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(Article begins on next page)

LOCAL BOUNDEDNESS FOR SOLUTIONS OF A CLASS OF NON-UNIFORMLY ELLIPTIC ANISOTROPIC PROBLEMS

STEFANO BIAGI, GIOVANNI CUPINI, ELVIRA MASCOLO

ABSTRACT. We consider a class of energy integrals, associated to nonlinear and non-uniformly elliptic equations, with integrands $f(x, u, \xi)$ satisfying anisotropic p_i, q -growth conditions of the form

$$\sum_{i=1}^n \lambda_i(x) |\xi_i|^{p_i} \leq f(x, u, \xi) \leq \mu(x) \{|\xi|^q + |u|^\gamma + 1\}$$

for some exponents $\gamma \geq q \geq p_i > 1$, and non-negative functions λ_i, μ subject to suitable summability assumptions. We prove the local boundedness of scalar local quasi-minimizers of such integrals.

*This paper is dedicated to Gioconda Moscariello in the occasion of her 70th birthday,
with friendship and esteem for her influential contributions on the areas of Partial
Differential Equations and of Calculus of Variations.*

1. INTRODUCTION

Given a local minimizer u of an integral functional, it is important to investigate the additional structure and integrability properties of the associated energy density in order to establish the regularity of u , at least in the interior of the domain Ω .

A powerful method in regularity theory is due to De Giorgi [14], who introduced an approach based on *Caccioppoli-type inequalities on level sets*. This method, through an iteration procedure, first yields local boundedness and then, by a delicate and ingenious argument, establishes the local Hölder

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continuity of weak solutions to the linear uniformly elliptic equation:

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_i} (a_{ij}(x) u_{x_j}) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n, \quad n \geq 2, \quad (1.1)$$

where the coefficients a_{ij} are measurable functions such that $a_{ij} = a_{ji}$ and they satisfying the uniform ellipticity condition

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \leq \mu |\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^n \text{ and a.e. } x \in \Omega,$$

with λ, μ positive constants such that $\lambda \leq \mu$ (in other words, the principal matrix $A(x) = (a_{ij}(x))_{i,j}$ is symmetric and positive definite in Ω).

In 1971, Trudinger [29] considered the same equation (1.1), but with measurable coefficients a_{ij} which satisfy (for every $\xi \in \mathbb{R}^n$ and a.e. $x \in \Omega$)

$$\lambda(x) |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \leq \mu(x) |\xi|^2$$

without assuming that $\lambda^{-1}, \mu \in L^\infty$, that is, *without the uniform ellipticity condition*. Trudinger proved that, if

$$\lambda^{-1} \in L_{\text{loc}}^r(\Omega), \quad \lambda^{-1} \mu^2 \in L_{\text{loc}}^s(\Omega),$$

for suitable exponents $r, s \geq 1$ satisfying the relation

$$\frac{1}{r} + \frac{1}{s} < \frac{2}{n}, \quad (1.2)$$

then the weak solutions are locally bounded. For related results, see [3, 7, 9, 12, 13, 15, 26]. Trudinger's result has been recently improved by Bella and Schäffer [2], who relaxed the above relation, proving that the right-hand side in (1.2) can be replaced by $\frac{2}{n-1}$.

In this paper, we consider integral functionals associated with non-uniformly elliptic problems. More precisely, we investigate functionals from the Calculus of Variations of the form:

$$\mathcal{F} : W_{\text{loc}}^{1,1}(\Omega) \rightarrow [0, \infty], \quad \mathcal{F}(u) := \int_{\Omega} f(x, u, Du) \, dx, \quad (1.3)$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded open set and f is a real-valued Carathéodory function on $\Omega \times \mathbb{R} \times \mathbb{R}^n$, satisfying the following *anisotropic* and *non-uniform* growth conditions:

For some exponents $1 < p_i \leq q \leq \gamma$, and for measurable functions $\lambda_i, \mu : \Omega \rightarrow [0, \infty)$, with $i = 1, \dots, n$, we assume

$$\sum_{i=1}^n \lambda_i(x) |\xi_i|^{p_i} \leq f(x, u, \xi) \leq \mu(x) (|\xi|^q + |u|^\gamma + 1) \quad (1.4)$$

for almost every $x \in \Omega$ and every $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, where

$$\lambda_i^{-1} \in L_{\text{loc}}^{r_i}(\Omega), \quad \mu \in L_{\text{loc}}^s(\Omega),$$

for some $r_i \in [1, \infty]$ and $s \in (1, \infty]$.

Under suitable relationships among the exponents (see (3.4)), we prove that local-minimizers (or, more generally, *quasi-minimizers*) u of (1.3) are locally bounded. Moreover, an explicit estimate for $\|u\|_{L^\infty}$ on a given compact subset of Ω is provided (see (3.5)). This result extends a previous work by the same authors, non-isotropic case where the case $p = p_i = q$ was considered, see [3]. In this setting of anisotropic growth and degenerate ellipticity conditions, we cite the very recent articles [16] (scalar framework) and [1, 30] (vectorial), where local boundedness results have been proved.

It is worth mentioning that, since the counterexamples in [19, 21], the foundational works by Marcellini [22, 23] and the papers [4, 10, 17, 18, 27], it is now well-known that *the regularity of the minimizers may be lost, even in the non-degenerate case* (i.e., when the coefficients λ_i 's are strictly positive), if the exponents p_i 's and q are spread out

From a physical perspective, anisotropy refers to the property of a material to exhibit different mechanical, electrical, thermal, or optical behaviors depending on the direction. This feature arises in many physical models, including magnetic theory (via density functionals), composite materials, convection-diffusion problems, and the analysis of sedimentary rocks. Since the aforementioned papers [22, 23], anisotropic problems have received considerable attention, and it is nearly impossible to provide an exhaustive list of references. We mention here the articles [5, 6, 8, 11]; see also the recent surveys [24, 25] for an overview of the subject.

Plan of the paper. The structure of the paper is as follows. In Section 2, we introduce the notation used throughout the work. Section 3 contains the precise assumptions on the integrand and the statement of the main regularity result. Section 4 is devoted to the proof of a new anisotropic Caccioppoli inequality, which is a key tool in the analysis. Finally, in Section 5, we develop the iteration scheme that leads to the local boundedness of minimizers and establish the L^∞ estimate.

2. NOTATION

Throughout the paper, we will tacitly exploit all the notation listed below; we thus refer the Reader to this list for any (non-standard) notation encountered.

- If $A, B \subseteq \mathbb{R}^n$ are arbitrary sets, we write $A \Subset B$ if A is *relatively compact in B* , that is, A is bounded and $\overline{A} \subset B$;
- Given any $n \in \mathbb{N}$ and any $1 \leq \beta < n$, we indicate by β^* the *Sobolev exponent* of β with respect to $\mathbb{R}^n$, that is,

$$\beta^* := \frac{n\beta}{n - \beta}. \quad (2.1)$$

- Given any $\beta \in [1, \infty]$, we indicate by β' the *conjugate exponent* of β in the classical Hölder inequality, that is,

$$\beta' = \begin{cases} \frac{\beta}{\beta-1} & \text{if } 1 < \beta < \infty \\ \infty & \text{if } \beta = 1 \\ 1 & \text{if } \beta = \infty \end{cases}$$

- Given any multi-index $\beta = (\beta_1, \dots, \beta_n)$ with $1 \leq \beta_1, \dots, \beta_n \leq \infty$, we indicate by $\bar{\beta}$ the harmonic average of the β_i 's, that is,

$$\bar{\beta} = \frac{n}{\sum_{i=1}^n 1/\beta_i} \quad (2.2)$$

(with the usual convention $1/\infty = 0$).

