k\,,\,\partial_y V_k)\cdot(\nabla(V_k-V))\,dx\,dy \\
&\geq \iint_{\mathcal{C}} y^a\left(\varepsilon|\nabla_x V_k|^2+|\partial_y V_k|^2\right)-\iint_{\mathcal{C}} y^a\left(\varepsilon|\nabla_x V|^2+|\partial_y V|^2\right)\,dx\,dy,
\end{aligned}
\tag{4.9}
$$

we deduce that

$$
\iint_{\mathcal{C}} y^a\left(\varepsilon|\nabla_x V|^2+|\partial_y V|^2\right)\,dx\,dy \geq \limsup_{k\to\infty}\iint_{\mathcal{C}} y^a\left(\varepsilon|\nabla_x V_k|^2+|\partial_y V_k|^2\right)\,dx\,dy. \tag{4.10}
$$

The reverse inequality

$$
\iint_{\mathcal{C}} y^a\left(\varepsilon|\nabla_x V|^2+|\partial_y V|^2\right)\,dx\,dy \leq \liminf_{k\to\infty}\iint_{\mathcal{C}} y^a\left(\varepsilon|\nabla_x V_k|^2+|\partial_y V_k|^2\right)\,dx\,dy \tag{4.11}
$$

just follows by lower–semicontinuity.
Combining (4.10) and (4.11), we obtain (4.3) and thus conclude that

$$
\|V_k-V\|_{\varepsilon,a}\to 0.
$$

Step 2: It is standard to verify that the energy functional satisfies the Mounatin pass geometry (see [2, Lemma 3.3]), i.e.:

- $J_{\varepsilon,a}(0)=0$;
- There exists some $\rho>0$ such that $J_{\varepsilon,a}(V)=\beta>0$ if $\|V\|_{\varepsilon,a}=\rho$. This follows by the trace-Sobolev inequality (see Remark 2.5), which gives that
$$
\int_\Omega |U(x,0)_+|^{p+1}\leq C\|V\|_{\varepsilon,a}^{p+1}=o(\|V\|_{\varepsilon,a}^2);
$$
- there exists $\widetilde{V}$ with $\|\widetilde{V}\|_{\varepsilon,a}>\rho$ such that $J_{\varepsilon,a}(\widetilde{V})<0$.

Step 3: For $\varepsilon>0$ sufficiently small, there is a non negative function $\Phi\in \mathrm{H}^\varepsilon(\mathcal{C},y^a)$ and positive constants t_0, C_0 such that

$$
J_{\varepsilon,a}(t_0\Phi)=0 \quad\text{and}\quad J_{\varepsilon,a}(t\Phi)\leq C_0\varepsilon^{\frac{n}{2}} \text{ for all } t\in[0,t_0].
$$

To prove this, we assume that $0\in\Omega$ and we define the functions

$$
\varphi(x)=\begin{cases}\varepsilon^{-\frac{n}{2}}\left(1-\varepsilon^{\frac12}|x|\right) & \text{if } |x|<\varepsilon^{\frac12}\\ 0 & \text{if } |x|\geq\varepsilon^{\frac12},\end{cases}
$$

and

$$
\Phi(x,y)=e^{-\frac{y}{2}}\varphi(x).
$$

It is easy to see that $\Phi\in\mathrm{H}^\varepsilon(\mathcal{C},y^a)$ and

$$
\iint_{\mathcal{C}}|\nabla_x\Phi|^2\,y^a dxdy=\Gamma(1+a)\int_\Omega|\nabla\varphi|^2dx,\qquad \iint_{\mathcal{C}}|\Phi_y|^2\,y^a dxdy=\Gamma(1+a)\frac14\int_\Omega\varphi^2dx,
$$

and

$$
\int_\Omega|\Phi(x,0)|^2\,dx=\int_\Omega\varphi^2dx.
$$

At this point the claim follows by [16, Lemma 2.4] (see also [23, Proof of Theorem 4.3]).

Step 4: Conclusion: If we define

$$\Gamma = \{\gamma \in C([0,1]; \mathrm{H}^{\varepsilon}(\mathcal{C}, y^a)) : \gamma(0) = 0, \gamma(1) = t_0\Phi\},$$

the Mountain–Pass Theorem implies the existence of a function $U^{\varepsilon,a}$ such that

$$J'_{\varepsilon,a}(U^{\varepsilon,a}) = 0 \quad \text{and} \quad 0 < m := J_{\varepsilon,a}(U_{\varepsilon,a}) = \min_{\gamma\in\Gamma} \max_{t\in[0,1]} J_{\varepsilon,a}(\gamma(t)).$$

We immediately have

$$J_{\varepsilon,a}(U_{\varepsilon,a}) \le C\varepsilon^{\frac{n}{2}}.$$

Let us show now that $U^{\varepsilon,a} > 0$ in $\mathcal{C}$ and $u^{\varepsilon,a}(x) = U^{\varepsilon,a}(x,0) > 0$ in Ω. Using the negative part of $U^{\varepsilon,a}$ as a test function in the weak formulation (4.2) we deduce that

$$\begin{aligned}\iint_{\mathcal{C}} (\varepsilon|\nabla_x U_-^{\varepsilon,a}|^2 + |\partial_y U_-^{\varepsilon,a}|^2) y^a \, dxdy + \int_\Omega |U_-^{\varepsilon,a}(x,0)|^2 \, dx \\ = \frac{1}{p+1}\int_\Omega \big(U^{\varepsilon,a}(x,0)\big)_+^{p+1} U^{\varepsilon,a}(x,0)_- \, dx = 0,\end{aligned}$$

which imples that $U^{\varepsilon,a} \ge 0$ in $\mathcal{C}$ and $u^{\varepsilon,a} \ge 0$ in Ω.

