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An Analytical Treatise on Endo-Atmospheric Fuel-Optimal Rocket Landings

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While there is a strong current interest in and an increasing body of latest numerical work on fuel-optimal powered descent inside an atmosphere, little recent analytical understanding of the problem has been reported. This paper analyzes the endo-atmospheric fuel-optimal rocket landing problem in three-dimensional motion. The necessary conditions for the problem, that account for both propulsive and aerodynamic forces, are derived. They include the full set of costate equations, and the optimality conditions of the body attitude in terms of the angle of attack and sideslip angle. In contrast to a long-standing assumption of coordinated flight in the literature, it is shown that in general coordinated flight is not optimal, if sideslip modulation is allowed. Special cases, such as small-angle and in-plane motion, are analyzed and harmonized with the previous results that are well known in the literature. The analytical conditions derived in this paper can also serve as independent verification for numerical solutions obtained by direct methods in optimal control, or can be exploited for trajectory-design purposes.

I. Introduction

The problem of fuel-optimal rocket landing inside an atmosphere is a strong current research interest because of its close connection to reusable launch vehicles and planetary landing missions. The problem of fuel-optimal rocket landing in vacuum is conducive to analytical investigation and enjoys a number of theoretical findings on the properties of the optimal solution [1–7]. These findings not only offer insight to the vacuum optimal landing problem, but also help design guidance algorithms. In contrast, the presence of the atmosphere in the problem formulation significantly complicates

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the dynamics, and analysis of the problem is much more difficult. As a result, the recent work on endo-atmospheric fuel-optimal rocket landing is exclusively limited to finding numerical solutions, both in 3-degrees-of-freedom (3-DOF) (e.g., [8]) and more recently 6-degrees-of-freedom (6-DOF) motion [9–15]. In some cases only aerodynamic drag is considered in the solution to simplify the problem [9, 11, 13, 15].

It is the intent of this paper to continue the fine tradition of analytically understanding the problem of endo-atmospheric fuel-optimal landing, exemplified by the pioneering work by the late Prof. N. X. Vinh in Ref. [16]. Compared to the previous works of Refs. [16–18] where the vehicle is assumed to be always in a coordinated flight where the sideslip angle is zero and only the angle of attack is modulated, here we allow the vehicle to use the sideslip angle as an additional trajectory control. Furthermore, the full aerodynamic force vector with components on all 3 body axes is considered. The problem is formulated in three-dimensional motion where the angle of attack and sideslip angle define the aerodynamic force vector as well as the direction of the rocket engine thrust vector, which is assumed to be fixed in the direction of longitudinal body axis. The complete necessary conditions for the optimal solution are provided. The necessary conditions are analyzed to determine their implication on the optimal solution. For instance, in the vacuum optimal rocket powered flight, it is well-known that the optimal thrust direction is along the primer vector [19]. But with the atmosphere, the conclusion is different, and the primer vector alone is not sufficient to define the optimal thrust direction. In coordinated flight, it is shown in Refs. [16, 20] by geometric arguments, and later by a theoretical proof in Ref. [17], that the vehicle symmetric plane is in the plane spanned by the relative velocity vector and the primer vector, and the direction of the optimal thrust vector is in this plane and determined by the angle of attack. However, when sideslip modulation is the additional degree of trajectory control, the analysis in this paper reveals that the optimal sideslip angle is generally non-zero, therefore resulting in yet again a different orientation of the vehicle's body and optimal thrust direction as compared to the situation in coordinated flight.

Another persistent question in rocket powered flight is whether there is an optimal *singular* thrust arc in the trajectory where the optimal thrust takes intermediate (and generally time-varying) values between its upper and lower bounds. In the vacuum fuel-optimal landing problem in a constant gravitational field, it can be theoretically established that no singular arc is optimal, and the optimal thrust will switch between the upper and lower bounds at most twice in three-dimensional motion [3]. When the aerodynamic force is present in the problem, no theoretical results are available but there are reports of singular arcs found in numerical solutions [21]. While the question of whether an optimal singular solution could exist in the problem is not completely resolved (and left as an open question), the complexity of the necessary conditions we will show for singular optimal thrust in endo-atmospheric non-coordinated rocket flight would lead to the conjecture that optimal singular solutions are unlikely.

Not only does this work provide a needed analytical insight and understanding to a problem of current interest, the conditions derived in the paper can also serve as the independent verification to the optimality of numerical solutions, typically generated today by a direct method. In addition, the necessary conditions can be actively employed to build

trajectory-design algorithms. The latter aspects of the application of this work will be demonstrated in this paper.

This paper is structured as follows. In Sec. II we introduce the landing problem in vacuum and in atmosphere, and we derive the necessary conditions for optimality in the presence of aerodynamic effects. Section III harmonizes the general formulation proposed here with some classical results coming from literature, especially for what regards the coordinated turn, and shows that in general the optimal sideslip angle is not zero. Moreover, in this section we investigate the possible presence of singular arcs for the endo-atmospheric problem. In Sec. IV we describe the derivation of the costate differential equations for the proposed landing problem formulation and retrieve the elements characterizing the two-point-boundary-value problem (TPBVP), including the Hamiltonian structure of the optimal solution, while we show some numerical results coming from the practical applications of the proposed necessary conditions for a fictitious rocket landing on Mars in Sec. V. Finally, conclusions about this work are drawn in Sec. VI.

II. Three-Dimensional Endo-Atmospheric Powered Landing Optimal Control Problem

A. Formulation of Cost Function and Equations of Motion

The fuel-optimal landing problem is typically formulated as an Optimal Control Problem (OCP). The cost function in the Mayer form we are interested in minimizing is

$$\min_{\mathbf{u}, t_F} J = -m(t_F) \quad (1)$$

where m is the mass of the rocket, t_F is the final time of the flight, and $\mathbf{u}$ the generic control vector, to be specified later. In the equivalent Lagrange form the cost may be expressed as

$$\min_{\mathbf{u}, t_F} J = \frac{T_{max}}{c} \int_{t_0}^{t_F} \eta(t) dt \quad (2)$$

with η representing the throttle level, T_{max} the maximum thrust generated by the engine at $\eta = 1$, c the corresponding exhaust speed, and t_0 the initial time of flight. Yet, an equivalent form, differing only for a multiplicative factor, is

$$\min_{\mathbf{u}, t_F} J = \int_{t_0}^{t_F} \eta(t) dt \quad (3)$$

In this work we will mostly Eq. (1) or Eq. (3), specifying from time to time which form we refer to depending on the specific use. Once we have defined the cost function we can include the dynamics of the system, meaning that the

solution must obey the following set of differential equations

$$\begin{cases} \dot{\mathbf{r}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \mathbf{g} + \frac{\mathbf{F}_e^T}{m} + \frac{\mathbf{F}_e^A}{m} \\ \dot{m} &= -\eta \frac{T_{max}}{c} \end{cases} \quad (4)$$

with the states $\mathbf{x} \in \mathbb{R}^7$

$$\mathbf{x} = [\mathbf{r}^T, \quad \mathbf{v}^T, \quad m]^T \quad (5)$$

where $\mathbf{r} = [x, y, z]^T \in \mathbb{R}^3$, $\mathbf{v} = [u, v, w]^T \in \mathbb{R}^3$ represent the position and the velocity in the external target-fixed, target-centered reference frame e [1, 22], assumed to be inertial, given the short time of flight. Its axes are defined such that the x, y, z components correspond to the locally defined *Up*, *East*, and *North* directions (see Fig. 1). The gravity acceleration vector is $\mathbf{g}$. The terms $\mathbf{F}_e^T$ and $\mathbf{F}_e^A$ are the propulsive and the aerodynamic force vectors, respectively. The overall control vector $\mathbf{u} \in \mathbb{R}^4$ for this problem is

$$\mathbf{u} = [\eta, \quad \hat{\mathbf{u}}^T]^T \quad (6)$$

with the throttle level η bounded by lower bound η_{min} and upper bound η_{max} , and $\hat{\mathbf{u}} = [u_x, \quad u_y, \quad u_z]$ is a unit vector representing the direction of the thrust expressed in the e (or, equivalently, $\mathcal{UEN}$) reference frame, hence

$$\|\hat{\mathbf{u}}\| = 1 \quad (7)$$

Clearly, the following equality holds.

$$\mathbf{F}_e^T \triangleq \eta T_{max} \hat{\mathbf{u}} \quad (8)$$

It is typically assumed that the initial state $\mathbf{x}(t_0)$ is known,

$$\mathbf{x}(t_0) = \mathbf{x}_0 \quad (9)$$

while the final state $\mathbf{x}(t_F)$ can have several flavours of constraints: in the classical *pinpoint landing* approach both final position and velocities are fixed [1],

$$\mathbf{r}(t_F) = \mathbf{r}_F, \quad \mathbf{v}(t_F) = \mathbf{v}_F \quad (10)$$

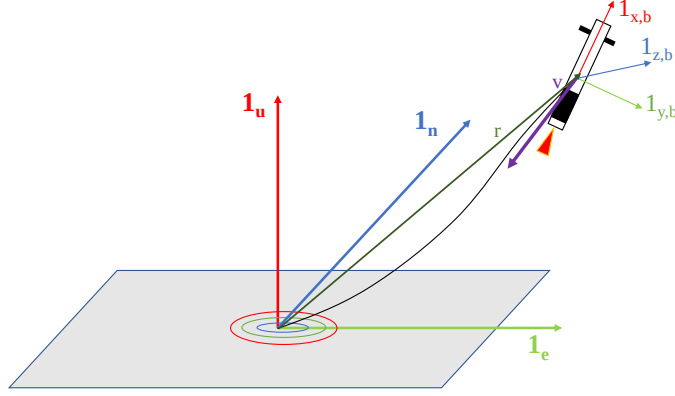


Fig. 1 UEN reference frame centered at the target landing point, and body axes representation.

while when we consider the *soft landing* [3] problem we only impose the final altitude, (here corresponding to the x component), together with the final velocity, as hard constraints, while tolerating some horizontal position errors

$$x(t_F) = x_F, \quad \mathbf{v}(t_F) = \mathbf{v}_F \quad (11)$$

A middle-ground between Eq. (10) and Eq. (11) is the *minimum-landing error* formulation, where the same conditions of Eq. (11) hold, but the horizontal components of position are upper-bounded by some slack variables that are penalized in the cost function [23, 24]. In this work we consider the set of boundary conditions given by Eqs. (9), (10), with the final mass left free to be determined, with the assumption that t_0 is known, and t_F needs in general to be determined as well. We will therefore focus on the *pinpoint landing* formulation.

B. A Different Control Parameterization

Independent of the specific landing conditions, the above 3-DoF formulation is convenient for vacuum flight, as it is compatible with further constraints, such as thrust pointing limitations and glideslope constraints [1]. When we deal with endo-atmospheric flight, (and therefore, the aerodynamic effects do play a role), the control representation in terms of thrust direction is potentially troublesome, because the aerodynamic forces intrinsically depend on the attitude of the body axes with respect to the wind axes, rather than on the target-related UEN reference frame. This is why in Ref. [17] the starting point is given by the body axes of the rocket. Moreover, the aerodynamic effects are generally just coped with rather than actively exploited. To allow a more effective exploitation of the aerodynamic effects we take here a similar route as in Ref. [17]. We define as controls the throttle η (or equivalently, the thrust magnitude T), the angle of attack α , and the sideslip angle β , which completely describe how the body axes are oriented with respect to the external

axes, (in the hypothesis of zero bank angle). The new controls must satisfy some corresponding box constraints.

$$\begin{cases} \eta_{min} \leq \eta \leq \eta_{max} \\ \alpha_{min} \leq \alpha \leq \alpha_{max} \\ \beta_{min} \leq \beta \leq \beta_{max} \end{cases} \quad (12)$$

Additional degrees of freedom could come from using the TVC deflections as extra controls. However, their use would complicate the formulation, and it would require the explicit consideration of their effect in terms of the torque as well, making a full 6-DoF analysis necessary. It was therefore decided to exclude them and to keep the 3-DoF formulation, which is more useful for the classical design of cascaded control systems. To formulate the OCP with respect to the controls defined in Eq. (12) it is necessary to build a connection between α and β and the body axes $\mathbf{1}_{x,e}$, $\mathbf{1}_{y,e}$, and $\mathbf{1}_{z,e}$ with respect to the external reference frame. Let us rewrite the terms $\mathbf{F}_e^T$ and $\mathbf{F}_e^A$. For the thrust we can write it as

$$\mathbf{F}_e^T = \eta T_{max} \cdot (\mathbf{R}_b^e \cdot \mathbf{1}_{x,b}) \quad (13)$$

where $\mathbf{R}_b^e$ is the Direction Cosine Matrix (DCM) from the body reference frame to the external reference frame. Note that in the following we will use ηT_{max} or T as equivalent terms to make the notation lighter when required. For the aerodynamic force $\mathbf{F}_e^A$ we can equivalently write

$$\mathbf{F}_e^A = \mathbf{R}_b^e \cdot (X\mathbf{1}_{x,b} + Y\mathbf{1}_{y,b} + Z\mathbf{1}_{z,b}) \quad (14)$$

with the terms X , Y , and Z representing the force components along the body axes $\mathbf{1}_{x,b}$, $\mathbf{1}_{y,b}$, $\mathbf{1}_{z,b}$, and given by

$$W = \frac{1}{2}\rho V^2 S_{ref} C_w, \quad W = X, Y, Z, \quad w = x, y, z \quad (15)$$

where ρ is the atmospheric density, $V^2 = \mathbf{v}^T \mathbf{v}$, S_{ref} the reference surface of the rocket, and C_w the corresponding aerodynamic coefficient expressed in the respective body axis $\mathbf{1}_{x,b}$, $\mathbf{1}_{y,b}$, and $\mathbf{1}_{z,b}$. By virtue of Eqs. (13) and (14) we can re-write the equations of motion as

$$\begin{cases} \dot{\mathbf{r}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \mathbf{g} + \frac{1}{m} \mathbf{R}_b^e \cdot [(\eta T_{max} + X)\mathbf{1}_{x,b} + Y\mathbf{1}_{y,b} + Z\mathbf{1}_{z,b}] \\ \dot{m} &= -\eta \frac{T_{max}}{c} \end{cases} \quad (16)$$

However, Eq. (16) does not show two of our three controls yet, i.e., α and β . To explicitly see the controls we can expand the term $\mathbf{R}_b^e$ as follows,

$$\mathbf{R}_b^e = \mathbf{R}_w^e \cdot \mathbf{R}_b^w \quad (17)$$

where the new two DCMs $\mathbf{R}_w^e$ and $\mathbf{R}_b^w$ represent the transformation from wind axes to external reference frame, and from body to wind axes, respectively. Let us see how these two matrices are represented. The transformation $\mathbf{R}_b^w$ can be expressed as (see Ref. [25])

$$\mathbf{R}_b^w = \begin{bmatrix} \cos \alpha \cos \beta & \sin \beta & \sin \alpha \cos \beta \\ -\cos \alpha \sin \beta & \cos \beta & -\sin \alpha \sin \beta \\ -\sin \alpha & 0 & \cos \alpha \end{bmatrix} \quad (18)$$

The columns of this matrix can be interpreted as the body axes expressed in the wind reference frame, that is,

$$\begin{cases} \mathbf{1}_{x,w} &= [\cos \alpha \cos \beta, \quad -\cos \alpha \sin \beta, \quad -\sin \alpha]^T \\ \mathbf{1}_{y,w} &= [\sin \beta, \quad \cos \beta, \quad 0]^T \\ \mathbf{1}_{z,w} &= [\sin \alpha \cos \beta, \quad -\sin \alpha \sin \beta, \quad \cos \alpha]^T \end{cases} \quad (19)$$

The relationship between the body axes with respect to the wind reference frame and the two aerodynamic controls α and β is therefore completely characterized. The next step is to extend this connection to the dynamics given by Eq. (16).