3. ASSUMPTIONS AND STATEMENT OF THE MAIN RESULTS

Consider the integral functional

$$F(u; \Omega) := \int_{\Omega} f(x, u, Du) dx, \quad (3.1)$$

where Ω is an open and bounded subset of $\mathbb{R}^n$ (with $n \geq 2$) and

$$f : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$$

is a Carathéodory function satisfying the following *structural assumptions*

(H1) $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is convex in the pair (s, ξ)

(H2) for a.e. $x \in \Omega$ and for every $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$ we have

$$\sum_{i=1}^n \lambda_i(x) |\xi_i|^{p_i} \leq f(x, u, \xi) \leq \mu(x) \{|\xi|^q + |u|^\gamma + 1\}, \quad (3.2)$$

where $1 \leq p_1, \dots, p_n \leq q \leq \gamma$, and $\lambda_1, \dots, \lambda_n, \mu : \Omega \rightarrow (0, +\infty)$ are non-negative, measurable functions such that

$$\lambda_i^{-1} \in L_{\text{loc}}^{r_i}(\Omega) \text{ (for } 1 \leq i \leq n), \quad \mu \in L_{\text{loc}}^s(\Omega), \quad (3.3)$$

for some exponents $r_i \in [1, \infty]$ and $s \in (1, \infty]$.

Recall the definition of quasi-minimizers of functional in (3.1).

Definition 3.1. A function $u \in W_{\text{loc}}^{1,1}(\Omega)$ is a *quasi-minimizer* of (3.1) if there exists some constant $Q \geq 1$ (depending on u) such that

- i) $f(x, u, Du) \in L_{\text{loc}}^1(\Omega)$;
- ii) for every $\varphi \in W^{1,1}(\Omega)$ with $\text{supp } \varphi \subset \Omega$ we have

$$F(u; \text{supp } \varphi) \leq Q F(u + \varphi; \text{supp } \varphi),$$

Let us now state our main regularity result.

Theorem 3.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and let f be a Carathéodory function satisfying assumptions **(H1)** and **(H2)** (of which we inherit the notation). Setting $\mathbf{p} = (p_1, \dots, p_n)$ and*

$$\boldsymbol{\sigma} := (\sigma_1, \dots, \sigma_n), \quad \text{with } \sigma_i := p_i \cdot \frac{r_i}{r_i + 1},$$

(with the convention that $\frac{r_i}{r_i+1} = 1$ if $r_i = \infty$), we assume that

$$(i) \ \bar{\sigma} < n \quad (ii) \ q < \frac{\bar{\sigma}^*}{s'} \quad (iii) \ q \leq \gamma < \frac{\bar{\sigma}^*}{s'} \cdot \frac{\bar{p}}{q} + q - \bar{p}. \quad (3.4)$$

Then, any quasi-minimizer $u \in W_{\text{loc}}^{1,1}(\Omega)$ of (3.1) is locally bounded in Ω .

More precisely, given any $x_0 \in \Omega$ and any $R_0 < 1$ such that $B_{R_0}(x_0) \Subset \Omega$, for every fixed $0 < R \leq R_0$ we have the following estimate

$$\|u\|_{L^\infty(B_{R/2}(x_0))} \leq \frac{c}{R^{\vartheta_2}} [1 + \|u\|_{L^{\bar{\sigma}^*}(B_R(x_0))}]^{\vartheta_1} \quad (3.5)$$

where ϑ_1, ϑ_2 are explicitly given by

$$\vartheta_1 := \frac{\bar{\sigma}^* \gamma - q s' \bar{p}}{\bar{p} \cdot \bar{\sigma}^* - q s' (\gamma - q + \bar{p})}$$

$$\vartheta_2 := \frac{q \bar{\sigma}^*}{\bar{p} \cdot \bar{\sigma}^* - q s' (\gamma - q + \bar{p})}$$

and $c > 0$ is a constant only depending on the data in assumption **(H2)**.