Moreover, by the strong maximum principle for degenerate elliptic equations [10, Corollary 2.3.10], we have that $U^{\varepsilon,a} > 0$ in $\mathcal{C}$.

In order to prove that $u^{\varepsilon,a} > 0$ in Ω, we argue by contradiction. Assume that there exists $x_0 \in \Omega$ such that $u^{\varepsilon,a}(x_0) = 0$. Then, we can choose $r > 0$ such that $B(x_0, r) \subset \Omega$, hence $U^{\varepsilon,a}$ solves

$$\begin{cases} \varepsilon\Delta_x U^{\varepsilon,a} + \frac{a}{y} U_y^{\varepsilon,a} + U_{yy}^{\varepsilon,a} = 0, & \text{in } \widetilde{B}^+((x_0,0),r) \\ U^{\varepsilon,a} > 0 & \text{in } \widetilde{B}^+((x_0,0),r) \\ -\lim_{y\to 0} y^a U_y^{\varepsilon,a}(x,y) + u^{\varepsilon,a}(x) = u^{\varepsilon,a}(x)_+^p & \text{in } B(x_0,r). \end{cases}$$

By Corollary 3.7, we can apply the Hopf principle of Cabrè and Sirè [4, Proposition 4.11] to deduce that

$$0 > \lim_{y\to 0} -y^a U_y^{\varepsilon,a}(x_0,y) = (u^{\varepsilon,a}(x_0))^p - u^{\varepsilon,a}(x_0) = 0,$$

which is a contradiction.

It remains only to show that $U_{\varepsilon,a}$ is not constant. Assume by contradiction that $U_{\varepsilon,a} = c_{\varepsilon,a}$ then we have

$$J_{\varepsilon,a}(c_{\varepsilon,a}) = |\Omega| \left(\frac{c_{\varepsilon,a}^2}{2} - \frac{(c_{\varepsilon,a})_+^{p+1}}{p+1} \right) \tag{4.12}$$

and

$$0 = \langle J'_{\varepsilon,a}(c_{\varepsilon,a}); c_{\varepsilon,a}\rangle = |\Omega| \left(c_{\varepsilon,a}^2 - (c_{\varepsilon,a})_+^{p+1} \right). \tag{4.13}$$

Since $p > 1$ and $c_{\varepsilon,a} > 0$, we deduce from (4.13) that $c_{\varepsilon,a} = 1$ and by (4.12) we have

$$J_{\varepsilon,a}(1) = |\Omega| \left(\frac{1}{2} - \frac{1}{p+1} \right) \le C\varepsilon^{\frac{n}{2}},$$

that is not possible if ε is small enough.

□

5. Non–existence of non–constant solutions for large ε

In this Section, we establish the nonexistence of non–constant solutions for ε large enough. To do this, a crucial ingredient will be a uniform L^∞ estimate, which is given in Theorem 5.3 below. Such an estimate will be obtained, following the approach by Gidas and Spruck [11], by the combination of a blow-up argument and the following Liouville-type result, which can be found in [15] (see also [24]).

Theorem 5.1. [15, Remark 1.9] *Let $1 < p < \frac{n+2s}{n-2s}$ and let $U \in H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+, y^a)$ be a weak solution of*

$$\begin{cases} \Delta_x U(x,y) - \frac{a}{y} U_y(x,y) + U_{yy}(x,y) = 0 & x \in \mathbb{R}^n, y > 0, \\ U(x,y) \geq 0 & x \in \mathbb{R}^n, y > 0, \\ -\lim_{y\to 0+} y^a U_y(x,y) = U(x,0)^p, & x \in \mathbb{R}^n. \end{cases}$$

Then, $U \equiv 0$.

The following Caccioppoli–type inequality will be useful later on. The proof can be found in [5, Lemma 3.2.].