C. Singularity-free Flight-Path Angle and Azimuth Angle for Rocket Landing

The matrix $\mathbf{R}_w^e$ identifies how the wind axes are oriented with respect to the external reference frame $e = \mathbf{UEN}$, centered in the target position. In aeronautics it is typical to characterize the orientation of the wind axes with respect to the local reference frame $\mathbf{NED}$ by using the heritage flight-path angle γ_h and the velocity azimuth angle ψ_h [26]. So, the transformation from wind coordinates to $\mathbf{NED}$ would simply be given by

$$\mathbf{R}_w^{NED} = \mathbf{R}_3(-\psi_h) \cdot \mathbf{R}_2(-\gamma_h) \quad (20)$$

where γ_h and ψ_h are defined as

$$\gamma_h = \tan^{-1} \left[-\frac{\tilde{w}}{(\tilde{u}^2 + \tilde{v}^2)^{\frac{1}{2}}} \right] \quad \psi_h = \tan^{-1} \left(\frac{\tilde{v}}{\tilde{u}} \right) \quad (21)$$

and with $\tilde{u}$, $\tilde{v}$, and $\tilde{w}$ representing the velocity components along *North*, *East*, and *Down*, respectively, and $\mathbf{R}_k(\theta)$ a rotation of an angle θ around the k axis. The transformation chain from wind axes to the external reference frame e

would then be

$$\mathbf{R}_w^e = \mathbf{R}_{UEN}^e \cdot \mathbf{R}_{NED}^{UEN} \cdot \mathbf{R}_w^{NED} \quad (22)$$

The right-hand side of Eq. (22) has now three terms. The last one, that is, $\mathbf{R}_w^{NED}$ is characterized by Eq. (20), while the middle term $\mathbf{R}_{NED}^{UEN}$ is a constant matrix.

$$\mathbf{R}_{NED}^{UEN} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad (23)$$

Finally, in the hypothesis that the *body*-centered (and not the target-centered) UEN can be approximated with the target-centered external frame e (in other words, neglecting the curvature of the Earth, which is a common assumption when target-centered reference frames are in use, due to the limited horizontal range that the rocket has to cover), We can say that $\mathbf{R}_{UEN}^e$ is equal to the 3×3 identity matrix $\mathbf{I}_3$. Equation (22) is therefore completely defined. However, there is a problem: although the reference frame transformations are formally defined, such transformation chains are not well-posed for a rocket-landing application (remember the aeronautical origin of such definitions, i.e., for horizontal flight). In fact, the landing profile will tend to have $\gamma_h = -\frac{\pi}{2}$, meaning that $\mathbf{R}_w^{NED}$ becomes singular. The practical consequence is a lack of meaning of the azimuth angle ψ_h , which will strongly vary for tiny variations in the horizontal components v , w of the velocity vector $\mathbf{v}$. A better strategy can be obtained by redefining two new angles: the *vertical flight-path angle* γ_v and the *vertical azimuth angle* ψ_v as follows:

$$\gamma_v \triangleq \tan^{-1} \left(\frac{w}{u} \right) \cong \frac{w}{u}, \quad \psi_v \triangleq \tan^{-1} \left(\frac{v}{(u^2+w^2)^{\frac{1}{2}}} \right) \cong \frac{v}{(u^2+w^2)^{\frac{1}{2}}} \quad (24)$$

These unusual definitions are conceived to work well with rocket-landing applications, since they are always well defined for a non-zero vertical component of the velocity (Fig. 2). With these definitions we can redefine Eq. (22) as

$$\mathbf{R}_w^e = \mathbf{R}_{UEN}^e \cdot \mathbf{R}_w^{UEN} \cong \mathbf{R}_w^{UEN} \quad (25)$$

with $\mathbf{R}_w^{UEN} = \mathbf{R}_2(\gamma_v) \cdot \mathbf{R}_3(-\psi_v)$ or more explicitly as follows,

$$\mathbf{R}_w^e \cong \mathbf{R}_w^{UEN} = \begin{bmatrix} \cos \gamma_v \cos \psi_v & -\sin \psi_v \cos \gamma_v & -\sin \gamma_v \\ \sin \psi_v & \cos \psi_v & 0 \\ \sin \gamma_v \cos \psi_v & -\sin \gamma_v \sin \psi_v & \cos \gamma_v \end{bmatrix} \quad (26)$$

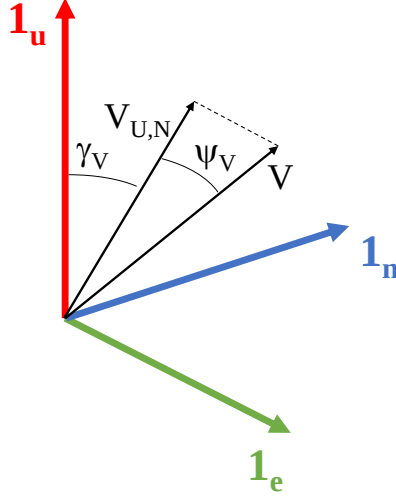


Fig. 2 Vertical angles γ_v and ψ_v to handle the transformation from the wind reference frame to the external one without singularity.

and, for almost-vertical flights the matrix could be further approximated as follows.

$$\mathbf{R}_w^e \cong \mathbf{R}_w^{UEN} \cong \begin{bmatrix} -1 & \psi_v & \gamma_v - \pi \\ \psi_v & 1 & 0 \\ \pi - \gamma_v & 0 & -1 \end{bmatrix} \quad (27)$$

It is worth noting that, while $\mathbf{R}_b^w$ contains explicitly our controls α and β , $\mathbf{R}_w^{UEN}$ depends only on the velocity vector $\mathbf{v}$ through the definitions of Eq. (24). We are now ready to reformulate the equations of motion in a the following operative form, completing the definition of our OCP.

$$\begin{cases} \dot{\mathbf{r}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \mathbf{g} + \frac{1}{m} \mathbf{R}_w^e \cdot [(\eta T_{max} + X) \mathbf{1}_{x,w} + Y \mathbf{1}_{y,w} + Z \mathbf{1}_{z,w}] \\ \dot{m} &= -\eta \frac{T_{max}}{c} \end{cases} \quad (28)$$

The system of Eq. (28) makes now explicit use of the body axes of the rocket expressed with respect to the wind reference frame, that is, the unit vectors $\mathbf{1}_{y,w}$, $\mathbf{1}_{y,w}$, and $\mathbf{1}_{y,w}$, as defined in Eq. (19).

D. Hamiltonian and Lagrangian for the Optimal Endo-atmospheric Landing Problem

With the equations of motion (Eq. (28)), the boundary conditions ((Eq. (10))), and the cost function (Eq. (3)) defined it follows that the OCP to be solved is completely characterized. We can therefore focus on the construction of

the necessary conditions for optimality. The first step is to build the Hamiltonian and the corresponding Lagrangian of the system by using Eq. (3) as cost function. Defined the costate vector $\mathbf{p} \triangleq [\mathbf{p}_r^T \ \mathbf{p}_v^T \ p_m]^T \in \mathbb{R}^7$, the Hamiltonian H is

$$H = \eta + \mathbf{p}_r^T \mathbf{v} + \mathbf{p}_v^T \mathbf{g} + \mathbf{p}_v^T \cdot \left\{ \frac{1}{m} \mathbf{R}_w^e \cdot [(\eta T_{max} + X) \mathbf{1}_{x,w} + Y \mathbf{1}_{y,w} + Z \mathbf{1}_{z,w}] \right\} - p_m \eta \frac{T_{max}}{c} \quad (29)$$

and the Lagrangian function L reads

$$L = H + \mu_1(\eta_{min} - \eta) + \mu_2(\eta - \eta_{max}) + \mu_3(\alpha_{min} - \alpha) + \mu_4(\alpha - \alpha_{max}) + \mu_5(\beta_{min} - \beta) + \mu_6(\beta - \beta_{max}) \quad (30)$$

with the complementary conditions affecting the non-negative Lagrange multipliers μ_i , $i \in [1, \dots, 6]$ given by the following expressions.

$$\mu_1 \begin{cases} > 0, & \eta = \eta_{min} \\ = 0, & \eta > \eta_{min} \end{cases}, \quad \mu_2 \begin{cases} > 0, & \eta = \eta_{max} \\ = 0, & \eta < \eta_{max} \end{cases} \quad (31)$$

$$\mu_3 \begin{cases} > 0, & \alpha = \alpha_{min} \\ = 0, & \alpha > \alpha_{min} \end{cases}, \quad \mu_4 \begin{cases} > 0, & \alpha = \alpha_{max} \\ = 0, & \alpha < \alpha_{max} \end{cases} \quad (32)$$

$$\mu_5 \begin{cases} > 0, & \beta = \beta_{min} \\ = 0, & \beta > \beta_{min} \end{cases}, \quad \mu_6 \begin{cases} > 0, & \beta = \beta_{max} \\ = 0, & \beta < \beta_{max} \end{cases} \quad (33)$$

With these definitions it is possible to derive the necessary conditions for optimality, that are developed in the next sections.

E. Fuel-Optimal Throttle/Thrust

Re-express the Hamiltonian as

$$H \triangleq H_0 + S\eta \quad (34)$$

with H_0 representing the collection of the terms independent of the thrust, and S being the switching function defined by

$$S \triangleq \frac{\partial H}{\partial \eta} = 1 + T_{max} \left[\frac{1}{m} \mathbf{p}_v^T \cdot (\mathbf{R}_w^e \cdot \mathbf{1}_{x,w}) - \frac{p_m}{c} \right] \quad (35)$$

and, since the thrust appears linearly in the Hamiltonian, we must invoke the Pontryagin Minimum Principle (PMP), which gives us the optimal control η^* as function of S :

$$\eta^* = \frac{\eta_{min} + \eta_{max}}{2} + \text{sign}(S) \left[\frac{\eta_{min} - \eta_{max}}{2} \right] \quad (36)$$

In accordance with the PMP, when S is positive, a minimum throttle $\eta^* = \eta_{min}$ will be issued, while for negative values of S we will have $\eta^* = \eta_{max}$. For the moment we assume that S is different from 0 almost everywhere (i.e., there is only a finite number of instantaneous times t^* at which S is 0). The analysis of potential singular arcs requires further differentiation of S , that will be carried out in Sec. III.D. Note however that, despite the different control parametrization, the switching function of Eq. (35) is identical to Eq. (A10) in Ref. [3], if we consider that $\mathbf{R}_w^e \cdot \mathbf{1}_{x,w} = \mathbf{1}_{x,e}$ ($\mathbf{1}_T$ in the original reference).

F. Energy-Optimal Throttle/Thrust

So far we considered fuel-optimal criteria in virtue of Eqs. (1)-(3). However, we might also be interested to energy-optimal solutions, whose cost function can be formulated as

$$\min_{\mathbf{u}, t_F} J = \int_{t_0}^{t_F} \eta^2(t) dt \quad (37)$$

By borrowing the idea from low-thrust optimization community [27], the cost functions Eq. (3) and Eq. (37) can be unified by introducing a parameter $\epsilon \in [0, 1]$, and defining the new cost function as follows.

$$\min_{\mathbf{u}, t_F} J = \int_{t_0}^{t_F} [(1 - \epsilon)\eta^2(t) + \epsilon\eta(t)] dt \quad (38)$$

The above cost function clearly reduces to Eq. (37) when $\epsilon = 0$ and Eq. (3) when $\epsilon = 1$. Any intermediate value of ϵ between 0 and 1 refers to energy-fuel optimal control problems, which are typically used in continuation schemes (e.g., [27, 28]). The use of energy-optimal formulations is also useful for two more reasons. First, the presence of η in quadratic form in Eq. (38) allows for an explicit solution of the throttle signal as a function of states and costates. Second, the energy-optimal formulation provides a milder testcase given that, at least for the results obtained here, the throttle profile is a continuous, smooth function. Let us derive the optimal throttle in this case. By imposing the stationary condition of the Hamiltonian with respect to the throttle η we obtain

$$\frac{\partial H}{\partial \eta} = 2(1 - \epsilon)\eta + \epsilon + \frac{T_{max}}{m} \mathbf{p}_v^T \cdot (\mathbf{R}_w^e \cdot \mathbf{1}_{x,w}) - p_m \frac{T_{max}}{c} = 0 \quad (39)$$

from which η can be solved as follows.

$$\eta = \frac{p_m \frac{T_{max}}{c} - \frac{T_{max}}{m} \mathbf{p}_v^T \cdot (\mathbf{R}_w^e \cdot \mathbf{1}_{x,w}) - \epsilon}{2(1 - \epsilon)} \quad (40)$$

Equation (40) holds for any $\epsilon \in [0, 1)$. When $\epsilon = 1$ the quadratic term in Eq. (38) disappears, and Eq. (40) goes to infinity. In this case the Hamiltonian depends linearly on η , and therefore the PMP in its hard form must be applied, i.e., by looking at Eqs. (35) and (36). Note however that the fuel-optimal and the energy-optimal problems, and any intermediate case between these two extremes obey exactly the same set of differential equations, whose solutions will only differ because of the different time histories of η , that will enter the costate differential equations and the necessary conditions for the optimality of the aerodynamic controls, as we will see.

Finally, we note that we did not consider thrust rate limits, because the theoretical fuel-optimal thrust is bang-bang. If the limits on the throttle rate need to be considered, one could use a standard control augmentation technique to treat thrust as a state and the thrust rate as a control, subject to upper and lower bound. But doing so would significantly complicate the analysis of the problem. In a most recent work in Ref. [29], it is shown that a slowly varying throttle (such as a constant throttle) in a fuel-optimal powered descent problem can yield a solution with a propellant performance close to that by the bang-bang solution. If limited throttle rate is a major concern, the approach in Ref. [29] may be used.

G. Optimal Angle of Attack

The analysis of the optimality in terms of angle of attack α can be performed by setting

$$\frac{\partial H}{\partial \alpha} = \frac{1}{m} \mathbf{p}_v^T \cdot \left\{ \mathbf{R}_w^e \cdot \left[X_\alpha \mathbf{1}_{x,w} + Y_\alpha \mathbf{1}_{y,w} + Z_\alpha \mathbf{1}_{z,w} + (\eta T_{max} + X) \frac{\partial \mathbf{1}_{x,w}}{\partial \alpha} + Y \frac{\partial \mathbf{1}_{y,w}}{\partial \alpha} + Z \frac{\partial \mathbf{1}_{z,w}}{\partial \alpha} \right] \right\} = 0 \quad (41)$$

with

$$\begin{cases} \frac{\partial \mathbf{1}_{x,w}}{\partial \alpha} &= [-\sin \alpha \cos \beta, \quad \sin \alpha \sin \beta, \quad -\cos \alpha]^T = -\mathbf{1}_{z,w} \\ \frac{\partial \mathbf{1}_{y,w}}{\partial \alpha} &= [0, \quad 0, \quad 0]^T \\ \frac{\partial \mathbf{1}_{z,w}}{\partial \alpha} &= [\cos \alpha \cos \beta, \quad -\cos \alpha \sin \beta, \quad -\sin \alpha]^T = \mathbf{1}_{x,w} \end{cases} \quad (42)$$

By plugging Eq. (42) into Eq. (41), and considering the fact that m is always finite, we get the more compact expression

$$\frac{\partial H}{\partial \alpha} = \mathbf{p}_v^T \cdot \{ \mathbf{R}_w^e \cdot [(X_\alpha + Z) \mathbf{1}_{x,w} + Y_\alpha \mathbf{1}_{y,w} + (Z_\alpha - \eta T_{max} - X) \mathbf{1}_{z,w}] \} = 0 \quad (43)$$

which is a nonlinear equation in the control components α and β , is a function of the state $\mathbf{x}$, the costate $\mathbf{p}$, and the throttle η , and represents a generalization of Eq. (32) in Ref. [17]. In fact, the form proposed here is also valid for $Y \neq 0$ and $\beta \neq 0$. In many case H may be reasonably approximated by a quadratic function of α , meaning that H is unimodal

with respect to α . Let α_{unc}^* be the angle of attack solution from Eq. (43). When α_{unc}^* exceeds one of the bounds, the unimodality of H implies that the admissible minimizing α is at the bound:

$$\alpha^* = \begin{cases} \alpha_{max}, & \alpha_{unc}^* > \alpha_{max} \\ \alpha_{unc}^*, & \alpha_{min} \leq \alpha_{unc}^* \leq \alpha_{max} \\ \alpha_{min}, & \alpha_{unc}^* < \alpha_{min} \end{cases} \quad (44)$$

H. Optimal Sideslip Angle

The corresponding analysis of the optimality in terms of sideslip angle β can be performed by setting

$$\frac{\partial H}{\partial \beta} = \frac{1}{m} \mathbf{p}_v^T \cdot \left\{ \mathbf{R}_w^e \cdot \left[X_\beta \mathbf{1}_{x,w} + Y_\beta \mathbf{1}_{y,w} + Z_\beta \mathbf{1}_{z,w} + (\eta T_{max} + X) \frac{\partial \mathbf{1}_{x,w}}{\partial \beta} + Y \frac{\partial \mathbf{1}_{y,w}}{\partial \beta} + Z \frac{\partial \mathbf{1}_{z,w}}{\partial \beta} \right] \right\} = 0 \quad (45)$$

or, equivalently,

$$\frac{\partial H}{\partial \beta} = \mathbf{p}_v^T \cdot \left\{ \mathbf{R}_w^e \cdot \left[X_\beta \mathbf{1}_{x,w} + Y_\beta \mathbf{1}_{y,w} + Z_\beta \mathbf{1}_{z,w} + (\eta T_{max} + X) \frac{\partial \mathbf{1}_{x,w}}{\partial \beta} + Y \frac{\partial \mathbf{1}_{y,w}}{\partial \beta} + Z \frac{\partial \mathbf{1}_{z,w}}{\partial \beta} \right] \right\} = 0 \quad (46)$$

with $\frac{\partial H}{\partial \beta}$ showing the same dependencies as Eq. (43), and

$$\begin{cases} \frac{\partial \mathbf{1}_{x,w}}{\partial \beta} &= [-\cos \alpha \sin \beta, -\cos \alpha \cos \beta, 0]^T \\ \frac{\partial \mathbf{1}_{y,w}}{\partial \beta} &= [\cos \beta, -\sin \beta, 0]^T \\ \frac{\partial \mathbf{1}_{z,w}}{\partial \beta} &= [-\sin \alpha \sin \beta, -\sin \alpha \cos \beta, 0]^T \end{cases} \quad (47)$$

Let β_{unc}^* be the sideslip angle solution from Eq. (46). Under the assumption that H is unimodal with respect to β for $\beta \in [\beta_{min}, \beta_{max}]$, we obtain a similar expression for the optimal sideslip angle β^* as for the angle of attack

$$\beta^* = \begin{cases} \beta_{max}, & \beta_{unc}^* > \beta_{max} \\ \beta_{unc}^*, & \beta_{min} \leq \beta_{unc}^* \leq \beta_{max} \\ \beta_{min}, & \beta_{unc}^* < \beta_{min} \end{cases} \quad (48)$$

Equation (46) is a second nonlinear equation in the control components α and β , that, together with Eq. (41), (and by considering Eq. (36), which is therefore also a function of α and β) can be solved for the optimal α and β . Note that the vectors $\mathbf{1}_{x,w}$, $\mathbf{1}_{y,w}$, $\mathbf{1}_{z,w}$, as well as their derivatives, are also function of α and β in virtue of Eqs. (19), (42), and (47).