Remark 3.3. A few remarks on Theorem 3.2 are in order.

- (1) The second inequality in assumption (3.4) ensures that the range of admissible values for γ in (3.4)-(iii) is non-empty: indeed, we have

$$\frac{\bar{\sigma}^*}{s'} \cdot \frac{\bar{p}}{q} + q - \bar{p} > \bar{p} + (q - \bar{p}) = q.$$

- (2) In the particular case when Assumption **(H2)** holds with $p_i = p = q$, then Assumption (3.4)-(ii) is equivalent to

$$\frac{1}{r} + \frac{1}{s} < \frac{p}{n},$$

which is consistent with condition (1.9) in [3]. If, moreover, we also have $r_i = s = \infty$, Assumption (3.4)-(iii) boils down to

$$p < n \quad \text{and} \quad \gamma < p^*$$

and we have

$$(*) \quad \vartheta_1 = \frac{p^* \gamma - p^2}{p(p^* - \gamma)}$$

$$(*) \quad \vartheta_2 = \frac{p^*}{p^* - \gamma}.$$

- (3) The boundedness result in Theorem 3.2 still holds if f satisfies assumption **(H2)** with $0 \leq \gamma < q$. Indeed, in this case we have

$$\begin{aligned} \mu(x) \{|\xi|^q + |u|^\gamma + 1\} &\leq \mu(x) \{|\xi|^q + (1 + |u|^q) + 1\} \\ &\leq 2\mu(x) \{|\xi|^q + |u|^q + 1\}, \end{aligned}$$

and this shows that f satisfies **(H2)** also with the choice $\gamma = q$. As a consequence, we can apply Theorem 3.2 with $\gamma = q$, thus obtaining the following specialized version of estimate (3.5):

$$\|u\|_{L^\infty(B_{R/2}(x_0))} \leq \frac{c}{R^{\frac{q \bar{\sigma}^*}{\bar{p}(\bar{\sigma}^* - q s')}}} [1 + \|u\|_{L^{\bar{\sigma}^*}(B_R(x_0))}]^{\frac{q(\bar{\sigma}^* - s' \bar{p})}{\bar{p}(\bar{\sigma}^* - q s')}}]$$

(which holds true for every quasi-minimizer u of (3.1), every $x_0 \in \Omega$ and every $0 < R \leq R_0 \leq 1$ with $B_{R_0}(x_0) \Subset \Omega$).

4. PRELIMINARY RESULTS

In order to keep the paper as self-contained as possible, we collect here below all the preliminary (and mostly well-known) results which will be used in the sequel. Throughout what follows, we tacitly inherit all the notation introduced in the previous sections (in particular, those appearing in the structural assumptions **(H1)** - **(H2)**).

1) Anisotropic Sobolev spaces. Let $\mathcal{O} \subseteq \mathbb{R}^n$ be an arbitrary open set, and let $\beta = (\beta_1, \dots, \beta_n)$, with $1 \leq \beta_1, \dots, \beta_n \leq \infty$. The *anisotropic Sobolev space* $W^{1,\beta}(\mathcal{O})$ is the space of functions defined as follows

$$W^{1,\beta}(\mathcal{O}) = \{u \in W^{1,1}(\mathcal{O}) : u_{x_i} \in L^{\beta_i}(\mathcal{O}) \text{ for all } 1 \leq i \leq n\},$$

endowed with the norm

$$\|u\|_{W^{1,\beta}(\mathcal{O})} := \|u\|_{L^1(\mathcal{O})} + \sum_{i=1}^n \|u_{x_i}\|_{L^{\beta_i}(\mathcal{O})}.$$

We also define

$$W_0^{1,\beta}(\mathcal{O}) = W_0^{1,1}(\mathcal{O}) \cap W^{1,\beta}(\mathcal{O}).$$

Finally, we say that $u \in W_{\text{loc}}^{1,\beta}(\mathcal{O})$ if

$$u \in W^{1,\beta}(\mathcal{V}) \text{ for every open set } \mathcal{V} \Subset \mathcal{O}.$$

In view of the growth condition in assumption **(H2)**, the anisotropic Sobolev spaces are naturally related with the functional F defined in (3.1); more precisely, we have the following proposition.