Let $\mathbf{B}(x)$ be a symmetric, uniformly elliptic and uniformly bounded $n \times n$ matrix, and let $\mathbf{A}(x)$ be of the form

$$\mathbf{A}(x) = \begin{bmatrix} \mathbf{B}(x) & 0 \\ 0 & 1 \end{bmatrix}. \tag{5.1}$$

Lemma 5.2. [5, Lemma 3.2.] *Let U be a solution of*

$$\begin{cases} \operatorname{div}(y^a \mathbf{A}(x) \nabla U) = 0 & \text{in } \Omega \times (0, +\infty) \\ -\lim_{y\to 0+} y^a U_y(x,0) = f(x) & \text{in } \Omega, \end{cases} \tag{5.2}$$

where the matrix $\mathbf{A}$ is as in (5.1)*. Then for every $R > 0$ such that $\widetilde{B}^+_{2R} \subseteq \Omega \times (0 + \infty)$ we have*

$$\iint_{\widetilde{B}_R{}^+} |\nabla U|^2 y^a dxdy \leq \frac{C}{R^2} \int_{\widetilde{B}^+_{2R}} U^2 y^a dxdy + C \int_{B_{2R}} |f(x) U(x,0)| dx.$$

We can now give the L^∞ a–priori estimate for positive solutions to (1.1), which will play a crucial role in our nonexistence result (Theorem 5.4 below).

Theorem 5.3. *There exists a positive constant $C > 0$, that does not depend on ε, such that, if u is a weak solution of* (1.1)*, then,*

$$\sup_{x\in\Omega} u(x) \leq C.$$

Proof. Step 1: First, we prove the statement for any $\varepsilon \leq \varepsilon_0$, where ε_0 is fixed. Assume by contradiction that there exists a sequence of functions u_k that solve problem (1.1) with parameters $0 \leq \varepsilon_k \leq \varepsilon_0$ and a sequence of points $P_k \in \overline{\Omega}$ such that

$$M_k := \sup_{\Omega} u_k = u_k(P_k) \to \infty, \quad \text{and} \quad P_k \to P \in \overline{\Omega}, \quad \text{as } k \to \infty.$$

Let U_k be the extension of u_k given in Theorem 2.2: we have that $U_k(x,0)=u_k(x)$ and U_k satisfies (4.1).

By the strong maximum principle for degenerate elliptic equations [10, Corollary 2.3.10] and Hopf's boundary Lemma (applied to points on the lateral boundary of the cylinder $\partial_L\mathcal{C}$), we deduce that the maximum of U_k is attained on $\overline{\Omega}\times\{0\}$; therefore

$$\sup_{\overline{\mathcal{C}}} U_k = U_k\left(P_k,0\right) = M_k.$$

We distinguish two cases, depending on where P is located.

Case 1: $P\in\partial\Omega$. Since $\partial\Omega$ is regular, we straighten the boundary of Ω near P with a local diffeomorphism Ψ such that $\Psi(P)=0$. If we call $z=\Psi(x)$ and we define $\widetilde{U}_k(z,y)=U_k(\Psi^{-1}(z),y)$ we find that $\widetilde{U}_k$ solves the system

$$\begin{cases} \varepsilon\,\mathrm{div}_z(\mathbf{B}(z)\nabla_z\widetilde{U}_k)+\frac{a}{y}(\widetilde{U}_k)_y+(\widetilde{U}_k)_{yy}=0, & \text{in } B_{2\delta}\cap\{z_n>0\}\times(0,+\infty),\\ \partial_\nu\widetilde{U}_k=0, & \text{on } B_{2\delta}\cap\{z_n=0\}\times[0,+\infty),\\ -\lim_{y\to0}y^a(\widetilde{U}_k)_y(z,y)+\widetilde{U}_k(z,0)=\widetilde{U}_k(z,0)_+^p, & \text{in } B_{2\delta}\cap\{z_n>0\}, \end{cases}$$

where $\delta>0$ is a small parameter and $\mathbf{B}$ is the matrix

$$\mathbf{B}(z)=\frac{J\Psi(\Psi^{-1}(z))J\Psi(\Psi^{-1}(z))^t}{|\det J\Psi(\Psi^{-1}(z))|}$$

that is uniformly elliptic and uniformly bounded. Let $(q_k',\alpha_k)=Q_k=\Psi(P_k)$ with $\alpha_k\geq0$. Since $Q_k\to0$ we can assume $|Q_k|<\delta$. Now we define

$$\lambda_k:=\frac{\varepsilon_k^{\frac12}}{M_k^{\frac{p-1}{2s}}},$$

and since ε_k is bounded and $M_k\to\infty$ we have that λ_k goes to 0. Now we distinguish two cases.

Sub–case 1.1 $\frac{\alpha_k}{\lambda_k}$ remains bounded as $k\to+\infty$. We can assume that $\frac{\alpha_k}{\lambda_k}\to\alpha\geq0$. If we define the rescaled function as

$$W_k(z,y):=\frac{1}{M_k}\widetilde{U}_k\left(\lambda_kz'+q_k',\lambda_kz_n,\frac{y}{M_k^{\frac{p-1}{2s}}}\right),\quad z=(z',z_n)\in B_{\frac{\delta}{\lambda_k}}\cap\{z_n>0\},\ y>0,$$

we can see that $0<W_k\leq1$ for any k and satisfies

$$\begin{cases} \mathrm{div}_z(\widetilde{\mathbf{B}}(z)\nabla_zW_k)+\frac{a}{y}(W_k)_y+(W_k)_{yy}=0, & \text{in } B_{\frac{\delta}{\lambda_k}}\cap\{z_n>0\}\times(0,+\infty),\\ \partial_\nu W_k=0, & \text{on } B_{\frac{\delta}{\lambda_k}}\cap\{z_n=0\}\times[0,+\infty),\\ -\lim_{y\to0}y^a(W_k)_y(z,y)+M_k^{-p+1}W_k(z,0)=W_k(z,0)_+^p, & \text{in } B_{\frac{\delta}{\lambda_k}}\cap\{z_n>0\}, \end{cases}\tag{5.3}$$

where $\widetilde{\mathbf{B}}(z)=\mathbf{B}(\lambda_kz'+q_k',\lambda_kz_n)$.