III. Special Cases

This section harmonizes the proposed formulation with some known results available in the literature on optimal endo-atmospheric rocket flight by exploring the cases, in particular, the small-angles simplification, and the coordinated-turn maneuvers. In addition, we will show that coordinated turn is generally not optimal if sideslip angle modulation is allowed. Finally, in Sec. III.D we explore the intricacies coming from the possibility to admit singular optimal thrust profiles during the endo-atmospheric rocket landing.

A. Small-Angles Model

With the hypothesis of small aerodynamic angles during the descent, that is

$$|\pi - \alpha| \ll 1, \quad |\beta| \ll 1 \quad (49)$$

we may use the following first order approximations near $\alpha = \pi$ and $\beta = 0$:

$$\begin{cases} \sin \alpha \cong \pi - \alpha & \cos \alpha \cong -1 \\ \sin \beta \cong \beta & \cos \beta \cong 1 \end{cases} \quad (50)$$

By replacing Eq. (50) into Eq. (19) and Eq. (42) we get

$$\begin{cases} \mathbf{1}_{x,w} \cong [-1, \beta, \alpha - \pi]^T \\ \mathbf{1}_{y,w} \cong [\beta, 1, 0]^T \\ \mathbf{1}_{z,w} \cong [\pi - \alpha, 0, -1]^T \\ \frac{\partial \mathbf{1}_{x,w}}{\partial \beta} \cong [\beta, 1, 0]^T = \mathbf{1}_{y,w} \\ \frac{\partial \mathbf{1}_{y,w}}{\partial \beta} \cong [1, -\beta, 0]^T = -\mathbf{1}_{x,w} + (\alpha - \pi)\hat{\mathbf{k}} \\ \frac{\partial \mathbf{1}_{z,w}}{\partial \beta} \cong [0, \alpha - \pi, 0]^T = (\alpha - \pi)\hat{\mathbf{j}} \end{cases} \quad (51)$$

with the unit vectors $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ fixed and given by the following expressions.

$$\hat{\mathbf{j}} = [0, 1, 0]^T, \quad \hat{\mathbf{k}} = [0, 0, 1]^T \quad (52)$$

We can replace Eq. (51) and Eq. (27) in the Hamiltonian defined in Eq. (29), to get the small-angles approximated expression of the Hamiltonian. First, let us write the dot products between $\mathbf{R}_w^e$ and the unit vectors $\mathbf{1}_{x,w}$, $\mathbf{1}_{y,w}$, $\mathbf{1}_{z,w}$.

$$\left\{ \begin{array}{lll} \mathbf{R}_w^e \cdot \mathbf{1}_{x,w} & = & [1 + \beta\psi_v + (\alpha - \pi)(\gamma_v - \pi), \quad \beta - \psi_v, \quad \gamma_v - \alpha]^T \cong [1, \quad \beta - \psi_v, \quad \gamma_v - \alpha]^T \\ \mathbf{R}_w^e \cdot \mathbf{1}_{y,w} & = & [\psi_v - \beta, \quad 1 + \beta\psi_v, \quad \beta(\pi - \gamma_v)]^T \cong [\psi_v - \beta, \quad 1, \quad 0]^T \\ \mathbf{R}_w^e \cdot \mathbf{1}_{z,w} & = & [\alpha - \pi + \pi - \gamma_v, \quad \psi_v(\pi - \alpha), \quad 1 + (\pi - \alpha)(\pi - \gamma_v)]^T \cong [\alpha - \gamma_v, \quad 0, \quad 1]^T \end{array} \right. \quad (53)$$

With these definitions, and the explicit use of the components of the primer vector $\mathbf{p}_v = [p_u, p_v, p_w]^T$ as well as the ones associated with the co-position $\mathbf{p}_r = [p_x, p_y, p_z]^T$ the small-angles approximated Hamiltonian is then:

$$\begin{aligned} H &= p_x u + p_y v + p_z w + p_u \left\{ g_x + \frac{1}{m} [(\eta T_{max} + X) + Y(\psi_v - \beta) + Z(\alpha - \gamma_v)] \right\} + \\ &\quad + p_v \left\{ g_y + \frac{1}{m} [(\eta T_{max} + X)(\beta - \psi_v) + Y] \right\} + p_w \left\{ g_z + \frac{1}{m} [(\eta T_{max} + X)(\gamma_v - \alpha) + Z] \right\} - p_m \eta \frac{T_{max}}{c} \end{aligned} \quad (54)$$

In this expression it might seem that the Hamiltonian depends linearly on the angles α and β . However this is in general not the case, since the aerodynamic force components X , Y , and Z depend on these angles as well. The partial derivatives of the Hamiltonian with respect to the angles α and β give

$$\left\{ \begin{array}{l} \frac{\partial H}{\partial \alpha} = q_{dyn} S \frac{p_u}{m} [C_{x,\alpha} + C_{z,\alpha}(\alpha - \gamma_v) + C_z] + \\ \quad + q_{dyn} S \frac{p_v}{m} C_{x,\alpha} + \frac{p_w}{m} [q_{dyn} S C_{x,\alpha}(\gamma_v - \alpha) - (\eta T_{max} + X) + q_{dyn} S C_{z,\alpha}] = 0 \\ \frac{\partial H}{\partial \beta} = q_{dyn} S \frac{p_u}{m} [C_{x,\beta} + C_{y,\beta}(\psi_v - \beta) - C_y] + \\ \quad + \frac{p_v}{m} [q_{dyn} S C_{x,\beta}(\beta - \psi_v) + (\eta T_{max} + X) + q_{dyn} S C_{y,\beta}] + q_{dyn} S \frac{p_w}{m} [C_{x,\beta}(\gamma_v - \alpha) + C_z] = 0 \end{array} \right. \quad (55)$$

and, by further assuming that the sideforce does not depend on α ($Y_\alpha = 0$), and that the normal force does not depend on β ($Z_\beta = 0$), we can rewrite Eqs. (41) and (45) as follows.

$$\left\{ \begin{array}{l} \frac{\partial H}{\partial \alpha} = \mathbf{p}_v^T \cdot \{ \mathbf{R}_w^e \cdot [(X_\alpha + Z)\mathbf{1}_{x,w} - (\eta T_{max} + X - Z_\alpha)\mathbf{1}_{z,w}] \} = 0 \\ \frac{\partial H}{\partial \beta} = \mathbf{p}_v^T \cdot \{ \mathbf{R}_w^e \cdot [(X_\beta - Y)\mathbf{1}_{x,w} + (\eta T_{max} + X + Y_\beta)\mathbf{1}_{y,w} + (\alpha - \pi)(Y\hat{\mathbf{k}} + Z\hat{\mathbf{j}})] \} = 0 \end{array} \right. \quad (56)$$

We could develop the dot products and solve numerically Eq. (56). However, since $\mathbf{R}_w^e$ does not depend on the controls, we can approach the problem differently: let us rewrite Eq. (56) as

$$\left\{ \begin{array}{l} \frac{\partial H}{\partial \alpha} = \mathbf{p}_v^T \cdot (\mathbf{R}_w^e \cdot \mathbf{q}_\alpha) = 0 \\ \frac{\partial H}{\partial \beta} = \mathbf{p}_v^T \cdot (\mathbf{R}_w^e \cdot \mathbf{q}_\beta) = 0 \end{array} \right. \quad (57)$$

where $\mathbf{q}_\alpha$ and $\mathbf{q}_\beta$ are defined as follows.

$$\begin{cases} \mathbf{q}_\alpha & \triangleq (X_\alpha + Z)\mathbf{1}_{x,w} - (\eta T_{max} + X - Z_\alpha)\mathbf{1}_{z,w} \\ \mathbf{q}_\beta & \triangleq (X_\beta - Y)\mathbf{1}_{x,w} + (\eta T_{max} + X + Y_\beta)\mathbf{1}_{y,w} + (\alpha - \pi)(Y\hat{\mathbf{k}} + Z\hat{\mathbf{j}}) \end{cases} \quad (58)$$

We can invoke the following scalar identity involving two general vectors $\mathbf{p}$, $\mathbf{q}$, and a rotation matrix $\mathbf{R}$, based on the observation that the norms of vectors and the angle between two vectors are invariant with respect to arbitrary rotations.

$$\mathbf{p}^T \cdot (\mathbf{R} \cdot \mathbf{q}) = (\mathbf{R}^T \cdot \mathbf{p})^T \cdot \mathbf{q} \quad (59)$$

This expression, applied to our case, only means that we are evaluating the same dot product expressed with respect to the wind reference frame, and not with respect to the external one. However, this expression allows us to rewrite Eq. (57) in a more compact form, that is

$$\begin{cases} \tilde{\mathbf{p}}_v^T \cdot [(X_\alpha + Z)\mathbf{1}_{x,w} - (\eta T_{max} + X - Z_\alpha)\mathbf{1}_{z,w}] & = \tilde{\mathbf{p}}_v^T \cdot \mathbf{q}_\alpha = 0 \\ \tilde{\mathbf{p}}_v^T \cdot [(X_\beta - Y)\mathbf{1}_{x,w} + (\eta T_{max} + X + Y_\beta)\mathbf{1}_{y,w} + (\alpha - \pi)(Y\hat{\mathbf{k}} + Z\hat{\mathbf{j}})] & = \tilde{\mathbf{p}}_v^T \cdot \mathbf{q}_\beta = 0 \end{cases} \quad (60)$$

with $\tilde{\mathbf{p}}_v$ implicitly defined as follows.

$$\tilde{\mathbf{p}}_v \triangleq \mathbf{R}_e^w \cdot \mathbf{p}_v \quad (61)$$

The optimal angles α^* and β^* will therefore be such that the vectors $\mathbf{q}_\alpha$ and $\mathbf{q}_\beta$ will always be orthogonal to $\tilde{\mathbf{p}}_v$ by virtue of Eq. (60). In the following we will show that $\mathbf{q}_\alpha$ and $\mathbf{q}_\beta$ cannot be parallel to each other. Therefore the two relationships in Eq. (60) are linearly independent, a necessary condition for the existence of their solution.

Theorem III.1. *The vectors $\mathbf{q}_\alpha$ and $\mathbf{q}_\beta$ in general are not parallel to each other, hence the conditions in Eq. (60) admit a unique solution for α and β .*

Proof. Let us suppose that $\mathbf{q}_\alpha$ and $\mathbf{q}_\beta$ are parallel. This assumption is equivalent to say that there exists a scalar constant $s \in \mathbb{R}$ such that the following conditions hold.

$$\begin{cases} X_\alpha + Z & = s(X_\beta - Y) \\ -(\eta T_{max} + X - Z_\alpha)\mathbf{1}_{z,w} & = s[(\eta T_{max} + X + Y_\beta)\mathbf{1}_{y,w} + (\alpha - \pi)(Y\hat{\mathbf{k}} + Z\hat{\mathbf{j}})] \end{cases} \quad (62)$$

First, let us consider the case $s \in \mathbb{R}_{\neq 0}$. Note that the second condition is a vector identity, which we can rewrite as

$$-s(\eta T_{max} + X + Y_\beta)\mathbf{1}_{y,w} - (\eta T_{max} + X - Z_\alpha)\mathbf{1}_{z,w} = (\alpha - \pi)(Y\hat{\mathbf{k}} + Z\hat{\mathbf{j}}) \quad (63)$$

This condition implies that at each instant of time the linear combination of the wind unit vectors $\mathbf{1}_{y,w}$ and $\mathbf{1}_{z,w}$, with coefficients $-s(\eta T_{max} + X + Y_\beta)$ and $-(\eta T_{max} + X - Z_\alpha)$, respectively, must be exactly equal to the linear combination of the fixed unit vectors $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ with coefficients Z and Y . For this condition to be possible, the two sets of unit vectors must belong to the same fixed plane formed by $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$. If we evaluate the dot product of the two sides of Eq. (63) once with $\hat{\mathbf{j}}$ and once with $\hat{\mathbf{k}}$ we obtain

$$\begin{cases} -s(\eta T_{max} + X + Y_\beta)\mathbf{1}_{y,w} \cdot \hat{\mathbf{j}} - (\eta T_{max} + X - Z_\alpha)\mathbf{1}_{z,w} \cdot \hat{\mathbf{j}} &= (\alpha - \pi)Z \\ -s(\eta T_{max} + X + Y_\beta)\mathbf{1}_{y,w} \cdot \hat{\mathbf{k}} - (\eta T_{max} + X - Z_\alpha)\mathbf{1}_{z,w} \cdot \hat{\mathbf{k}} &= (\alpha - \pi)Y \end{cases} \quad (64)$$

and, taking Eq. (51) into account, we can rewrite Eq. (64) as

$$\begin{cases} -s(\eta T_{max} + X + Y_\beta) &= s(\alpha - \pi)Z \\ (\eta T_{max} + X - Z_\alpha) &= s(\alpha - \pi)Y \end{cases} \quad (65)$$

These two expressions tell us that the only s satisfying these conditions is

$$s = \frac{\eta T_{max} + X - Z_\alpha}{(\alpha - \pi)Y} \quad (66)$$

and, at the same time, it must hold that α , which affects X , Z_α and appears explicitly in the right-hand side of the second expression in Eq. (65), and β (here embedded in Y and X) must satisfy the following relationship.

$$-(\eta T_{max} + X + Y_\beta) = (\alpha - \pi)Z \quad (67)$$

Now, while it is not possible to algebraically exclude that Eq. (67) can be satisfied, we want to highlight a series of arguments showing why Eq. (67) cannot in general hold physically. In first place, for a given rocket, the aerodynamic behavior is characterized a priori once its shape is frozen, and such a behavior represents an input to the flight dynamics equations. Consequently, we argue that, from the aerodynamic perspective, it is with probability zero that the aerodynamic coefficients of the rockets will satisfy Eq. (67). In second place, if we freeze the shape and the size of the rocket, and assume Eq. (67) to be true, one could argue that a different engine (and therefore a different T_{max}) would then violate such a condition, making it therefore generally not true. Lastly, from the point of view of the order of magnitude of the forces under analysis, in first approximation it is reasonable to assume a quadratic dependence of X on α . If we consider Y to be linearly dependent on β , it follows that Y_β is independent from β itself. On the right-hand side, it is reasonable to assume Z mainly a function of α , and independent from β . Finally, ηT_{max} can be seen as a discrete variable (assuming T_{min} or T_{max} , in the absence of singular extremals) of states and costates, and therefore, an indirect

piecewise-constant function of α and β . In conclusions, we have all functional dependencies of the terms within Eq. (67) from α and β which are different from each other, making Eq. (67) impossible to hold all the time. Hence, We conclude that Eq. (67) is in general not true, and therefore the vectors $\mathbf{q}_\alpha$ and $\mathbf{q}_\beta$ cannot be parallel. $\square$

The procedure to obtain the optimal angle of attack and the optimal sideslip angle reduces therefore to the resolution of a system of two equations given by Eq. (60) into two unknowns, which can be performed very efficiently with a Newton-Raphson method. The initial guess can be supplied assuming that the rocket body $\mathbf{1}_{b,x}$ is antiparallel to the velocity vector (i.e., $\alpha_{ig} = \pi$, $\beta_{ig} = 0$). In case one of the two controls exceed the maximum or minimum value, the corresponding stationary equation will be discarded and replaced by the associated saturation condition, reducing the algorithm to a one-parameter search (the non-saturated one). In case both the controls exceed the maximum (or the minimum) both the stationary conditions are replaced by the corresponding boundary values. In these special cases the multipliers μ_i , $i = 3, \dots, 6$ can be obtained by imposing the stationary conditions of the Lagrangian rather than the Hamiltonian, completing the determination of all the unknowns. If no saturation appears, they are simply equal to 0 in virtue of the complementary conditions of Eqs. (32)-(33).

B. Optimal Angle of Attack in Coordinated Turn

A commonly used condition in flight mechanics is coordinated turn, that is, the flight is assumed such that the velocity vector is always in the symmetric plane of the vehicle, therefore $\beta \equiv 0$. It is known that in this case the optimal symmetric plan of the vehicle is the plane formed by the relative velocity vector and the prime vector $\mathbf{p}_v$ [16, 17]. The remaining degree-of-freedom of attitude modulation concerns the determination of the optimal value of α . To this end Eq. (41) may be rearranged to get

$$\frac{\partial H}{\partial \alpha} = \mathbf{p}_v^T \cdot \{ \mathbf{R}_w^e \cdot [(X_\alpha + Z)\mathbf{1}_{x,w} + (Z_\alpha - \eta T_{max} - X)\mathbf{1}_{z,w}] \} = 0 \quad (68)$$

and, by considering the unit vector $\hat{\mathbf{p}}_v = \mathbf{p}_v / \|\mathbf{p}_v\|$, we get the following equivalent expression.

$$\frac{\partial H}{\partial \alpha} = \hat{\mathbf{p}}_v^T \cdot \{ \mathbf{R}_w^e \cdot [(X_\alpha + Z)\mathbf{1}_{x,w} + (Z_\alpha - \eta T_{max} - X)\mathbf{1}_{z,w}] \} = 0 \quad (69)$$

By observing that $\mathbf{1}_{x,e} = \mathbf{R}_w^e \cdot \mathbf{1}_{x,w}$ and $\mathbf{1}_{z,e} = \mathbf{R}_w^e \cdot \mathbf{1}_{z,w}$, and by defining in the same spirit of Ref. [17]

$$\cos(\Phi - \alpha) = \hat{\mathbf{p}}_v^T \cdot \mathbf{1}_{x,e}, \quad \sin(\Phi - \alpha) = \hat{\mathbf{p}}_v^T \cdot \mathbf{1}_{z,e} \quad (70)$$

we can rewrite Eq. (69) as

$$\tan(\Phi - \alpha)(\eta T_{max} + X - Z_\alpha) - (X_\alpha + Z) = 0 \quad (71)$$

which, keeping in mind that $\eta T_{max} = T$, $X = -A$ and $Z_\alpha = -N_\alpha$ due to the different direction of motion, is fully consistent with Eq. (32) in Ref. [17]:

$$\tan(\Phi - \alpha)(T - A + N_\alpha) - (A_\alpha + N) = 0 \quad (72)$$

The solution to Eq. (71) will therefore give the optimal α at each time step.