Proposition 4.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and let f be a Carathéodory function satisfying assumption **(H2)**. We set*

$$\sigma = (\sigma_1, \dots, \sigma_n), \quad \text{with } \sigma_i = p_i \cdot \frac{r_i}{r_i + 1} \quad (1 \leq i \leq n)$$

(here, p_i and r_i are the exponents introduced in (3.3)).

Then the following estimate holds

$$\frac{1}{n} \sum_{i=1}^n \|\lambda_i^{-1}\|_{L^{r_i}(\Omega')}^{-1} \|v_{x_i}\|_{L^{\sigma_i}(\Omega')}^{p_i} \leq \int_{\Omega'} f(x, v, Dv) dx \quad (4.1)$$

for every $v \in W_{\text{loc}}^{1,1}(\Omega)$ and every bounded open set $\Omega' \Subset \Omega$.

As a consequence, if $u \in W_{\text{loc}}^{1,1}(\Omega)$ is a quasi-minimizer of the functional F associated with f and defined in (3.1), we have

$$u \in W_{\text{loc}}^{1,\sigma}(\Omega). \quad (4.2)$$

Proof. Let Ω', v be as in the statement. We distinguish two cases.

CASE I): $r_i < \infty$. In this case, by exploiting the classical Hölder inequality with exponents $\frac{r_i+1}{r_i}$ and $r_i + 1$, we have

$$\begin{aligned} \int_{\Omega'} |v_{x_i}|^{\frac{p_i r_i}{r_i+1}} dx &= \int_{\Omega'} \lambda_i^{-\frac{r_i}{r_i+1}} \lambda_i^{\frac{r_i}{r_i+1}} |v_{x_i}|^{\frac{p_i r_i}{r_i+1}} dx \\ &\leq \|\lambda_i^{-1}\|_{L^{r_i}(\Omega')}^{\frac{r_i}{r_i+1}} \left(\int_{\Omega'} \lambda_i |v_{x_i}|^{p_i} dx \right)^{\frac{r_i}{r_i+1}}. \end{aligned} \quad (4.3)$$

By the growth condition (3.2), we have

$$\int_{\Omega'} \lambda_i |v_{x_i}|^{p_i} dx \leq \int_{\Omega'} f(x, v, Dv) dx;$$

as a consequence, we obtain

$$\left(\int_{\Omega'} |v_{x_i}|^{\frac{p_i r_i}{r_i+1}} dx \right)^{\frac{r_i+1}{r_i}} \leq \|\lambda_i^{-1}\|_{L^{r_i}(\Omega')} \int_{\Omega'} f(x, v, Dv) dx, \quad (4.4)$$

CASE II): $r_i = \infty$. In this case, the convention $\frac{r_i}{r_i+1} = 1$ applies. Thus,

$$\begin{aligned} \int_{\Omega'} |v_{x_i}|^{\frac{p_i r_i}{r_i+1}} dx &= \int_{\Omega'} |v_{x_i}|^{p_i} dx = \int_{\Omega'} \lambda_i^{-1} \lambda_i |v_{x_i}|^{p_i} dx \\ &\leq \|\lambda_i^{-1}\|_{L^\infty(\Omega')} \int_{\Omega'} \lambda_i |v_{x_i}|^{p_i} dx. \end{aligned}$$

that is (4.3) holds true also for $r_i = \infty$. Therefore, reasoning as above, we conclude that (4.4) holds also in this case.