Now we want to investigate the limit of W_k as k goes to infinity. In order to do this, we set

$$f_k(x) = W_k(z,0)_+^p - M_k^{-p+1} W_k(z,0).$$

Since $\frac{\delta}{\lambda_k} \to \infty$ we can choose R such that $\widetilde{B}_{4R}^+ \subseteq \widetilde{B}^+_{\frac{\delta}{\lambda_k}}$. By (ii) of Theorem 3.3 and (i) of Theorem 3.4 we have that f_k are equi–bounded in $C^{0,\alpha}(\overline{B_{4R}})$.

We can use Lemma 5.2 to infer that ∇W_k are equi–bounded in $L^2(\overline{\widetilde{B}_R^+}, y^a)$. Therefore, up to subsequence, the functions W_k converges to a function W weakly in $H^1(y^a, \widetilde{B}_R^+)$. Now let R_j be a sequence such that $R_j \to \infty$ (and $\widetilde{B}_{4R_j}^+ \subseteq \widetilde{B}^+_{\frac{\delta}{\lambda_k}}$), arguing as before we can find, for every j, a subsequence of W_k such that $W_{k_i(j)}$ weakly converges to a function W_j in $H^1(y^a, \overline{\widetilde{B}_{R_j}^+})$. By a diagonal argument we can find a subsequence W_{k_i} that weakly converges to W in $H^1_{\text{loc}}(y^a, \mathbb{R}^{n+1}_+)$. Passing to the limit in the weak formulation of (5.3) it is easy to see that the limit function W is a non-negative solution to

$$\begin{cases} \Delta_x W + \frac{a}{y} W_y + W_{yy} = 0, & \text{in } \mathbb{R}^n_+ \times (0,+\infty), \\ \partial_\nu W = 0 & \text{on } \partial\mathbb{R}^n_+ \times (0+\infty) \\ -\lim\limits_{y\to 0} y^a W_y(z,y) = W(z,0)^p & \text{in } \mathbb{R}^n_+. \end{cases} \tag{5.4}$$

Let now consider W^* to be the even reflection of W through $\{z_n = 0\}$, then W^* satisfies

$$\begin{cases} \Delta_x W^* + \frac{a}{y} W^*_y + W^*_{yy} = 0, & \text{in } \mathbb{R}^{n+1}_+, \\ -\lim\limits_{y\to 0} y^a W^*_y(z,y) = W(z,0)^p & \text{in } \mathbb{R}^n. \end{cases} \tag{5.5}$$

By Theorem 5.1, we get that W^* is identically 0, but this is not possible since

$$W^*(0', \alpha, 0) = \lim_{k\to\infty} W_k\left(0', \frac{\alpha_k}{\lambda_k}, 0\right) = \lim_{k\to\infty} \frac{1}{M_k} \widetilde{U}_k(Q_k, 0) = 1.$$

Sub–case 1.2 $\frac{\alpha_k}{\lambda_k}$ is unbounded. We can assume that $\frac{\alpha_k}{\lambda_k} \to \infty$. If we define

$$W_k(z,y) = \frac{1}{M_k} \widetilde{U}_k \left(\lambda_k z + Q_k, \frac{y}{M_k^{\frac{p-1}{2s}}} \right)$$

we find that W_k solves a problem that is similar to (5.3) but with Neumann boundary conditions on $\{z_n = -\frac{\alpha_k}{\lambda_k}\}$. With the same compactness argument, we find that W_k converges to a solution W of (5.4), and again, after considering the even reflection W^*, we deduce that $W^* = 0$ which gives a contradiction as before.

Case 2: $P \in \Omega$. In this case we do not need to straighten the boundary. If we perform the scaling

$$W_k(x,y) = \frac{1}{M_k} U_k \left(\lambda_k x + P_k, \frac{y}{M_k^{\frac{p-1}{2s}}} \right)$$

the argument follows that of the sub–case 1.2.

Step 2: If $\varepsilon \geq \varepsilon_0$ we can apply *Step 1* to the function $u\varepsilon^{-\frac{s}{(p-1)}}$ to find that there exists a constant C (independent on ε) such that

$$\sup_{\Omega} u \leq C\varepsilon^{\frac{s}{(p-1)}}. \tag{5.6}$$

In what follows, C will denote possibly different positive constants, which does not depend on ε.