C. The Optimal Sideslip Angle Is Generally Non-Zero

Given the nice cleanness of coordinated-turn flight, a tempting conjecture is that coordinated turn may be optimal, even if the modulation of the sideslip angle is allowed. Can $\beta \equiv 0$ be the solution to the optimality condition in Eq. (46) (reproduced here for ease of reference)?

$$\frac{\partial H}{\partial \beta} = \mathbf{p}_v^T \cdot \left\{ \mathbf{R}_w^e \cdot \left[X_\beta \mathbf{1}_{x,w} + Y_\beta \mathbf{1}_{y,w} + Z_\beta \mathbf{1}_{z,w} + (\eta T_{max} + X) \frac{\partial \mathbf{1}_{x,w}}{\partial \beta} + Y \frac{\partial \mathbf{1}_{y,w}}{\partial \beta} + Z \frac{\partial \mathbf{1}_{z,w}}{\partial \beta} \right] \right\} = 0 \quad (73)$$

We have the following answer to this question:

Theorem III.2. *If sideslip angle modulation is allowed as a trajectory control, coordinated-turn flight where $\beta \equiv 0$ is in general not optimal.*

Proof. To prove this assertion, let us substitute $\beta \equiv 0$ in Eq. (73), as well as in Eqs. (19) and (47), and reuse the definition of $\tilde{\mathbf{p}}_v$ introduced in Eq. (61) to arrive at

$$\frac{\partial H}{\partial \beta} = \tilde{\mathbf{p}}_v^T \cdot \left\{ X_\beta \begin{bmatrix} \cos \alpha \\ 0 \\ -\sin \alpha \end{bmatrix} + Y_\beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + Z_\beta \begin{bmatrix} \sin \alpha \\ 0 \\ \cos \alpha \end{bmatrix} + (\eta T_{max} + X) \begin{bmatrix} 0 \\ -\cos \alpha \\ 0 \end{bmatrix} + Y \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + Z \begin{bmatrix} 0 \\ -\sin \alpha \\ 0 \end{bmatrix} \right\} = 0 \quad (74)$$

and defined $\mathbf{p}_v \triangleq [p_u, \quad p_v, \quad p_w]^T$, we can rewrite Eq. (74) as

$$\frac{\partial H}{\partial \beta} = p_u(X_\beta \cos \alpha) + p_v[Y_\beta - (\eta T_{max} + X) \cos \alpha - Z \sin \alpha] - p_w X_\beta \sin \alpha = 0 \quad (75)$$

where we have adopted the hypothesis that $Y = 0$ at $\beta = 0$ deg, and that $Z_\beta = 0$. By grouping the terms in $\cos \alpha$ and $\sin \alpha$ we can write

$$\frac{\partial H}{\partial \beta} = [p_u X_\beta - p_v(\eta T_{max} + X)] \cos \alpha - (p_v Z + p_w X_\beta) \sin \alpha + p_v Y_\beta = 0 \quad (76)$$

which can only be true either if $p_u = p_v = p_w = 0$ or if there is a specific α satisfying Eq. (76). Let us analyse the two

cases:

case 1: $p_u = p_v = p_w = 0$

This case can be excluded. In fact, if $\mathbf{p}_v$ were identically equal to $\mathbf{0}$, this would imply that $\mathbf{p}_r$ would be equal to $\mathbf{0}$, and $p_m = 0$ too, due to the transversality condition on $p_m(t_F) = 0$ (the reader can refer to Eq. (102) that will be presented later for the details). This conclusion would violate the PMP which states that the costate vector is non-zero.

case 2: α^* is a root of the nonlinear expression in Eq. (76)

If we assume that we can find an α^* , such that $(\alpha^*, 0)$ are the optimal controls we are looking for, then α^* must also satisfy the optimality condition $\frac{\partial H}{\partial \alpha} = 0$. By performing the same substitutions into Eq. (69) we get

$$\frac{\partial H}{\partial \alpha} = [p_u(X_\alpha + Z) + p_w(Z_\alpha - \eta T_{max} - X)] \cos \alpha + [p_u(Z_\alpha - \eta T_{max} - X) - p_w(X_\alpha + Z)] \sin \alpha = 0 \quad (77)$$

The only possibility for which one solution satisfies two equations in Eqs. (75) and (77) is when the two equations are not independent. In other words, the coefficients of $\sin \alpha$ and $\cos \alpha$, as well as the term(s) not dependent on $\sin \alpha$ and $\cos \alpha$ in these two equations must be a multiple of each other. This would imply that there exists a scalar function ζ such the following equalities have to be satisfied:

$$\left\{ \begin{array}{lcl} p_u X_\beta - p_v(\eta T_{max} + X) & = & \zeta [p_u(X_\alpha + Z) + p_w(Z_\alpha - \eta T_{max} - X)] \\ (p_v Z + p_w X_\beta) & = & \zeta [p_w(X_\alpha + Z) - p_u(Z_\alpha - \eta T_{max} - X)] \\ p_v Y_\beta & = & 0 \end{array} \right. \quad (78)$$

The last condition would imply that at least Y_β or p_v are equal to 0. Since the former is in general not true, it must follow that $p_v = 0$, which implies that the first two equations become

$$\left\{ \begin{array}{lcl} p_u X_\beta & = & \zeta [p_u(X_\alpha + Z) + p_w(Z_\alpha - \eta T_{max} - X)] \\ p_w X_\beta & = & \zeta [p_w(X_\alpha + Z) - p_u(Z_\alpha - \eta T_{max} - X)] \end{array} \right. \quad (79)$$

or, equivalently,

$$\left\{ \begin{array}{lcl} p_u(X_\beta - \zeta X_\alpha - \zeta Z) & = & \zeta p_w(Z_\alpha - \eta T_{max} - X) \\ p_w(X_\beta - \zeta X_\alpha - \zeta Z) & = & -\zeta p_u(Z_\alpha - \eta T_{max} - X) \end{array} \right. \quad (80)$$

By solving with respect to p_u/p_w we can rewrite them as

$$\begin{cases} \frac{p_u}{p_w} = \zeta \frac{Z_\alpha - \eta T_{max} - X}{X_\beta - \zeta X_\alpha - \zeta Z} \\ \frac{p_u}{p_w} = -\zeta \frac{X_\beta - \zeta X_\alpha - kZ}{Z_\alpha - \eta T_{max} - X} \end{cases} \quad (81)$$

By equating the right-hand side of the first and the second equation, we obtain

$$(Z_\alpha - \eta T_{max} - X)^2 = -(X_\beta - \zeta X_\alpha - \zeta Z)^2 \quad (82)$$

Since both quantities are squared, and have different signs, the only possibility for which Eq. (82) can be satisfied is when both sides of the equation are zero, leading to

$$\zeta = \frac{X_\beta}{Z + X_\alpha} \quad (83)$$

and, to a second expression, that is,

$$Z_\alpha = \eta T_{max} + X \quad (84)$$

The first condition tells us the functional form that the scalar ζ should assume. More importantly, Eq. (84) is not true in general, for the same arguments moved at the end of Sec. III.1. Hence this case is not an option either.

case 3: the coefficients multiplying $\cos \alpha$ and $\sin \alpha$, as well as the last term in Eq. (76) are all equal to 0.

This last case implies that the following conditions hold:

$$\begin{cases} p_u X_\beta = p_v (\eta T_{max} + X) \\ p_v Z = p_w X_\beta \\ p_v Y_\beta = 0 \end{cases} \quad (85)$$

The last expression implies, since Y_β is in general different from 0, that $p_v = 0$, and therefore

$$p_w X_\beta = 0 \quad (86)$$

Since X_β is also in general different from 0, when we look at the first expression in Eq. (85) it follows that it must be $p_u = 0$. Consequently, we obtain $p_u = p_v = p_w = 0$. These conditions let case 3 degenerate to case 1. In conclusion, $\beta \equiv 0$ does not satisfy the optimality condition, therefore is not optimal. $\square$

D. Singular Thrust Extremals

The fact that the engine throttle η appears linearly in the Hamiltonian in Eq. (29) solicits the possibility of singular thrust arcs in the optimal solution. To gain a quick understanding about the possibility of singular thrust extremals, let us rewrite the switching function, this time by retrieving the Mayer formulation based on Eq. (1), meaning that the Hamiltonian becomes

$$H = \mathbf{p}_r^T \mathbf{v} + \mathbf{p}_v^T \mathbf{g} + \mathbf{p}_v^T \cdot \left\{ \frac{1}{m} [(\eta T_{max} + X)\mathbf{1}_{x,e} + Y\mathbf{1}_{y,e} + Z\mathbf{1}_{z,e}] \right\} - p_m \eta \frac{T_{max}}{c} \quad (87)$$

Note that we absorbed the rotation matrix $\mathbf{R}_w^e$ within the brackets, and consequently had the unit vectors expressed with respect to the external reference frame e , $\mathbf{1}_{x,e}$, $\mathbf{1}_{y,e}$, and $\mathbf{1}_{z,e}$. We obtain a slightly different switching function

$$S \triangleq \frac{\partial H}{\partial (\eta T_{max})} = \frac{1}{m} \mathbf{p}_v^T \cdot \mathbf{1}_{x,e} - \frac{p_m}{c} \quad (88)$$

where the term T_{max} was absorbed in the partial derivative of H , since it always appears together with η , and it does not change the functional behavior of the switching function itself. The existence of a singular extremal of thrust (or, equivalently, of throttle) requires that in a finite interval $[t_1, t_2] \subseteq [t_0, t_F]$ the switching function S is equal to 0. By differentiating S with respect to the time we get

$$\dot{S} = -\frac{\dot{m}}{m^2} (\mathbf{p}_v^T \cdot \mathbf{1}_{x,e}) + \frac{1}{m} (\dot{\mathbf{p}}_v^T \cdot \mathbf{1}_{x,e}) + \frac{1}{m} (\mathbf{p}_v^T \cdot \dot{\mathbf{i}}_{x,e}) - \frac{\dot{p}_m}{c} \quad (89)$$

By retrieving the differential equations for the mass (the last of Eq. (16)) and for the mass costate (Eq. (99)), and substituting them into Eq. (89) we get

$$\dot{S} = \frac{\eta T_{max}}{cm^2} (\mathbf{p}_v^T \cdot \mathbf{1}_{x,e}) + \frac{1}{m} (\dot{\mathbf{p}}_v^T \cdot \mathbf{1}_{x,e}) + \frac{1}{m} (\mathbf{p}_v^T \cdot \dot{\mathbf{i}}_{x,e}) - \frac{\eta T_{max}}{cm^2} \mathbf{p}_v^T \cdot \mathbf{1}_{x,e} - \mathbf{p}_v^T \cdot \frac{1}{cm^2} \mathbf{F}_e^A \quad (90)$$

which, by setting S in Eq. (88) to be equal to 0, gives the following expression for the switching function rate.

$$\dot{S} = \frac{1}{m} (\dot{\mathbf{p}}_v^T \cdot \mathbf{1}_{x,e}) + \frac{1}{m} (\mathbf{p}_v^T \cdot \dot{\mathbf{i}}_{x,e}) - \mathbf{p}_v^T \cdot \frac{1}{cm^2} \mathbf{F}_e^A \quad (91)$$

Equation (91) does not contain the throttle η anymore, as expected. Note however that $\dot{S}$ depends not only on α and β , which are embedded in $\mathbf{1}_{x,w}$, but also on their derivatives $\dot{\alpha}$ and $\dot{\beta}$, which come from the term $\dot{\mathbf{i}}_{x,w}$. The examination of a singular throttle necessitates the computation of the second derivative of the switching function with respect to the time, leading to the functional form

$$\ddot{S} = f_S(\mathbf{x}, \mathbf{p}, \eta, \alpha, \beta, \dot{\alpha}, \dot{\beta}, \ddot{\alpha}, \ddot{\beta}) = 0 \quad (92)$$

where f_S signifies the functional dependence of $\ddot{S}$ on the states, costates, angles α and β , their first derivatives $\dot{\alpha}$, and $\dot{\beta}$, as well as their second derivatives $\ddot{\alpha}$ and $\ddot{\beta}$. In addition, Eqs. (41) and (46) still hold, and they depend on the throttle value η . These terms are deeply intertwined, and form a differential-algebraic system of equations (DAE), which is significantly more complex to solve, when compared with the previous results available in literature. The existing investigations on singular thrust extremals in the presence of aerodynamic controls (e.g., Refs. [30], [31]) rely on the closed-form connection between the aerodynamic angles and the states and costates from the optimality conditions to explicitly express $\dot{\alpha}$, $\dot{\beta}$, $\ddot{\alpha}$, and $\ddot{\beta}$ in terms of states and / or costates' derivatives. This is only possible due to the simplifications coming from the in-plane motion formulation and relatively simple aerodynamic models that they adopt. When such simplifications are not available, as is the case in this work, where we have shown that an optimal three-dimensional trajectory in general requires a non-zero β and the aerodynamic models are based on tabulated data, further analytical exploration of singular thrust extremals does not appear feasible. As a consequence, the possibility of existence of a singular thrust arc may only be numerically investigated as in Ref. [21], but the hints represented by Eq. (92) suggest that they are quite unlikely in the general case.

IV. Dualization of Optimal Control Problem

In this section we will briefly build the symbolic expressions for the dual components of our OCP, that is, the costate differential equations in vectorial form, and the augmented terminal cost function. The Hamiltonian function was already introduced in Eq. (29), and here we will simply review the properties for the class of problems under analysis.

A. Costates Differential Equations

Let's restart from the Hamiltonian defined in Eq. (29). Since neither state constraints, nor state-control mixed constraints are considered, using the Hamiltonian function is equivalent to using the associated Lagrangian expression. Moreover, jumps in the costates as well as in the Hamiltonian can be excluded [32] for the case we are analyzing. Let us rewrite the Hamiltonian in a slightly different form.

$$H = \eta + \mathbf{p}_r^T \mathbf{v} + \mathbf{p}_v^T \mathbf{g} + \frac{1}{m} \mathbf{p}_v^T \cdot \left(\mathbf{R}_w^e \cdot \mathbf{F}_w^{A,T} \right) - p_m \eta \frac{T_{max}}{c} \quad (93)$$

with the aeropropulsive force $\mathbf{F}_w^{A,T}$ which is

$$\mathbf{F}_w^{A,T} \triangleq (\eta T_{max} + X) \mathbf{1}_{x,w} + Y \mathbf{1}_{y,w} + Z \mathbf{1}_{z,w} \quad (94)$$

and let us derive the costate differential equations. Specifically we have

$$\dot{\mathbf{p}}_r = -\frac{\partial H}{\partial \mathbf{r}} = -\mathbf{p}_v^T \nabla_{\mathbf{r}} \mathbf{g} - \frac{1}{m} \mathbf{p}_v^T \cdot \mathbf{R}_w^e \cdot [X_r \mathbf{1}_{x,w} + Y_r \mathbf{1}_{y,w} + Z_r \mathbf{1}_{z,w}] \triangleq \mathbf{p}_v^T \cdot \mathbf{C}_r = (\mathbf{C}_r^T \cdot \mathbf{p}_v)^T \quad (95)$$

with

$$\mathbf{C}_r \triangleq - \left\{ \nabla_{\mathbf{r}} \mathbf{g} + \frac{1}{m} \mathbf{R}_w^e \cdot [X_{\mathbf{r}} \mathbf{1}_{x,w} + Y_{\mathbf{r}} \mathbf{1}_{y,w} + Z_{\mathbf{r}} \mathbf{1}_{z,w}] \right\} \quad (96)$$

For the primer vector we can write

$$\begin{aligned} \dot{\mathbf{p}}_v &= -\frac{\partial H}{\partial \mathbf{v}} = -\mathbf{p}_r^T - \frac{1}{m} \mathbf{p}_v^T \cdot \left\{ \mathbf{R}_w^e \cdot [X_{\mathbf{v}} \mathbf{1}_{x,w} + Y_{\mathbf{v}} \mathbf{1}_{y,w} + Z_{\mathbf{v}} \mathbf{1}_{z,w}] + \mathbf{R}_{w,\mathbf{v}}^e \cdot \mathbf{F}_w^{A,T} \right\} \\ &\triangleq \mathbf{p}_r^T \cdot \mathbf{C}_{v,r} + \mathbf{p}_v^T \mathbf{C}_{v,v} = \mathbf{C}_{v,r}^T \mathbf{p}_r + \mathbf{C}_{v,v}^T \mathbf{p}_v \end{aligned} \quad (97)$$

with the matrices $\mathbf{C}_{v,r}^T$ and $\mathbf{C}_{v,v}^T$ defined in the following expression.

$$\begin{cases} \mathbf{C}_{v,r} &= -\mathbf{I} \\ \mathbf{C}_{v,v} &= -\frac{1}{m} \cdot \left\{ \mathbf{R}_w^e \cdot [X_{\mathbf{v}} \mathbf{1}_{x,w} + Y_{\mathbf{v}} \mathbf{1}_{y,w} + Z_{\mathbf{v}} \mathbf{1}_{z,w}] + \mathbf{R}_{w,\mathbf{v}}^e \cdot \mathbf{F}_w^{A,T} \right\} \end{cases} \quad (98)$$

Note that $\mathbf{R}_{w,\mathbf{v}}^e$ is a three-dimensional tensor, having dimensions equal to $[3 \times 3 \times 3]$. Its product with $(\mathbf{F}_w^T + \mathbf{F}_w^A)$ has to be intended as an *n-mode tensor product* [33], that generates a $[3 \times 3]$ matrix. Finally, for the mass costate we have

$$\dot{p}_m = -\frac{\partial H}{\partial m} = \frac{1}{m^2} \mathbf{p}_v^T \cdot \mathbf{R}_w^e \cdot [(T+X) \mathbf{1}_{x,w} + Y \mathbf{1}_{y,w} + Z \mathbf{1}_{z,w}] \triangleq \mathbf{p}_v^T \cdot \mathbf{C}_m = (\mathbf{C}_m^T \cdot \mathbf{p}_v)^T \quad (99)$$

with

$$\mathbf{C}_m = \frac{1}{m^2} \cdot \mathbf{R}_w^e \cdot \mathbf{F}_w^{A,T} \quad (100)$$

While Eqs. (95) and (97) show the functional dependence of the costate differential equations in vector forms, from the computational point of view the component-wise version of the costate equations are preferable for easiness of implementation, and have been provided explicitly in the Appendix.