From (4.4), adding with respect to i we readily derive the desired (4.1).

Let now $u \in W_{\text{loc}}^{1,1}(\Omega)$ be a quasi-minimizer of (3.1) (in the sense of Definition 3.1). Since, by definition we know that

$$f(x, u, Du) \in L_{\text{loc}}^1(\Omega),$$

from (4.1) we immediately derive (4.2). $\square$

As for *classical case* $\beta_1 = \dots = \beta_n = \beta$ considered in [3] (and in view of Proposition 4.1), the following Embedding Theorem for $W^{1,(\beta_1, \dots, \beta_n)}(\mathcal{O})$ will play key role in our argument; for a proof we refer, e.g., to [5, 28].

Theorem 4.2. *Let $\mathcal{O} \subseteq \mathbb{R}^n$ be a bounded open set, and let $\beta = (\beta_1, \dots, \beta_n)$, with $1 \leq \beta_1 \leq \dots \leq \beta_n \leq \infty$. We suppose that*

$$\bar{\beta} < n.$$

Then, the following facts hold.

- 1) *There exists a constant $c > 0$, only depending on n, β , such that*

$$\|u\|_{L^{\bar{\beta}^*}(\mathcal{O})} \leq c \left(\prod_{i=1}^n \|u_{x_i}\|_{L^{\beta_i}(\mathcal{O})} \right)^{\frac{1}{n}}$$

for every $u \in W_0^{1,\beta}(\mathcal{O})$.

- 2) *Assume that $\max_i \beta_i < \bar{\beta}^*$. Then, for every open set $\mathcal{V} \Subset \mathcal{O}$ we have the following continuous embedding of $W^{1,\beta}(\mathcal{O})$:*

$$W^{1,\beta}(\mathcal{O}) \hookrightarrow L^{\bar{\beta}^*}(\mathcal{V}).$$

By combining the previous Proposition 4.1 with the “anisotropic” embedding in Theorem 4.2, we derive the following simple yet crucial “higher-integrability” result for quasi-minimizers of (3.1).

Proposition 4.3. *Let the assumptions and the notation of Theorem 3.2 be in force, and let $u \in W_{\text{loc}}^{1,1}(\Omega)$ be a quasi-minimizer of (3.1). Then,*

$$u \in L_{\text{loc}}^{\bar{\sigma}^*}(\Omega).$$

Hence, in particular, $u \in L_{\text{loc}}^{qs'}(\Omega)$.

Proof. First of all we observe that, since u is a quasi-minimizer of (3.1) and since f satisfies assumption **(H2)**, we know from Proposition 4.1 that

$$u \in W_{\text{loc}}^{1,\sigma}(\Omega).$$

On the other hand, owing to (3.4)-(ii), we have

$$\sigma_i = p_i \cdot \frac{r_i}{r_i + 1} \leq p_i \leq q \leq qs' < \bar{\sigma}^* \quad \text{for all } 1 \leq i \leq n. \quad (4.5)$$

In view of this fact, we are then entitled to apply the anisotropic Embedding Theorem in Theorem 4.2-2), ensuring that $u \in L_{\text{loc}}^{\bar{\sigma}^*}(\Omega)$. From this, since

$$qs' < \bar{\sigma}^*,$$

we also derive that $u \in L_{\text{loc}}^{qs'}(\Omega)$, as desired. $\square$

The next proposition is a Poincaré-Sobolev - type inequality for the anisotropic setting. From now on, we adopt the notation already introduced in Proposition 4.1: with reference to assumption **(H2)**, we set

$$\sigma = (\sigma_1, \dots, \sigma_n), \quad \text{with } \sigma_i = p_i \cdot \frac{r_i}{r_i + 1} \quad (1 \leq i \leq n);$$

accordingly, $\bar{\sigma}$ is the harmonic mean of the σ_i 's as defined in (2.2), and $\bar{\sigma}^*$ is the Sobolev exponent of $\bar{\sigma}$ as defined in (2.1).