Let U be th extension of u satisfying (2.4) and let $\beta \geq 1$ to be chosen later. We choose $\Phi = U^{2\beta-1}$ as a test function in the weak formulation (4.2) to get

$$\begin{aligned} \varepsilon\frac{2\beta-1}{\beta^2}\iint_{\mathcal{C}} |\nabla_x(U^\beta)|^2 y^a dxdy + \frac{2\beta-1}{\beta^2}\iint_{\mathcal{C}} |\partial_y U^\beta|^2 y^a dxdy + \int_\Omega |U^\beta(x,0)|^2 dx \\ = \int_\Omega |U(x,0)|^{p-1+2\beta}\, dx. \end{aligned} \tag{5.7}$$

Thanks to (5.6) we can estimate the right hand side of (5.7) as

$$\int_\Omega |U(x,0)|^{p-1+2\beta}\, dx \leq C^{p-1}\varepsilon^s \int_\Omega u^{2\beta} dx. \tag{5.8}$$

If we use Proposition 2.4 and equation (2.3), we get

$$C_s \iint_{\mathcal{C}} \left(\varepsilon \left|\nabla_x U^\beta\right|^2 + \left|\partial_y U^\beta\right|^2\right) y^a dxdy \geq \left\|(-\varepsilon\Delta)^{\frac{s}{2}} u^\beta\right\|^2_{L^2(\Omega)} = \varepsilon^s [u^\beta]^2_{H^s(\Omega)}. \tag{5.9}$$

From equations (5.7), (5.8) and (5.9), and the fact that $\frac{\beta^2}{2\beta-1} \leq \beta$ for $\beta \geq 1$, we get

$$\|u^\beta\|^2_{H^s(\Omega)} \leq C\beta \int_\Omega u^{2\beta}.$$

By the fractional Sobolev embedding $H^s(\Omega) \hookrightarrow L^{2^*_s}(\Omega)$ (where $2^*_s = 2n/(n-2s)$), we get

$$\left(\int_\Omega u^{\beta 2^*_s} dx\right)^{\frac{2}{2^*_s}} \leq C\beta \int_\Omega u^{2\beta} dx,$$

and arguing as in [16, Pages 21–22] (see also [23, Pages 1038–1039]), we finally deduce that

$$\|u\|_{L^\infty(\Omega)} \leq C|\Omega|^{1/p},$$

which concludes the proof. □

We can now prove our non-existence result for ε sufficiently large.

Theorem 5.4. *There exists $\varepsilon^* > 0$ such that if $\varepsilon > \varepsilon^*$ then the unique solution of problem (1.1) is $u \equiv 1$.*

Proof. Assume by contradiction that u is a non constant solution of (1.1). Writing $u = \phi + u_\Omega$, where $u_\Omega := \frac{1}{|\Omega|}\int_\Omega u$, we can see that ϕ solves the following equation

$$(-\varepsilon\Delta)^s \phi + \phi - \left(\int_0^1 p\,(u_\Omega + t\phi)^{p-1}\,dt\right)\phi = u_\Omega^p - u_\Omega.$$

Let Φ be the extension of ϕ (given in Theorem 2.2), which satisfies

$$\begin{cases} \varepsilon\Delta_x\Phi + \frac{a}{y}\Phi_y + \Phi_{yy} = 0, & \text{in } \mathcal{C}, \\ \partial_\nu\Phi = 0, & \text{on } \partial_L\mathcal{C}, \\ -\lim_{y\to 0} y^a\Phi_y = \left(\int_0^1 p\,(u_\Omega + t\phi)^{p-1}\,dt\right)\phi - \phi + u_\Omega^p - u_\Omega & \text{in } \Omega. \end{cases} \tag{5.10}$$

Testing the weak formulation for problem (5.10) with Φ itself, and taking into account that ϕ has zero average we find:

$$\varepsilon\iint_{\mathcal{C}} |\nabla_x\Phi|^2\,y^a dxdy + \iint_{\mathcal{C}} |\Phi_y|^2\,y^a dxdy + \int_\Omega \phi^2 dx = \int_\Omega\left(\int_0^1 p\,(u_\Omega + t\phi)^{p-1}\,dt\right)\phi^2 dx.$$

By the uniform L^∞ estimate of Theorem 5.3, we deduce

$$\varepsilon\iint_{\mathcal{C}} |\nabla_x\Phi|^2\,y^a dxdy + \iint_{\mathcal{C}} |\Phi_y|^2\,y^a dxdy + \int_\Omega \phi^2 dx \le pC^{p-1}\int_\Omega \phi^2 dx,$$

and by Proposition 2.4 and equation (2.3) we get

$$\frac{\varepsilon^s}{C_s}[\phi]^2_{H^s(\Omega)} + \int_\Omega \phi^2 dx \le pC^{p-1}\int_\Omega \phi^2 dx.$$

The Poincarè inequality for fractional Sobolev spaces leads to

$$\varepsilon^s\int_\Omega \phi^2 dx \le \widetilde{C}_s(pC^{p-1} - 1)\int_\Omega \phi^2 dx,$$

that is not possible if ε is large enough. □

6. Shape of solutions

In this Section we prove that, if ε is sufficiently small, any solution $u^{\varepsilon,a}$ of (1.1) given by Theorem 4.2 concentrates around some points and tends to zero in measure as $\varepsilon \to 0$. This is the content of Theorem 6.6 below. A crucial ingredient in its proof is the following Harnack inequality.