B. Transversality Conditions

The transversality conditions of the costates can be easily obtained by combining the Lagrangian function defined in Eq. (30), and by defining the augmented terminal cost function Φ as follows.

$$\Phi = \nu_{\mathbf{r}}^T (\mathbf{r}(t_F) - \mathbf{r}_F) + \nu_{\mathbf{v}}^T (\mathbf{v}(t_F) - \mathbf{v}_F) \quad (101)$$

The application of the transversality conditions [34] gives

$$\begin{cases} \mathbf{p}_r(t_0) = \text{free}, & \mathbf{r}(t_F) = \mathbf{v}_r \\ \mathbf{p}_v(t_0) = \text{free}, & \mathbf{v}(t_F) = \mathbf{v}_v \\ p_m(t_0) = \text{free}, & p_m(t_F) = 0 \end{cases} \quad (102)$$

with the two unknown multiplier vectors $\mathbf{v}_r \in \mathbb{R}^3$, $\mathbf{v}_v \in \mathbb{R}^3$ to be determined. These conditions must be imposed when solving the corresponding TPBVP in addition to the boundary conditions associated with the primal dynamics, that is Eqs. (9) and (10).

The Hamiltonian in Eq.(29) does not explicitly depend on time. It is therefore constant along the optimal trajectory. Moreover, since we deal with a free-final time problem, we can apply the following transversality condition for the Hamiltonian [34].

$$H(t_F) = 0 \quad (103)$$

Therefore, we have that the optimal Hamiltonian along the entire trajectory is

$$H(t) = 0, \quad t \in [t_0, t_F] \quad (104)$$

with the final time t_F that has to be determined as well, while we, with no loss of generality, assume $t_0 = 0$ s.

V. Numerical Examples

This section illustrates dual uses of the derived necessary conditions. First they can be used to independently verify *a posteriori* the solution of the optimal landing problem obtained by a direct method. In this case we take from the direct solution the initial states and costates, and employ them to propagate the differential equations of Eqs. (28), (95)-(99) by a Runge-Kutta-45 integration scheme. In each time step the trajectory controls used in the propagation are obtained from the necessary conditions in Eqs. (36), (44), and (48) as functions of states and costates. In particular, the numerical procedure used to solve α and β from Eqs. (44) and (48) is described in Algorithm 1. This approach is preferred to simply interpolating the controls in the direct-method solution, as doing so typically leads to larger residuals of the necessary conditions.

All the optimal solutions to be verified in the rest of this section were obtained with a pseudospectral off-the-shelf transcription method [35]. The tool returns the supposedly optimal states, controls and costates used for the verification. The necessary conditions obtained in this paper can therefore be used as an independent analysis of the solutions to verify how accurately they satisfy the PMP.

The second use of the necessary conditions is for design purposes: they are embedded within a TPBVP representing

the energy-optimal solution. The scheme is then combined with a simple continuation approach for obtaining fuel-optimal trajectories.

All the results are based on the model for a notional reusable vertical landing rocket. The main parameters adopted for the numerical simulations are summarized in Table 1. The only caveat is related to the aerodynamic model: although the model in Ref. [36] was adopted, the look-up tables were transformed into continuous functions by using complete bivariate polynomials of degree 3 to model both the normal and the axial coefficients C_N and C_A , which are function of the angle of attack and the Mach number. The following mirroring was then adopted to transform the coefficients into a 3-D dataset:

$$C_x = C_A(\alpha_T, M) \quad C_y = C_N(\beta, M) \quad C_z = C_N(-\alpha, M) \quad (105)$$

where α_T is the total angle of attack defined by

$$\alpha_T = \sqrt{\alpha^2 + \beta^2} \quad (106)$$

This choice keeps consistency with the original purpose of the benchmark proposed in Ref. [36]. On the other hand, it provides analytical expressions to work with that make their evaluation faster, in a similar fashion to what Liu proposed in Ref. [12].

Table 1 Rocket Benchmark Parameters

Parameter	Value [Unit]	Parameter	Value [Unit]
gravity acceleration g	3.7114 m s^{-2}	exhaust velocity c	1966.1 m s^{-1}
atmospheric model	Gram model [37]	aerodynamic model	Simplicio benchmark [36]
reference Surface S	3.1415 m^2	maximum thrust T_{max}	16573 N
minimum throttle η_{min}	0.3	maximum throttle η_{max}	0.8
Initial position $x(t_0)$	2000 m	Final position $x(t_F)$	0.5 m
Initial position $y(t_0)$	1050 m	Final position $y(t_F)$	0 m
Initial position $z(t_0)$	1500 m	Final position $z(t_F)$	0 m
Initial velocity $u(t_0)$	-10 m s^{-1}	Final velocity $u(t_F)$	-1 m s^{-1}
Initial velocity $v(t_0)$	-70 m s^{-1}	Final velocity $v(t_F)$	0 m s^{-1}
Initial velocity $w(t_0)$	-85 m s^{-1}	Final velocity $w(t_F)$	0 m s^{-1}
Initial mass $m(t_0)$	1905 kg	Final mass $m(t_F)$	free

A. Numerical Procedure to Solve the Necessary Conditions

We have seen in Sec. II that the trajectory controls η , α , and β need to be solved numerically from the necessary conditions. To do so, we implement a modified Newton-Raphson method that can also handle upper and lower bound constraints by including control projections to ensure that the angle of attack and the sideslip angle stay within the prescribed boundaries. The method is described in Algorithm 1 below. At each time step we evaluate the switching

function and the throttle η (through Eqs. (35) and (36), or directly through Eq. (40) if we are dealing with energy or fuel/energy OCPs. After that, we can perform a first evaluation of the necessary conditions by using the initial guess of the aerodynamic angles. The initialization of the algorithm is complete, and the iterative loop can start. Lines 10–11 in the Algorithm represent the core of the Newton-Raphson method: the numerical Jacobian matrix associated with the necessary conditions is built, and used to update the values of α and β . A modification of the original algorithm occurs at lines 12–13. Here we perform a *projection* of the current values of α and β onto the feasible set, before re-evaluating the residuals of the necessary conditions in line 14. This operation represents the main modification to the Newton-Raphson algorithm and allows to effectively handle the constrained case. At this point there are three possibilities: the first is that the residuals, represented by $\mathbf{F}$, are smaller than our tolerance ϵ_{tol} . This means that the stationary conditions were satisfied, and therefore, no residual is left. The interpretation of this behavior is that the optimal solution is within the upper and lower bounds represented by $[\alpha_{min}, \alpha_{max}]$ and $[\beta_{min}, \beta_{max}]$. It follows that the corresponding Lagrange multipliers of Eqs. (32), (33) are equal to 0. If the residuals are larger than the prescribed tolerance, there are two possible scenarios: the first is that there are no more changes in the solution, (line 19), meaning that one or both control variables hit the saturation limits due to lines 13-14. In that case, one or both the components of the residuals $\mathbf{F}$ will be different from zero, and their absolute value represent our non-negative Lagrange multipliers. If that condition does not occur the algorithm can continue and the loop restarts.

Algorithm 1: Newton-Raphson algorithm with control projection

Input: $\mathbf{t}, \mathbf{x}, \mathbf{p}, \alpha_{ig}, \beta_{ig}$ defined in N discrete time steps

Output: $\eta, \alpha, \beta, \mu_i$

Parameters: $\alpha_{min}, \alpha_{max}, \beta_{min}, \beta_{max}, \epsilon_{tol}, \epsilon_{sat}, iter_{max}$

```
1  $[\alpha, \beta]^T \leftarrow [\alpha_{ig}, \beta_{ig}]^T$ 
2 for  $k = 1, 2, \dots, N$  do
3    $S \leftarrow$  Evaluate Eq. (35) // get the switching function value
4    $\eta \leftarrow$  Evaluate Eq. (36) or Eq. (40) // compute the throttle value for fuel / energy-optimal problems
5    $\eta \leftarrow \max[\eta_{min}, \min[\eta, \eta_{max}]]$  // apply saturation
6    $\mathbf{F} = \left[ \frac{\partial H}{\partial \alpha}, \frac{\partial H}{\partial \beta} \right]^T \leftarrow$  Evaluate Eqs. (43), (46) // evaluate the necessary conditions
7   counter = 0 // reset the iteration counter
8   for  $counter \leq iter_{max}$  do
9      $[\alpha_{old}, \beta_{old}]^T \leftarrow [\alpha, \beta]^T$ 
10    counter = counter + 1
11     $\mathbf{J} \leftarrow$  Numerical Jacobian
12     $[\alpha, \beta]^T \leftarrow [\alpha_{old}, \beta_{old}]^T - \mathbf{J}^{-1}\mathbf{F}$  // update the aerodynamic angles
13     $\alpha \leftarrow \max[\alpha_{min}, \min[\alpha, \alpha_{max}]]$  // apply saturation
14     $\beta \leftarrow \max[\beta_{min}, \min[\beta, \beta_{max}]]$  // apply saturation
15     $\mathbf{F} = \left[ \frac{\partial H}{\partial \alpha}, \frac{\partial H}{\partial \beta} \right]^T \leftarrow$  Evaluate Eqs. (43), (46) // evaluate the necessary conditions
16  if  $\max|\mathbf{F}| \leq \epsilon_{tol}$  then
17     $\mu = [0, 0]^T$  break // convergence reached, unconstrained solution found
18  else
19    if  $\max(|\alpha - \alpha_{old}|, |\beta - \beta_{old}|) < \epsilon_{sat}$  then
20       $\mu = |\mathbf{F}|$  // saturation detected, the residuals correspond to the Lagrange multipliers of Eq.
21      (30)
22      break // progress stalled, constrained solution found
```

B. A Posteriori Verification

In this first part of results we will verify a posteriori the optimality of the solutions obtained with a direct pseudospectral transcription method [35] combined with a nonlinear programming problem solver, in this case Ipopt [38]. We show the results obtained for three different cases: Non-saturated energy-optimal (NSE), Non-saturated fuel-optimal (NSF), and saturated fuel-optimal (SF) solutions. Here “non-saturated” and “saturated” refer to whether

the box constraints on α and/or β are active.

Energy-Optimal Case *without Saturation*

First we considered an energy-optimal trajectory, depicted in Fig. 3. The optimal controls coming from a direct method are compared with the ones extracted by solving the necessary conditions of Eqs. (43), (46). In addition, the throttle signals obtained numerically are compared with Eq. (40). The trajectory correctly ends at the prescribed final position with the corresponding final velocity. Note that the scenario here has been created to have non-negligible horizontal motion, and being therefore representative of a more general class of flights if compared with almost-vertical flights. When we look at Fig. 4(a) one may notice that, as expected according to Eq. (40), the throttle profile is smooth. Moreover, the angle of attack and the sideslip angle obtained by solving the necessary conditions of Eqs. (43) and (46) perfectly match the ones obtained with the direct method, confirming the validity of the conditions derived. Since the direct method adopted benefits from the Covector Mapping Principle [39], we can retrieve the costates as an additional output of the methodology employed here. Therefore we can, on the one hand, build the Hamiltonian function by using Eq. (29). On the other hand we can inject states and costates to see whether the necessary conditions of optimality in terms of controls α and β are satisfied. This is indeed what we can observe in Fig. 4(b): the Hamiltonian is quite close to 0, as expected by analyzing the structure of this problem. Moreover, the necessary conditions residuals in terms of partial derivatives with respect to α and β are in the order of 10^{-13} , fundamentally confirming the optimality of the solution. Note that the middle plot of Fig. 4(a) (and all the subsequent results shown in this section) do not represent directly the angle of attack α , but the angle of attack with respect to the tip of the vehicle, α_T , given by $\alpha_T = \alpha - \pi$. This choice is made for visualization reasons, to avoid having very different ranges for the two original aerodynamic controls α and β . Finally, we take the initial states (assigned) and costates (retrieved from the application of the covector mapping theorem) and use them to propagate the augmented equations of motion given by Eqs. (16), (95)-(99). We then compare the results with the states and costates obtained by the direct method. Results are depicted in Fig. 5, where it can be seen how the propagated trajectories match quite well with the optimized ones, confirming the validity of the necessary conditions, together with the costate differential equations, that have been derived in this work.

Fuel-Optimal Case *without Saturation*

A second test-case is represented by the validation of the NSF problem. The only difference with respect to the previous case is the cost function (with Eq. (3) instead of Eq. (37)). Results are shown in Figs. 6(a) through 7. The trajectory looks very similar to the previous one, and is therefore omitted here. However, the throttle, depicted in the top-plot of Fig. 6(a), shows the well-known and expected *max-min-max* behavior [3], and the switching times perfectly match the time instances at which the switching function, built by using Eq. (35), changes its sign. In this example we adopted on purpose large values for the aerodynamic angles, that is

$$\alpha_{T,max} = -\alpha_{T,min} = \beta_{max} = -\beta_{min} = 50 \text{ deg} \quad (107)$$

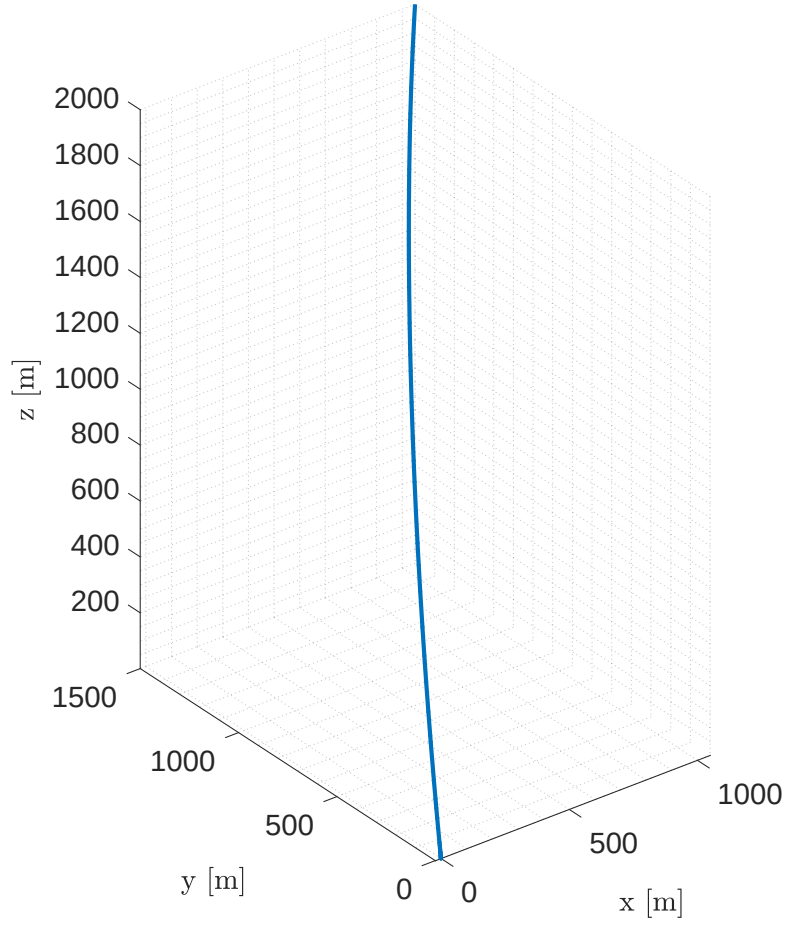
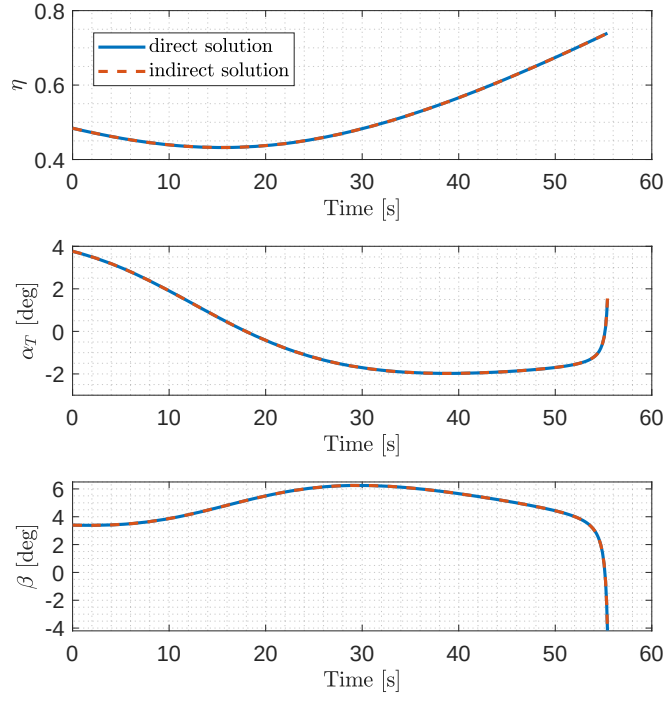
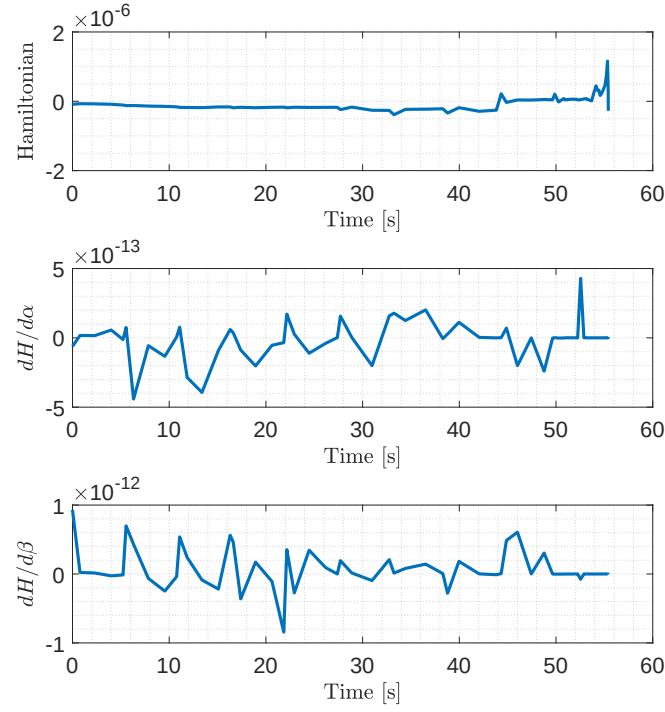


Fig. 3 Energy-optimal solution: trajectory

to avoid saturation on controls. This is visible in the smooth, unsaturated profiles of Fig. 6(a), and is reflected by the fact that the necessary conditions are fully satisfied, with a residual error in the order of 10^{-13} for both terms, and a maximum violation of the Hamiltonian condition equal to 10^{-3} . Also in this case we see a very good agreement between the output of the direct method and the outcome of the propagated augmented states (Fig. 7).