Proposition 4.4. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and let f be a Carathéodory function satisfying assumption **(H2)**. Moreover, let Ω' be a bounded open set such that $\Omega' \Subset \Omega$.*

Then, there exists a positive constant $c > 0$ such that

$$\begin{aligned} & \left(\int_{\Omega'} |v|^{\bar{\sigma}^*} dx \right)^{\frac{1}{\bar{\sigma}^*}} \\ & \leq c \left\{ \prod_{i=1}^n \left[\|\lambda_i^{-1}\|_{L^{r_i}(\Omega')} \int_{\Omega'} \lambda_i |v_{x_i}|^{p_i} dx \right]^{\frac{1}{p_i}} \right\}^{\frac{1}{n}}, \end{aligned}$$

for every $v \in W_0^{1,\sigma}(\Omega')$.

Proof. The inequality follows by collecting Theorem 4.2 and (4.3). $\square$

2) Some algebraic inequalities. In the next lemma we collect some useful algebraic inequalities among the parameters involved in Theorem 3.2, which are consequences of assumption (3.4).

Lemma 4.5. *Let the assumptions and the notation of Theorem 3.2 be in force. Then, the following assertions hold.*

- (1) $1 \leq \bar{p} \leq q$;
- (2) for every $1 \leq i \leq n$, we have $\lambda_i \leq 2\mu$ a.e. in Ω ;

Proof of (1). Since, by assumption, $1 \leq p_1, \dots, p_n \leq q$, we get

$$1 = \frac{1}{n} \cdot n \geq \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i} \geq \frac{1}{n} \cdot \frac{n}{q} = \frac{1}{q};$$

thus, by definition of $\bar{p}$ we immediately get $1 \leq \bar{p} \leq q$.

Proof of (2). By choosing $\xi = e_i$ (where e_i denotes the i -th vector of the canonical basis of $\mathbb{R}^n$) and $u = 0$ in (3.2), we immediately obtain

$$\begin{aligned}\lambda_i(x) &= \lambda_i(x)|\xi_i|^{p_i} = \sum_{j=1}^n \lambda_j(x)|\xi_j|^{p_i} \\ &\leq f(x, 0, \xi) \leq \mu(x)\{|\xi|^q + 1\} = 2\mu(x).\end{aligned}$$

This ends the proof. $\square$

3) Real-analysis lemmas. To prove the regularity result in Theorems 3.2 we will use the following two classical lemmas (see, e.g., [20] for a proof).

Lemma 4.6. *Let $\phi : [\tau_0, \tau_1] \rightarrow \mathbb{R}$ be a non-negative and bounded function satisfying, for every $\tau_0 \leq s < t \leq \tau_1$, the following estimate*

$$\phi(s) \leq \theta\phi(t) + \frac{A}{(t-s)^\alpha} + B,$$

where A, B, α are non-negative constants and $0 < \theta < 1$.

Then, for all ρ and R , such that $\tau_0 \leq \rho \leq R \leq \tau_1$, we have

$$\phi(\rho) \leq C \left\{ \frac{A}{(R-\rho)^\alpha} + B \right\}.$$

Lemma 4.7. *Let $\alpha > 0$, and let (J_h) be a sequence of positive real numbers satisfying, for some $A > 0$ and $\lambda > 1$, the following estimate*

$$J_{h+1} \leq A\lambda^h J_h^{1+\alpha} \quad \text{for all } h \geq 0.$$

If $J_0 \leq A^{-\frac{1}{\alpha}} \lambda^{-\frac{1}{\alpha^2}}$, then

$$J_h \leq \lambda^{-\frac{h}{\alpha}} J_0.$$

Hence, in particular, we have

$$\lim_{h \rightarrow \infty} J_h = 0.$$

5. A CACCIOPPOLI-TYPE INEQUALITY FOR (3.1)

In this section we establish a Caccioppoli-type inequality for the quasi-minimizers of the functional F in (3.1); as it is natural to expect, this result will be a key tool for proving Theorem 3.2.