Proposition 6.1 (Harnack inequality)**.** *Let u be a non–negative solution of*

$$\begin{cases} (-\varepsilon\Delta_N)^s u + c(x)u = 0 & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

where $c \in C^0(\overline{\Omega})$. Then, there exists a constant $C_0 = C_0\left(n, s, \Omega, \frac{\|c\|_\infty R^{2s}}{\varepsilon^s}\right)$ such that for every $x_0 \in \overline{\Omega}$ and any $R > 0$ we have

$$\sup_{B(x_0,R)\cap\Omega} u \le C_0 \inf_{B(x_0,R)\cap\Omega} u. \tag{6.2}$$

Proof. Let U be the extension of u, which satisfies

$$\begin{cases} \Delta_x U + \frac{a}{y} U_y + U_{yy} = 0 & \text{in } \mathcal{C} \\ \partial_\nu U = 0, & \text{on } \partial_L \mathcal{C} \\ -\lim_{y\to 0} y^a U_y(x,y) + \varepsilon^{-s} c(x) U(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{6.3}$$

Clearly it is enough to prove the existence of a constant $C_0 = C_0\left(n, s, \Omega, \frac{\|c\|_\infty R^{2s}}{\varepsilon^s}\right)$ such that, for every $x_0 \in \overline{\Omega}$

$$\sup_{\widetilde{B}^+((x_0,0),R)\cap\mathcal{C}} U \le C \inf_{\widetilde{B}^+((x_0,0),R)\cap\mathcal{C}} U.$$

If $x_0 \in \Omega$ and R is sufficiently small, then (6.2) follows from the Harnack inequality of Cabrè and Sire [4, Lemma 4.9]. Instead, if $x_0 \in \partial\Omega$ we can follow the argument in [16, Proof of Lemma 4.3]: we straighten the boundary near x_0 with a local diffeomorphism Ψ such that $\Psi(P) = 0$. If we call $z = \Psi(x)$ and we define $\widetilde{U}(z,y) = U(\Psi^{-1}(z), y)$ we find that $\widetilde{U}$ solves

$$\begin{cases} \operatorname{div}_z\left(\mathbf{B}(z)\nabla_z \widetilde{U}\right) + \frac{a}{y}\widetilde{U}_y + \widetilde{U}_{yy} = 0, & \text{in } (B_\delta \cap \{z_n > 0\}) \times \mathbb{R}^+, \\ \partial_\nu \widetilde{U}_k = 0, & \text{on } (B_\delta \cap \{z_n = 0\}) \times \mathbb{R}^+, \\ -\lim\limits_{y\to 0} y^a \widetilde{U}_y(z,y) + \varepsilon^{-s}\widetilde{d}(z)\widetilde{U}(z,0) = 0, & \text{in } B_\delta \cap \{z_n > 0\}. \end{cases}$$

where $d(z) = c(\Psi^{-1}(z))$. Since $\partial_\nu \widetilde{U} = 0$ on $B_\delta \cap \{z_n = 0\}$, we can extend $\widetilde{U}$ (by even reflection) to a function U^* that is a solution of

$$\begin{cases} \operatorname{div}_z\left(\widetilde{\mathbf{B}}(z)\nabla_z U^*\right) + \frac{a}{y}{U^*}_y + U^*_{yy} = 0, & \text{in } \widetilde{B}^+_\delta, \\ -\lim\limits_{y\to 0} y^a {U^*}_y(z,y) + \varepsilon^{-s} d^*(z) U^*(z,0) = 0, & \text{in } B_\delta, \end{cases} \tag{6.4}$$

where d^* is the even reflection of $\widetilde{d}$ through $\{x_n = 0\}$. Hence, one can argue as in [16, Proof of Lemma 4.3] and use the Harnack inequality of Lemma 6.2 below for the extended problem with bounded measurable coefficients (just depending on the horizontal variable), to conclude. □

Lemma 6.2. *Let $d \in L^\infty(B_{4R})$ and let $\widetilde{\mathbf{B}}(z)$ be a uniformly elliptic and uniformly bounded symmetric matrix with bounded measurable coefficients. Let $\varphi \in H^1(\widetilde{B}^+_{4R}, y^a)$ be a non–negative weak solution to*

$$\begin{cases} \operatorname{div}_z\left(\widetilde{\mathbf{B}}(z)\nabla_z U\right) + \frac{a}{y}U_y + U_{yy} = 0, & \text{in } \widetilde{B}^+_{4R}, \\ -\lim\limits_{y\to 0} y^a U_y(z,y) + d(z) U(z,0) = 0, & \text{in } B_{4R}. \end{cases} \tag{6.5}$$

Then,

$$\sup_{\widetilde{B}^+_R} U \le C \inf_{\widetilde{B}^+_R} U,$$

where C is a constant depending only on n, s, $R^{2s}\|d\|_{L^\infty(B_{4R})}$ and the ellipticity of the matrix B.