(a) Controls



(b) Optimality conditions

Fig. 4 Energy-optimal solution: controls and optimality conditions.

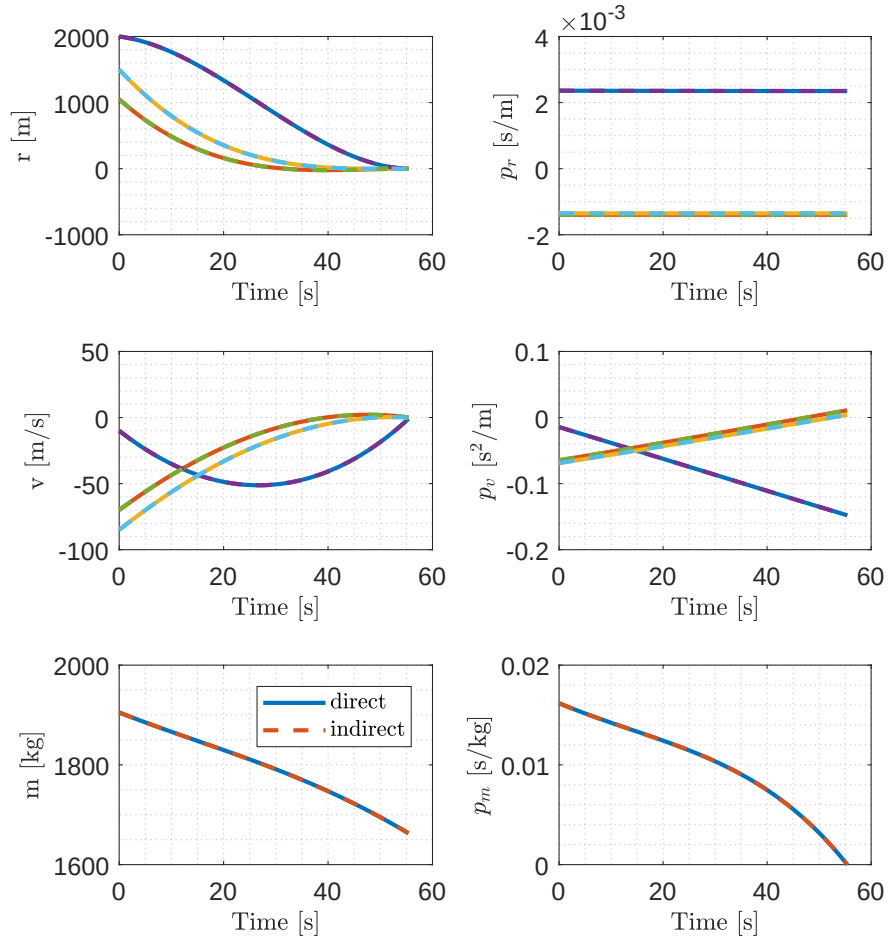
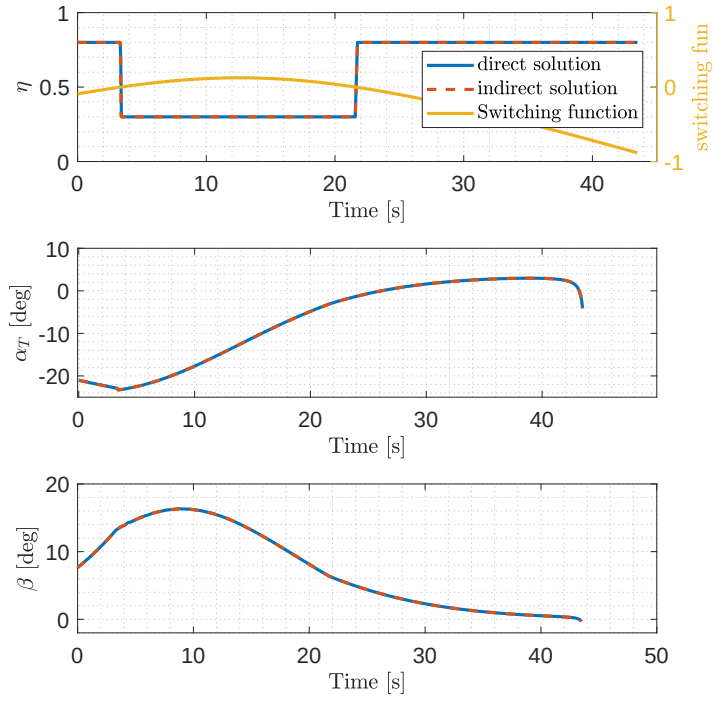
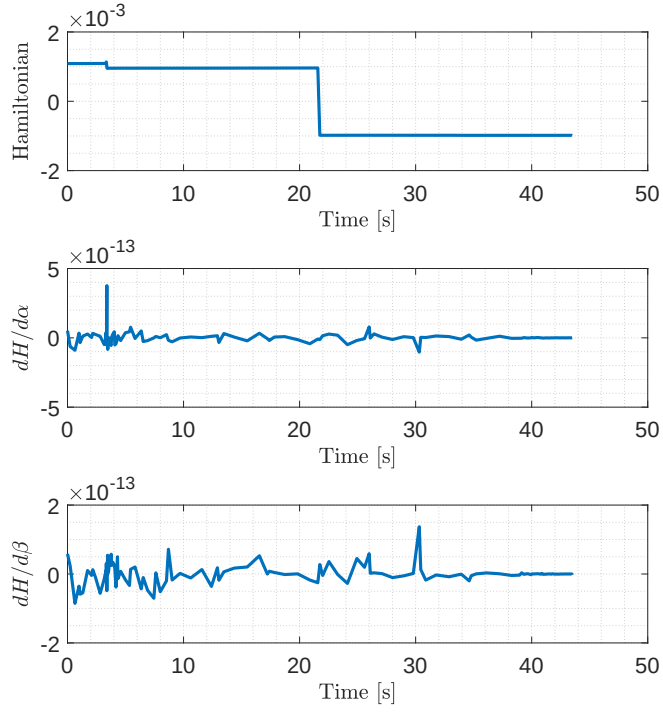


Fig. 5 Energy-optimal solution: comparison of states and costates propagation with a direct solution.



(a) Controls



(b) Optimality conditions

Fig. 6 Non-saturated fuel-optimal solution: controls and optimality conditions.

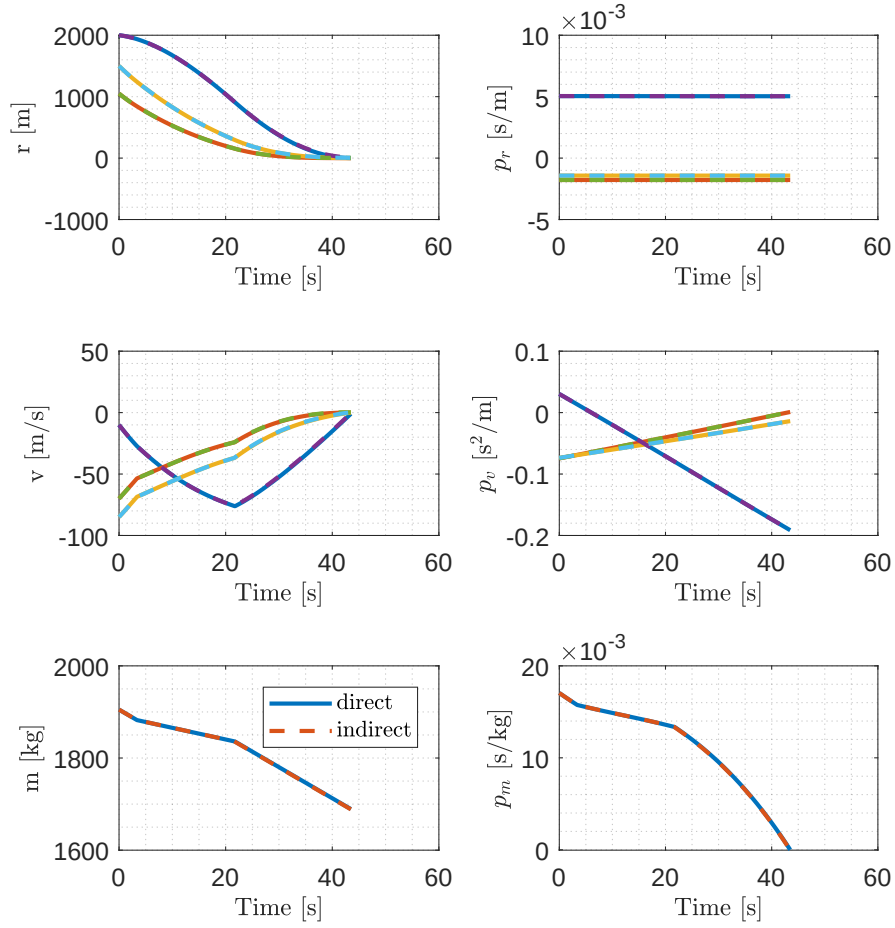


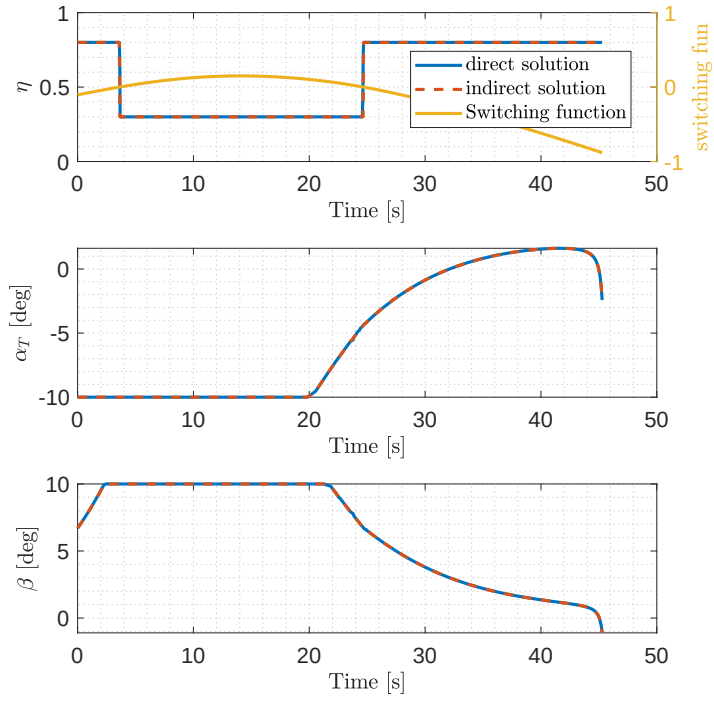
Fig. 7 Non-saturated fuel-optimal solution: comparison of states and costates propagation with a direct solution.

Fuel-Optimal Case *with Saturation*

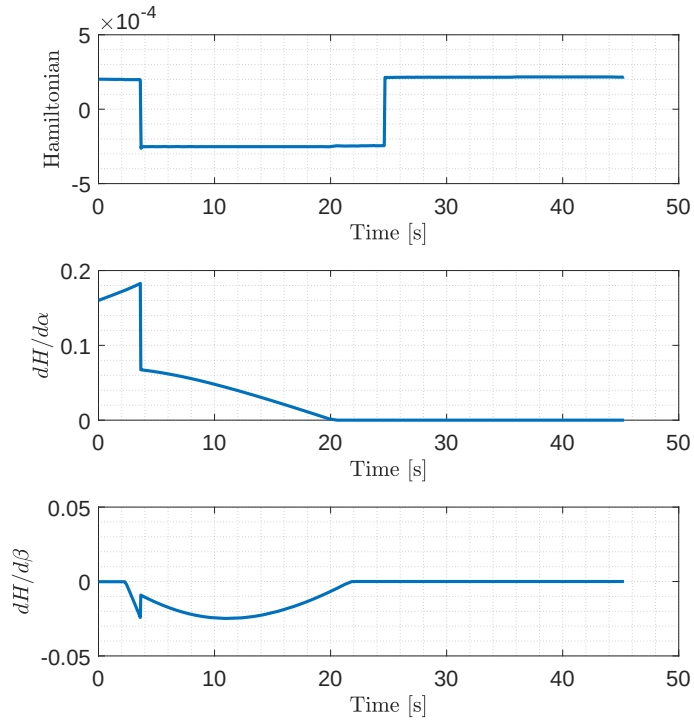
The final validation testcase we present refers to the SF problem, i.e., in presence of stricter upper and lower bounds which are more realistic, and lead to saturation of one or both the aerodynamic controls. Results are depicted in Figs. 8(a) - 9. We can now note an interesting difference in the behavior of the controls, shown in Fig. 8(a). Given the stricter upper and lower bounds, equal in this case to

$$\alpha_{T,max} = -\alpha_{T,min} = \beta_{max} = -\beta_{min} = 10 \text{ deg} \quad (108)$$

we observe that at the very beginning the angle of attack saturates, while the sideslip angle β does not. This translates into a positive Lagrange multiplier μ associated with the angle of attack that is detected by the routine to solve the necessary conditions (Fig. 8(b)). After about 2.5 s also the sideslip angle saturates, and therefore its corresponding Lagrange multiplier assumes a positive value as well, equal to $\left| \frac{\partial H}{\partial \beta} \right|$. This condition holds until $t = 20$ s, where the angles are no longer saturating and their corresponding residuals go to 0 again. The Hamiltonian function is in this case in the order of $2 \cdot 10^{-4}$, that is, quantitatively in line with the previous case. Finally, we show for this case as well the comparison of propagated states and costates (Fig. 9). Also in this last case the results agree well with each other.



(a) Controls



(b) Optimality conditions

Fig. 8 Saturated fuel-optimal solution: controls and optimality conditions.

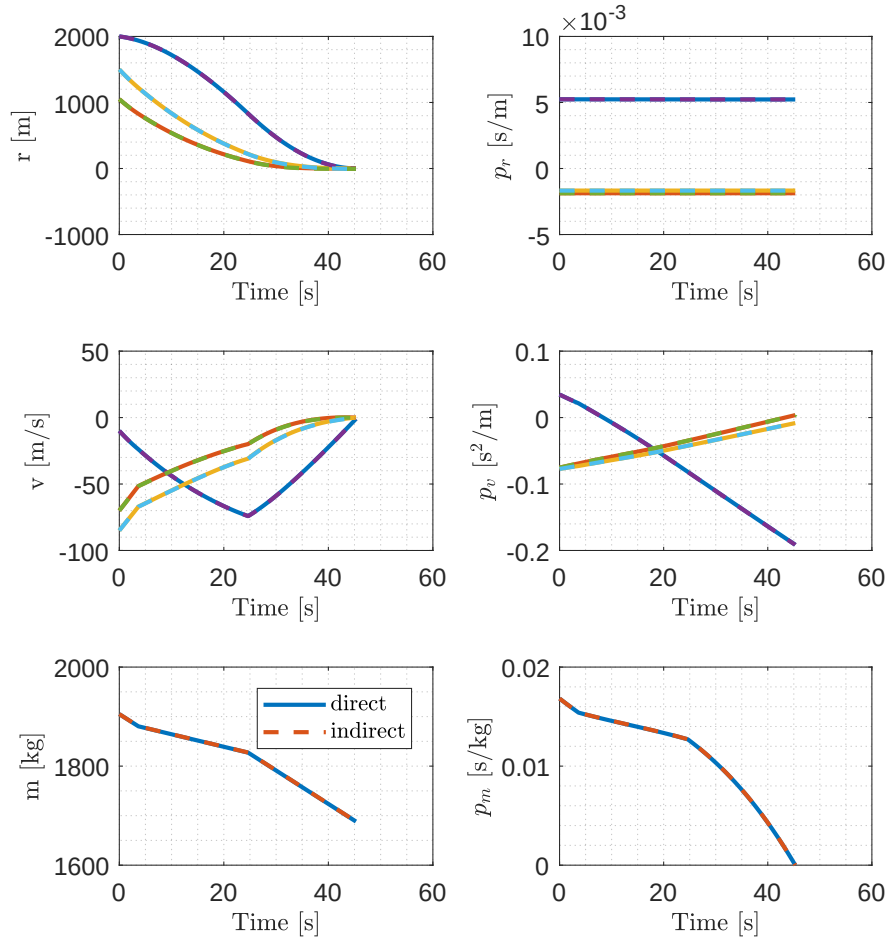


Fig. 9 Saturated fuel-optimal solution: comparison of states and costates propagation with a direct solution.

C. Trajectory Design

In this last section we show two examples of optimal trajectory design through the use of derived necessary conditions. First, we compute the NSE solution. Then, we adopt a simple continuation method to transform it into the corresponding NSF solution.

Energy-Optimal Problem

The first example of solution is the NSE, that is the energy-optimal one. We formulate the TPBVP given by Eqs. (9), (10), (16), (43), (46), (102), and (103). The problem is solved with the Matlab `bvp5c` solver [40]. The initial guess is provided by solving the direct OCP by using Sequential Pseudospectral Convex Programming, as detailed in Refs. [15, 24]. Note that, despite being able to quickly solve the problem with a good accuracy, the direct convex method provides a good estimation of the costates in virtue of the Covector Mapping Theorem [39], but not accurate enough to be considered as a solution to the TPBVP, as it is possible to observe in Fig. 10. The algorithm proposed here,

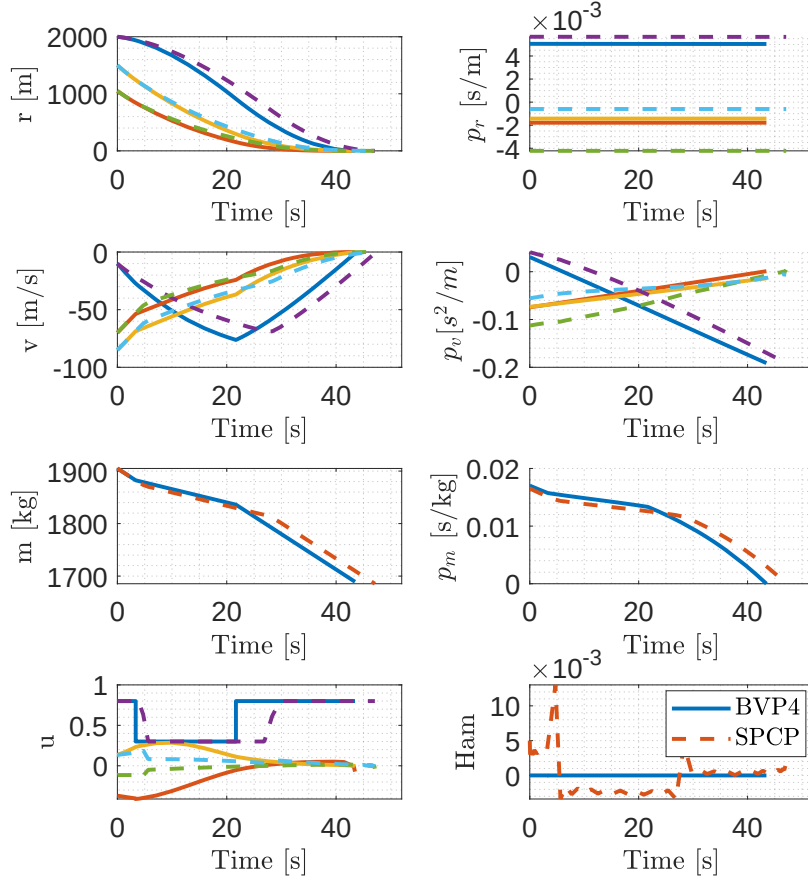


Fig. 10 Comparison of fuel-optimal solution obtained with a direct sequential convex method and with the proposed indirect methodology.

while being much slower in terms of computational speed, allows to obtain the complete solution to the associated TPBVP. The two methods can be therefore seen as complementary to each other, and even merged together in a more sophisticated fashion, as done for instance in Ref. [28]. The corresponding states, costates, controls and Hamiltonian functions are visible in Fig. 11. The solution agrees quite well with the one obtained by employing direct methods, and

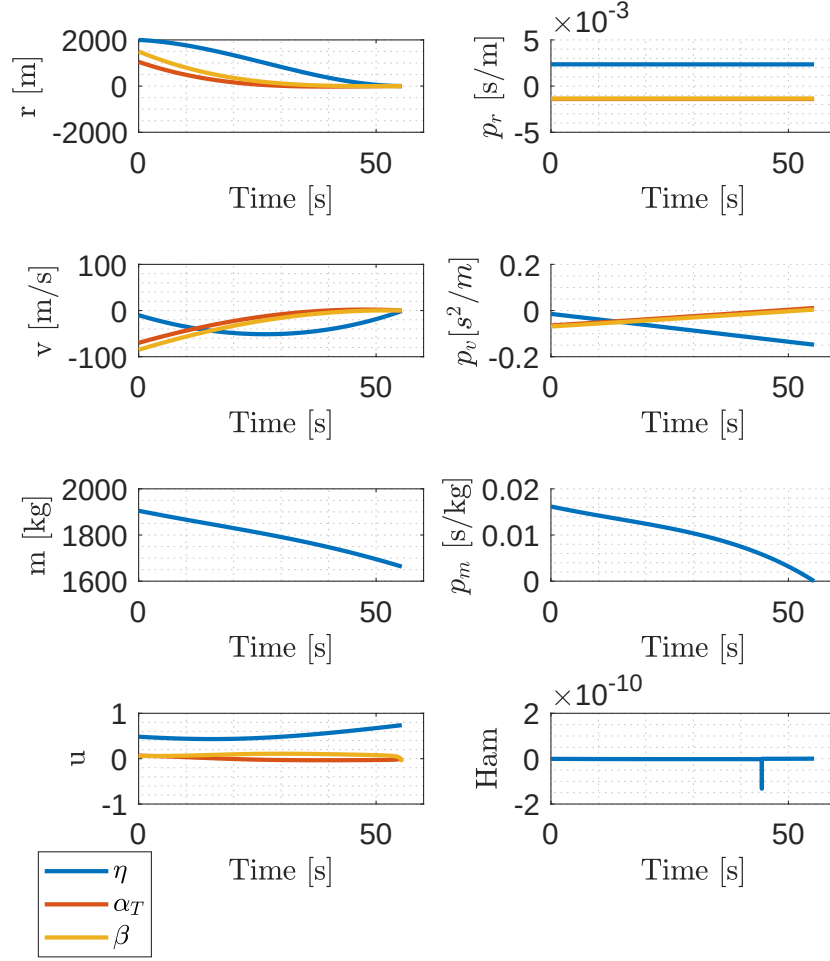


Fig. 11 Energy-optimal trajectory generation coming from the solution of the TPBVP.

exhibits a higher accuracy for what concerns the Hamiltonian, whose violations are here in the order of 10^{-10} .

Fuel-Optimal Problem

Starting from the energy-optimal solution, a very simple continuation scheme is adopted to slowly warp the solution from its energy-optimal to the fuel-optimal one. Given the demonstration purposes of this section, we adopted the simplest possible continuation scheme, meaning, a fixed step-size for the homotopic parameter ϵ in Eq. (38), varying from 0 to 0.99999, which is considered in this context close enough to the fuel-optimal case. Note that the true value of $\epsilon = 1$ would lead Eq. (40) to go to ∞ , breaking the smooth approximation to the optimal throttle signal. Nonetheless, it

is easy to see that there are no visible differences with the fuel-optimal case, and the bang-bang feature is reconstructed very accurately, as it possible to observe in Fig. 12. This continuation scheme has some shortfalls, specifically, it does

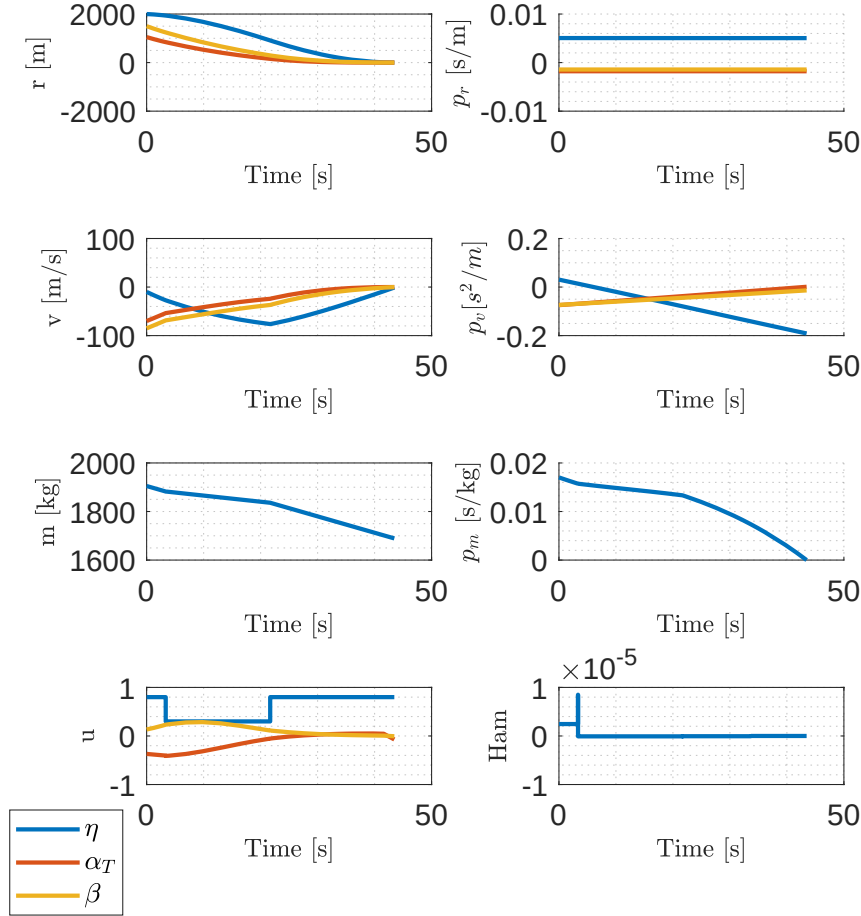
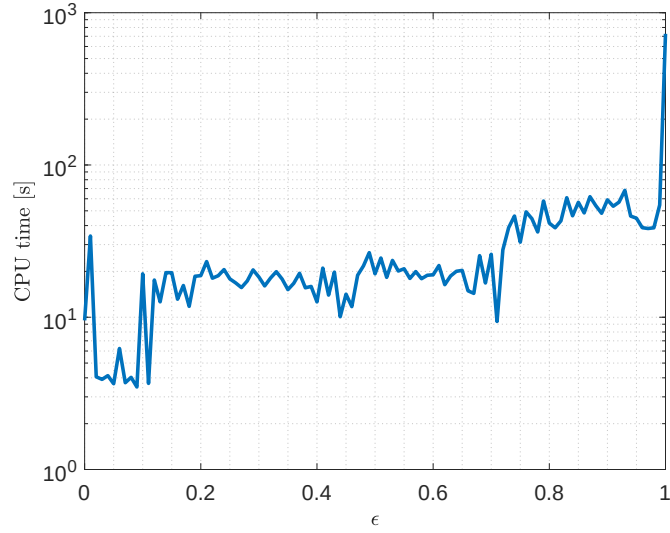


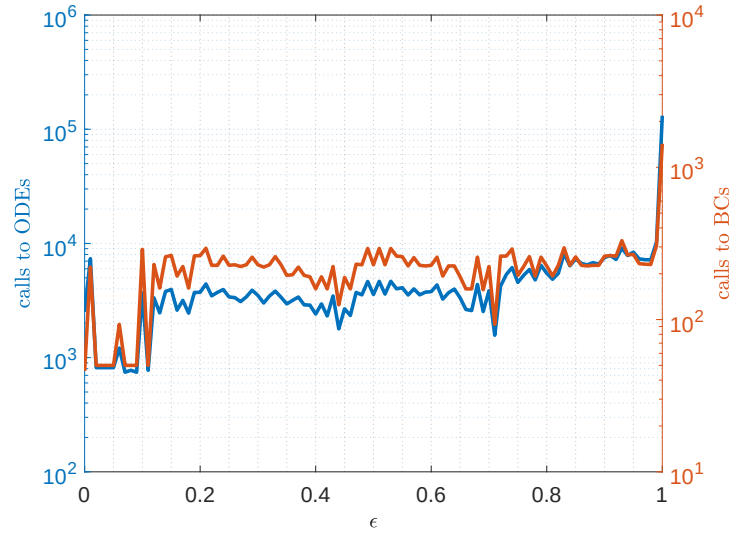
Fig. 12 Fuel-optimal trajectory generation coming from the solution of the TPBVP.

not ensure an always accurate solution of all the intermediate steps along the continuation. In addition, we noticed the discretization error of the final solution to be equal to 32.97 m for the position, and 1.42 m s⁻¹ for the velocity, which might be larger than some prescribed requirements. However, a high-accuracy implementation of the homotopy scheme goes beyond the scope of the present work, and we remind the reader to review state-of-the-art continuation schemes, that can alleviate such shortcomings, i.e., see Refs. [41–44]. The results obtained are qualitatively correct, with all the features which are correctly captured, and the Hamiltonian exhibiting a maximum violation in the order of 10⁻⁵, demonstrating the successful application of the necessary conditions derived here. In terms of computational behavior, as mentioned, the algorithm is not meant to be as fast as convex-based methods are. In fact, the current algorithm for generating the fuel-optimal solution employed 3241.6 s on a laptop with a 1.8 GHz i7-processor with 16 GB of RAM.

An overview of the homotopic path from the computational point of view is visible in Fig. 13, with Fig. 13(a) showing the computational time associated with each homotopic step. For what concerns the iterations, the method `bvpc5` does not really have an iteration counter, but it stores information about the number of calls to the ODE function and to the Boundary conditions function. These calls are depicted in Fig. 13(b). The two figures tell the same story: there are three sub-behaviors to observe: the first, ranging from $\epsilon = 0$ to approximately $\epsilon = 0.1$ for which the problem is relatively simple, and takes less iterations and consequently less computational time. From 0.1 to about 0.7 the number of iterations remains relatively flat, meaning no specific difficulties are encountered in the resolution of the problem. Starting from 0.7 onward the CPU time becomes large, and so do the calls to the ODE functions. In other words, the solver needs to refine more the solution to reach the prescribed convergence criterion. This aspect is particularly visible in the last iteration (that is, to change ϵ from 0.99 to 0.99999), which by far is the more computationally expensive step, and requires 726.8 s, that is, about 22.4% of the entire process.



(a) CPU times for each homotopic step



(b) Number of calls to ODE and Boundary condition functions

Fig. 13 Computational behavior of the homotopic path applied to the developed necessary conditions.

VI. Conclusions

In this work we offer an analytic treatise to the optimal rocket endo-atmospheric powered descent and landing problem, focusing on the role of the aerodynamic terms. The formulation of the problem is more general than in the existing literature by allowing simultaneous modulation of both angle of attack and sideslip angle. The full set of necessary conditions are derived and analyzed. We show that, contrary to a long-standing custom in flight mechanics, in general the optimal three-dimensional trajectory is not one in coordinated turn, that is, the optimal sideslip angle is not constantly zero. When the coordinated turn assumption is made, we further show that the conditions derived

in this paper reduce to the known results in the literature. Moreover, we observe, based on the derived necessary conditions, that singular thrust extremals for endo-atmospheric rocket landings seem to be at best a pathological case, if not impossible, due to the strong coupling between the state, costate, and the first- and higher-order derivatives of the aerodynamic controls in the required equations.

The conditions obtained lend a systematic means for independent verification of the numerical solutions from direct methods which are predominantly the techniques employed nowadays. Applications in this regard are demonstrated relatively thoroughly in this paper. On the other hand, the necessary conditions derived serve to be the foundation for the development of an indirect method to solve the problem of optimal endo-atmospheric rocket landing, a potential use also illustrated in the paper.