Lemma 6.2 is the analog of [4, Lemma 4.9] for weak solutions of (6.4), with the difference that here we have some coefficients $\widetilde{\mathbf{B}}(z)$. We omit the proof since, as one can easily verify, it is exactly the same as the one in [4], being $\widetilde{\mathbf{B}}(z)$ bounded elliptic and depending only on the horizontal variable z.

The following Lemma provides L^2 and L^p estimates (in terms of ε) for the solutions to (4.1) given by Theorem 4.2, which will be useful in the sequel.

Lemma 6.3. *Let $U^{\varepsilon,a} \in \mathrm{H}^{\varepsilon}(\mathcal{C}, y^a)$ be any solution of problem (4.1) given by Theorem 4.2 and let $u^{\varepsilon,a}$ be its trace over Ω.*

Then, there exists a constant $C > 0$ independent on ε such that

$$\varepsilon \iint_{\mathcal{C}} |\nabla_x U^{\varepsilon,a}|^2 y^a dxdy + \iint_{\mathcal{C}} |U^{\varepsilon,a}_y|^2 y^a dxdy + \int_\Omega |u^{\varepsilon,a}|^2 dx = \int_\Omega |u^{\varepsilon,a}|^{p+1} dx \le C\varepsilon^{\frac{n}{2}}.$$

Proof. If we use $U^{\varepsilon,a}$ as a test function in the weak formulation of problem (4.1), we find

$$\varepsilon \iint_{\mathcal{C}} |\nabla_x U^{\varepsilon,a}|^2 y^a dxdy + \iint_{\mathcal{C}} |U^{\varepsilon,a}_y|^2 y^a dxdy + \int_\Omega |u^{\varepsilon,a}|^2 dx = \int_\Omega |u^{\varepsilon,a}|^{p+1} dx.$$

Since $U^{\varepsilon,a}$ is a solution given by Theorem 4.2, it satisfies the energy estimate

$$\frac{\varepsilon}{2} \iint_{\mathcal{C}} |\nabla_x U^{\varepsilon,a}|^2 y^a dxdy + \frac{1}{2} \iint_{\mathcal{C}} |U^{\varepsilon,a}_y|^2 y^a dxdy + \frac{1}{2}\int_\Omega |u^{\varepsilon,a}|^2 dx - \frac{1}{p+1}\int_\Omega |u^{\varepsilon,a}|^{p+1} dx \le C\varepsilon^{\frac{n}{2}}.$$

Hence, we obtain

$$\int_\Omega |u^{\varepsilon,a}|^{p+1} dx \le C\varepsilon^{\frac{n}{2}} + \frac{2}{p+1}\int_\Omega |u^{\varepsilon,a}|^{p+1} dx,$$

and the conclusion follows observing that $\frac{2}{p+1} < 1$ for any $p > 1$. □

By using an interpolation argument, we have the following:

Lemma 6.4. *Let $u^{\varepsilon,a}$ and $U^{\varepsilon,a}$ be as in the previous Lemma. Then, for every $q \in [2, \frac{2n}{n-2s}]$ there exits a constant C_q independent on ε such that*

$$\int_\Omega |u^{\varepsilon,a}|^q dx \le C_q \varepsilon^{\frac{n}{2}}.$$

Proof. By Lemma 6.3, the statement is true for $q = 2$; hence, by a standard interpolation argument, we only need to show the estimate for $q = \frac{2n}{n-2s}$. Let then $q = \frac{2n}{n-2s}$, using the fractional Sobolev inequality (2.9) and Theorem 4.2, we have

$$\int_\Omega |u^{\varepsilon,a}|^q dx \le \left(C\varepsilon^{-\frac{s}{2}}\|U\|_{\varepsilon,a}\right)^q \le C\varepsilon^{-\frac{sq}{2}}\varepsilon^{\frac{nq}{4}} = C\varepsilon^{q(\frac{n}{4}-\frac{s}{2})}.$$

Finally, being $q = \frac{2n}{n-2s}$, we have

$$q\left(\frac{n}{4} - \frac{s}{2}\right) = \frac{n}{2},$$

hence the conclusion follows. □

The following Lemma provides an L^1 estimate, which will be crucial in the proof of Theorem 6.6 below.

Lemma 6.5. *Let $u^{\varepsilon,a}$ be as in the previous Lemma. Then, there exists a constant C_1 independent on ε such that*

$$\int_\Omega u^{\varepsilon,a}dx \leq C_1\varepsilon^{\frac{n}{2}}.$$

Proof. If we use the constant function 1 (that belongs to $\mathrm{H}^\varepsilon(\mathcal{C},y^a)$) as a test function in the weak formulation (4.2) we find that

$$\int_\Omega u^{\varepsilon,a}dx = \int_\Omega (u^{\varepsilon,a})^p dx.$$

Let $t=\frac{1}{p}$ and write $p=t+(1-t)(p+1)$.