Appendix

A. Some Useful Derivatives

Here we build some derivatives required for the derivation of the costates' equations. The derivative of the aerodynamic force components X , Y , and Z with respect to the position $\mathbf{r}$ and the velocity $\mathbf{v}$ are given by

$$\left\{ \begin{array}{l} X_{\mathbf{r}} = \frac{1}{2} \nabla_{\mathbf{r}}(\rho) V^2 S C_x + \frac{1}{2} \rho V^2 S \nabla_{\mathbf{r}}(C_x) = X \left(\frac{\nabla_{\mathbf{r}}(\rho)}{\rho} + \frac{\nabla_{\mathbf{r}}(C_x)}{C_x} \right) \\ Y_{\mathbf{r}} = \frac{1}{2} \nabla_{\mathbf{r}}(\rho) V^2 S C_y + \frac{1}{2} \rho V^2 S \nabla_{\mathbf{r}}(C_y) = q S (\nabla_{\mathbf{r}}(\rho) C_y + \nabla_{\mathbf{r}}(C_y)) \\ Z_{\mathbf{r}} = \frac{1}{2} \nabla_{\mathbf{r}}(\rho) V^2 S C_z + \frac{1}{2} \rho V^2 S \nabla_{\mathbf{r}}(C_z) = q S (\nabla_{\mathbf{r}}(\rho) C_z + \nabla_{\mathbf{r}}(C_z)) \\ X_{\mathbf{v}} = \rho S C_x \mathbf{v}^T + \frac{1}{2} \rho V^2 S \nabla_{\mathbf{v}}(C_x) = \rho S \left(C_x \mathbf{v}^T + \frac{V^2}{2} \nabla_{\mathbf{v}}(C_x) \right) \\ Y_{\mathbf{v}} = \rho S C_y \mathbf{v}^T + \frac{1}{2} \rho V^2 S \nabla_{\mathbf{v}}(C_y) = \rho S \left(C_y \mathbf{v}^T + \frac{V^2}{2} \nabla_{\mathbf{v}}(C_y) \right) \\ Z_{\mathbf{v}} = \rho S C_z \mathbf{v}^T + \frac{1}{2} \rho V^2 S \nabla_{\mathbf{v}}(C_z) = \rho S \left(C_z \mathbf{v}^T + \frac{V^2}{2} \nabla_{\mathbf{v}}(C_z) \right) \end{array} \right. \quad (109)$$

Where $q = \frac{1}{2} \rho V^2$ is the dynamic pressure. We need some further terms. Specifically, we have

$$\left\{ \begin{array}{ll} \nabla_{\mathbf{r}}(\rho) = \left[\frac{\partial h}{\partial \rho}, 0, 0 \right]^T & \nabla_{\mathbf{r}}(Temp) = \left[\frac{\partial h}{\partial Temp}, 0, 0 \right]^T \\ M_{Temp} = -\frac{M}{2Temp} & \nabla_{\mathbf{v}}(M) = 2\mathbf{v}/a^2 \\ \nabla_{\mathbf{r}}(C_x) = C_{x,M} M_{Temp} \nabla_{\mathbf{r}}(Temp) & \nabla_{\mathbf{v}}(C_x) = C_{x,M} \nabla_{\mathbf{v}}(M) \\ \nabla_{\mathbf{r}}(C_y) = C_{y,M} M_{Temp} \nabla_{\mathbf{r}}(Temp) & \nabla_{\mathbf{v}}(C_y) = C_{y,M} \nabla_{\mathbf{v}}(M) \\ \nabla_{\mathbf{r}}(C_z) = C_{z,M} M_{Temp} \nabla_{\mathbf{r}}(Temp) & \nabla_{\mathbf{v}}(C_z) = C_{z,M} \nabla_{\mathbf{v}}(M) \end{array} \right. \quad (110)$$

with a representing the speed of sound. Finally, we need some matrix derivatives. For the gravity gradient, we can write

$$\left\{ \begin{array}{l} \nabla_{\mathbf{r}}(\mathbf{g}) = \begin{bmatrix} g_{xx} & g_{xy} & g_{xz} \\ g_{yx} & g_{yy} & g_{yz} \\ g_{yz} & g_{yz} & g_{zz} \end{bmatrix} \\ g_{pp} = \mu_{\oplus} \left(\frac{3r_p^2}{\|\mathbf{r}+\mathbf{r}_L\|^{5/2}} - \frac{1}{\|\mathbf{r}+\mathbf{r}_L\|^{3/2}} \right) \\ g_{qr} = 3\mu_{\oplus} \frac{r_q r_r}{\|\mathbf{r}+\mathbf{r}_L\|^{5/2}} \\ p, q, r = x, y, z \end{array} \right. \quad (111)$$

while for the derivative with respect to the velocity vector of the rotation matrix $\mathbf{R}_{w,v}^e$ we can write, by considering how the matrix $\mathbf{R}_w^e$ given in Eq. (26) depends on u , v , and w , the following expressions.

$$\mathbf{R}_w^e \cong \mathbf{R}_w^{UEN} = \begin{bmatrix} \cos \gamma \cos \psi & -\sin \psi \cos \gamma & -\sin \gamma \\ \sin \psi & \cos \psi & 0 \\ \sin \gamma \cos \psi & -\sin \gamma \sin \psi & \cos \gamma \end{bmatrix} \quad (112)$$

$$\mathbf{R}_{w,\diamond}^e = \begin{bmatrix} -\gamma_{\diamond} \sin \gamma \cos \psi - \psi_{\diamond} \cos \gamma \sin \psi & \gamma_{\diamond} \sin \gamma \sin \psi - \psi_{\diamond} \cos \gamma \cos \psi & -\gamma_{\diamond} \cos \gamma \\ \psi_{\diamond} \cos \psi & -\psi_{\diamond} \sin \psi & 0 \\ \gamma_{\diamond} \cos \gamma \cos \psi - \psi_{\diamond} \sin \gamma \sin \psi & -\gamma_{\diamond} \cos \gamma \sin \psi - \psi_{\diamond} \sin \gamma \cos \psi & -\gamma_{\diamond} \sin \gamma \end{bmatrix} \quad (113)$$

with $\diamond \in [u, v, w]$. The terms $\gamma_{\diamond}$ and $\psi_{\diamond}$ appearing in the previous matrix are simply obtained by differentiating the exact expressions of Eq. (24).

$$\left\{ \begin{array}{ll} \gamma_u \triangleq \frac{\partial \gamma}{\partial u} = -\frac{w}{u^2+w^2}, & \psi_u \triangleq \frac{\partial \psi}{\partial u} = -\frac{uv}{(u^2+v^2+w^2)(u^2+w^2)^{1/2}} \\ \gamma_v \triangleq \frac{\partial \gamma}{\partial v} = 0 & \psi_v \triangleq \frac{\partial \psi}{\partial v} = \frac{(u^2+w^2)^{1/2}}{(u^2+v^2+w^2)} \\ \gamma_w \triangleq \frac{\partial \gamma}{\partial w} = \frac{u}{u^2+w^2}, & \psi_w \triangleq \frac{\partial \psi}{\partial w} = -\frac{vw}{(u^2+v^2+w^2)(u^2+w^2)^{1/2}} \end{array} \right. \quad (114)$$

B. Costate Equations in Scalar Form

Let us rewrite the Hamiltonian of Eq. (29) by explicitly defining all the scalar products.

$$\begin{aligned}
H = & p_{r,x}u + p_{r,y}v + p_{r,z}w + p_u g_x + p_v g_y + p_w g_z + \frac{p_u}{m} R_{w,11}^e [(\eta T_{max} + X)\mathbf{1}_{x,w,1} + Y\mathbf{1}_{y,w,1} + Z\mathbf{1}_{z,w,1}] + \\
& + \frac{p_u}{m} R_{w,12}^e [(\eta T_{max} + X)\mathbf{1}_{x,w,2} + Y\mathbf{1}_{y,w,2} + Z\mathbf{1}_{z,w,2}] + \frac{p_u}{m} R_{w,13}^e [(\eta T_{max} + X)\mathbf{1}_{x,w,3} + Y\mathbf{1}_{y,w,3} + Z\mathbf{1}_{z,w,3}] + \\
& + \frac{p_v}{m} R_{w,21}^e [(\eta T_{max} + X)\mathbf{1}_{x,w,1} + Y\mathbf{1}_{y,w,1} + Z\mathbf{1}_{z,w,1}] + \frac{p_v}{m} R_{w,22}^e [(\eta T_{max} + X)\mathbf{1}_{x,w,2} + Y\mathbf{1}_{y,w,2} + Z\mathbf{1}_{z,w,2}] + \\
& + \frac{p_v}{m} R_{w,23}^e [(\eta T_{max} + X)\mathbf{1}_{x,w,3} + Y\mathbf{1}_{y,w,3} + Z\mathbf{1}_{z,w,3}] + \frac{p_w}{m} R_{w,31}^e [(\eta T_{max} + X)\mathbf{1}_{x,w,1} + Y\mathbf{1}_{y,w,1} + Z\mathbf{1}_{z,w,1}] + \\
& + \frac{p_w}{m} R_{w,32}^e [(\eta T_{max} + X)\mathbf{1}_{x,w,2} + Y\mathbf{1}_{y,w,2} + Z\mathbf{1}_{z,w,2}] + \frac{p_w}{m} R_{w,33}^e [(\eta T_{max} + X)\mathbf{1}_{x,w,3} + Y\mathbf{1}_{y,w,3} + Z\mathbf{1}_{z,w,3}] + \\
& - p_m \frac{\eta T_{max}}{c}
\end{aligned} \tag{115}$$

The scalar costate equations for the position, $p_\square$, with $\square \in [x, y, z]$ then read:

$$\begin{aligned}
\dot{p}_\square = & -\frac{\partial H}{\partial \square} = -p_u g_{x,\square} - p_v g_{y,\square} - p_w g_{z,\square} - \frac{p_u}{m} \left[R_{w,11}^e (X_\square \mathbf{1}_{x,w,1} + Y_\square \mathbf{1}_{y,w,1} + Z_\square \mathbf{1}_{z,w,1}) \right] + \\
& - \frac{p_u}{m} \left[R_{w,12}^e (X_\square \mathbf{1}_{x,w,2} + Y_\square \mathbf{1}_{y,w,2} + Z_\square \mathbf{1}_{z,w,2}) \right] - \frac{p_u}{m} \left[R_{w,13}^e (X_\square \mathbf{1}_{x,w,3} + Y_\square \mathbf{1}_{y,w,3} + Z_\square \mathbf{1}_{z,w,3}) \right] + \\
& - \frac{p_v}{m} \left[R_{w,21}^e (X_\square \mathbf{1}_{x,w,1} + Y_\square \mathbf{1}_{y,w,1} + Z_\square \mathbf{1}_{z,w,1}) \right] - \frac{p_v}{m} \left[R_{w,22}^e (X_\square \mathbf{1}_{x,w,2} + Y_\square \mathbf{1}_{y,w,2} + Z_\square \mathbf{1}_{z,w,2}) \right] + \\
& - \frac{p_v}{m} \left[R_{w,23}^e (X_\square \mathbf{1}_{x,w,3} + Y_\square \mathbf{1}_{y,w,3} + Z_\square \mathbf{1}_{z,w,3}) \right] - \frac{p_w}{m} \left[R_{w,31}^e (X_\square \mathbf{1}_{x,w,1} + Y_\square \mathbf{1}_{y,w,1} + Z_\square \mathbf{1}_{z,w,1}) \right] + \\
& - \frac{p_w}{m} \left[R_{w,32}^e (X_\square \mathbf{1}_{x,w,2} + Y_\square \mathbf{1}_{y,w,2} + Z_\square \mathbf{1}_{z,w,2}) \right] - \frac{p_w}{m} \left[R_{w,33}^e (X_\square \mathbf{1}_{x,w,3} + Y_\square \mathbf{1}_{y,w,3} + Z_\square \mathbf{1}_{z,w,3}) \right]
\end{aligned} \tag{116}$$

Analogously, the scalar differential equations for the primer vectors, $p_\diamond$, with $\diamond \in [u, v, w]$ are

$$\begin{aligned}
\dot{p}_\diamond = & -\frac{\partial H}{\partial \diamond} = -p_{r,\square} - \frac{p_u}{m} R_{w,11,\diamond}^e [(\eta T_{max} + X) \mathbf{1}_{x,w,1} + Y \mathbf{1}_{y,w,1} + Z \mathbf{1}_{z,w,1}] + \\
& -\frac{p_u}{m} R_{w,12,\diamond}^e [(\eta T_{max} + X) \mathbf{1}_{x,w,2} + Y \mathbf{1}_{y,w,2} + Z \mathbf{1}_{z,w,2}] - \frac{p_u}{m} R_{w,13,\diamond}^e [(\eta T_{max} + X) \mathbf{1}_{x,w,3} + Y \mathbf{1}_{y,w,3} + Z \mathbf{1}_{z,w,3}] + \\
& -\frac{p_v}{m} R_{w,21,\diamond}^e [(\eta T_{max} + X) \mathbf{1}_{x,w,1} + Y \mathbf{1}_{y,w,1} + Z \mathbf{1}_{z,w,1}] - \frac{p_v}{m} R_{w,22,\diamond}^e [(\eta T_{max} + X) \mathbf{1}_{x,w,2} + Y \mathbf{1}_{y,w,2} + Z \mathbf{1}_{z,w,2}] + \\
& -\frac{p_v}{m} R_{w,23,\diamond}^e [(\eta T_{max} + X) \mathbf{1}_{x,w,3} + Y \mathbf{1}_{y,w,3} + Z \mathbf{1}_{z,w,3}] - \frac{p_w}{m} R_{w,31,\diamond}^e [(\eta T_{max} + X) \mathbf{1}_{x,w,1} + Y \mathbf{1}_{y,w,1} + Z \mathbf{1}_{z,w,1}] + \\
& -\frac{p_w}{m} R_{w,32,\diamond}^e [(\eta T_{max} + X) \mathbf{1}_{x,w,2} + Y \mathbf{1}_{y,w,2} + Z \mathbf{1}_{z,w,2}] - \frac{p_w}{m} R_{w,33,\diamond}^e [(\eta T_{max} + X) \mathbf{1}_{x,w,3} + Y \mathbf{1}_{y,w,3} + Z \mathbf{1}_{z,w,3}] + \\
& -\frac{p_u}{m} R_{w,11}^e [X_\diamond \mathbf{1}_{x,w,1} + Y_\diamond \mathbf{1}_{y,w,1} + Z_\diamond \mathbf{1}_{z,w,1}] - \frac{p_u}{m} R_{w,12}^e [X_\diamond \mathbf{1}_{x,w,2} + Y_\diamond \mathbf{1}_{y,w,2} + Z_\diamond \mathbf{1}_{z,w,2}] + \\
& -\frac{p_u}{m} R_{w,13}^e [X_\diamond \mathbf{1}_{x,w,3} + Y_\diamond \mathbf{1}_{y,w,3} + Z_\diamond \mathbf{1}_{z,w,3}] - \frac{p_v}{m} R_{w,21}^e [X_\diamond \mathbf{1}_{x,w,1} + Y_\diamond \mathbf{1}_{y,w,1} + Z_\diamond \mathbf{1}_{z,w,1}] + \\
& -\frac{p_v}{m} R_{w,22}^e [X_\diamond \mathbf{1}_{x,w,2} + Y_\diamond \mathbf{1}_{y,w,2} + Z_\diamond \mathbf{1}_{z,w,2}] - \frac{p_v}{m} R_{w,23}^e [X_\diamond \mathbf{1}_{x,w,3} + Y_\diamond \mathbf{1}_{y,w,3} + Z_\diamond \mathbf{1}_{z,w,3}] + \\
& -\frac{p_w}{m} R_{w,31}^e [X_\diamond \mathbf{1}_{x,w,1} + Y_\diamond \mathbf{1}_{y,w,1} + Z_\diamond \mathbf{1}_{z,w,1}] - \frac{p_w}{m} R_{w,32}^e [X_\diamond \mathbf{1}_{x,w,2} + Y_\diamond \mathbf{1}_{y,w,2} + Z_\diamond \mathbf{1}_{z,w,2}] + \\
& -\frac{p_w}{m} R_{w,33}^e [X_\diamond \mathbf{1}_{x,w,3} + Y_\diamond \mathbf{1}_{y,w,3} + Z_\diamond \mathbf{1}_{z,w,3}]
\end{aligned} \tag{117}$$

Note that the explicit terms of $R_{w,ij,\diamond}^e$ are defined in Eq. (113). Finally, for the mass costate, we can write

$$\begin{aligned}
\dot{p}_m = & -\frac{\partial H}{\partial m} = +\frac{p_u}{m^2} R_{w,11}^e [(\eta T_{max} + X) \mathbf{1}_{x,w,1} + Y \mathbf{1}_{y,w,1} + Z \mathbf{1}_{z,w,1}] + \\
& +\frac{p_u}{m^2} R_{w,12}^e [(\eta T_{max} + X) \mathbf{1}_{x,w,2} + Y \mathbf{1}_{y,w,2} + Z \mathbf{1}_{z,w,2}] + \frac{p_u}{m^2} R_{w,13}^e [(\eta T_{max} + X) \mathbf{1}_{x,w,3} + Y \mathbf{1}_{y,w,3} + Z \mathbf{1}_{z,w,3}] + \\
& +\frac{p_v}{m^2} R_{w,21}^e [(\eta T_{max} + X) \mathbf{1}_{x,w,1} + Y \mathbf{1}_{y,w,1} + Z \mathbf{1}_{z,w,1}] + \frac{p_v}{m^2} R_{w,22}^e [(\eta T_{max} + X) \mathbf{1}_{x,w,2} + Y \mathbf{1}_{y,w,2} + Z \mathbf{1}_{z,w,2}] + \\
& +\frac{p_v}{m^2} R_{w,23}^e [(\eta T_{max} + X) \mathbf{1}_{x,w,3} + Y \mathbf{1}_{y,w,3} + Z \mathbf{1}_{z,w,3}] + \frac{p_w}{m^2} R_{w,31}^e [(\eta T_{max} + X) \mathbf{1}_{x,w,1} + Y \mathbf{1}_{y,w,1} + Z \mathbf{1}_{z,w,1}] + \\
& +\frac{p_w}{m^2} R_{w,32}^e [(\eta T_{max} + X) \mathbf{1}_{x,w,2} + Y \mathbf{1}_{y,w,2} + Z \mathbf{1}_{z,w,2}] + \frac{p_w}{m^2} R_{w,33}^e [(\eta T_{max} + X) \mathbf{1}_{x,w,3} + Y \mathbf{1}_{y,w,3} + Z \mathbf{1}_{z,w,3}] = \\
& = \mathbf{p}_v \cdot [\mathbf{R}_w^e (\mathbf{F}_w^T + \mathbf{F}_w^A)]
\end{aligned} \tag{118}$$

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