Using Hölder inequality and Lemma 6.4, we obtain

$$\begin{aligned}\int_\Omega (u^{\varepsilon,a})^p dx &\leq \left(\int_\Omega u^{\varepsilon,a}dx\right)^t\left(\int_\Omega (u^{\varepsilon,a})^{p+1}dx\right)^{1-t}\\ &\leq \left(\int_\Omega u^{\varepsilon,a}dx\right)^t\left(C\varepsilon^{\frac{n}{2}}\right)^{1-t} = \left(\int_\Omega (u^{\varepsilon,a})^p dx\right)^t\left(C\varepsilon^{\frac{n}{2}}\right)^{1-t},\end{aligned}$$

that implies

$$\int_\Omega u^{\varepsilon,a}dx = \int_\Omega (u^{\varepsilon,a})^p dx \leq C_1\varepsilon^{\frac{n}{2}}.$$

□

In order to state our last result, let us introduce some notation. For $K=(k_1\ldots,k_n)\in\mathbb{Z}^n$ and $l>0$, define the cube of $\mathbb{R}^n$

$$Q_{K,l}=\left\{(x_1,\ldots,x_n)\in\mathbb{R}^n : |x_i-lk_i|\leq\frac{l}{2}, 1\leq i\leq n\right\}.$$

Moreover, if $u^{\varepsilon,a}$ is a solution of (1.1) given by Theorem 4.2, for every $\eta>0$ we define the super–level set of $u^{\varepsilon,a}$ as

$$\Omega_\eta=\{x\in\Omega : u^{\varepsilon,a}(x)>\eta\}.$$

We can now give the main result of this Section.

Theorem 6.6. *For ε small enough, let $u^{\varepsilon,a}\in\mathcal{H}^s_\varepsilon(\Omega)$ be a solution of (1.1) given by Theorem 4.2. Then, for every $\eta>0$, there exists a constant C that depends on n,Ω,s and η but not on ε, such that Ω_η can be covered by at most C of the cubes $Q_{K,\sqrt{\varepsilon}}$.*

Proof. First of all we have that $u^{\varepsilon,a}$ solves (1.1) if and only if it solves (6.1) with $c(x)=1-(u^{\varepsilon,a}(x))^{p-1}$. Observe that, thanks to Theorem 3.6, $c\in C^0(\overline{\Omega})$ and it is uniformly bounded in ε by Theorem 5.3. If we choose $R=\sqrt{n\varepsilon}$ we have $\varepsilon^{-s}R^{2s}=n^s$, hence the Harnack constant of Proposition 6.1 (with such a choice of the radius R) is independent on ε. If $z\in\Omega_\eta$ we can use Proposition 6.1 to infer that

$$\inf_{B(z,\sqrt{n\varepsilon})\cap\Omega} u^{\varepsilon,a}\geq C_0^{-1}\sup_{B(z,\sqrt{n\varepsilon})\cap\Omega} u^{\varepsilon,a}>C_0^{-1}\eta.$$

Thus the $\sqrt{n\varepsilon}$ neighborhood of Ω_η is contained in Ω_δ where $\delta = C_0^{-1}\eta$. Let $\{Q_{K^j,\sqrt{\varepsilon}}\}_{j=1}^{j=m}$ be the family of cubes whose intersection with Ω_η is non–empty; then we have

$$\bigcup_{j=1}^{m} Q_{K^j,\sqrt{\varepsilon}} \subset \Omega_\delta.$$

Such inclusion, together with the fact that, by definition, the cubes $Q_{K^j,\sqrt{\varepsilon}}$ are pairwise disjoint up to a set of measure 0, implies that

$$m\varepsilon^{\frac{n}{2}} = \left|\bigcup_{j=1}^{m} Q_{K^j,\sqrt{\varepsilon}}\right| \leq |\Omega_\delta|.$$

If we use Lemma 6.5 we have

$$C_0^{-1}\eta|\Omega_\delta| = \delta|\Omega_\delta| = \int_{\Omega_\delta}\delta dx \leq \int_{\Omega_\delta} u^{\varepsilon,a}dx + \int_{\Omega\setminus\Omega_\delta} u^{\varepsilon,a}dx = \int_{\Omega} u^{\varepsilon,a}dx \leq C_1\varepsilon^{\frac{n}{2}},$$

and combining the two above inequalities we find

$$m \leq C_0C_1\eta^{-1},$$

where C_0 is the constant in the Harnack inequality of Proposition 6.1 and C_1 is the constant in the L^1-estimate of Lemma 6.5. □

Remark 6.7. *Since $u^{\varepsilon,a} \in C^{1,\alpha}(\overline{\Omega})$ we have that $u^{\varepsilon,a}$ admits at least one maximum point. Assume that P_ε and Q_ε are two maximum points, if ε is small enough we can use Theorem 6.6 to infer that*

$$|P_\varepsilon - Q_\varepsilon| \leq \sqrt{\varepsilon}m \to 0 \quad \text{as } \varepsilon \to 0.$$

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