In order to properly state and prove the aforementioned Caccioppoli-type inequality, we first fix a notation: given any function $u \in W_{\text{loc}}^{1,1}(\Omega)$ and any Euclidean ball $B_R(x_0) \subseteq \Omega$, we define the super-level sets:

$$A_{k,R} := \{x \in B_R(x_0) : u(x) > k\}, \quad k \in \mathbb{R}.$$

Proposition 5.1. *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a Carathéodory function satisfying (H1) and the following one-side version of (H2), that is,*

$$0 \leq f(x, u, \xi) \leq \mu(x) \{|\xi|^q + |u|^\gamma + 1\}, \quad (5.1)$$

with $1 \leq q \leq \gamma$ and $\mu \in L_{\text{loc}}^s(\Omega)$ (for some $s \in (1, \infty]$). Moreover, let

$$u \in W_{\text{loc}}^{1,1}(\Omega) \cap L_{\text{loc}}^{q'}(\Omega) \quad (5.2)$$

be a quasi-minimizer of F , and let $B = B_R(x_0) \subset \Omega$.

Then, for every $0 < \rho < R$ and every $k \geq 1$ we have

$$\begin{aligned} & \int_{A_{k,\rho}} f(x, u, Du) dx \\ & \leq C \left\{ \frac{1}{(R-\rho)^q} \int_{A_{k,R}} \mu(x) ((u-k)^q + k^\gamma) dx \right. \\ & \quad \left. + \|\mu\|_{L^s(B_R(x_0))} |A_{k,R}|^{1/s'} \right\}, \end{aligned} \quad (5.3)$$

where $c > 0$ is a constant only depending on n, q, Q .

Proof. Let ρ, k be as in the statement of the proposition, and let $s, t > 0$ be such that $\rho \leq s < t \leq R$. Moreover, let $\eta \in C_0^\infty(B_t)$ be a cut-off function satisfying the following assumptions:

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ in } B_s(x_0), \quad |D\eta| \leq \frac{2}{t-s}. \quad (5.4)$$

Accordingly, we define

$$w := \max(u - k, 0) \quad \text{and} \quad \varphi = -\eta^q w.$$

We explicitly observe that, by definition, we have

- a) $\varphi \in W^{1,1}(\Omega)$;
- b) $\varphi = -(u - k) = k - u < 0$ on $A_{k,s} = B_s(x_0) \cap \{u > k\}$;
- c) $\text{supp}(\varphi) \subseteq B_t(x_0) \cap \{u > k\} = A_{k,t} \subset \Omega$.

In view of these facts, by Definition 3.1 we get

$$\begin{aligned} \int_{A_{k,s}} f(x, u, Du) dx & \leq \int_{\text{supp}(\varphi)} f(x, u, Du) dx \\ & = F(u; \text{supp}(\varphi)) \leq Q F(u + \varphi; \text{supp}(\varphi)) \\ & = Q \int_{A_{k,t}} f(x, u + \varphi, Du + D\varphi) dx = (*). \end{aligned} \quad (5.5)$$

We now observe that, for a.e. $x \in A_{k,t} \cap \{\eta = 0\}$, we have

$$f(x, u + \varphi, Du + D\varphi) = f(x, u, Du) = (1 - \eta^q) f(x, u, Du).$$

Moreover, by the convexity of f , for a.e. $x \in A_{k,t} \cap \{\eta > 0\}$, we have

$$\begin{aligned} & f(x, u + \varphi, Du + D\varphi) \\ & = f(x, (1 - \eta^q)u + \eta^q k, (1 - \eta^q)Du + q\eta^{q-1}(k - u)D\eta) \\ & \leq (1 - \eta^q) f(x, u, Du) + \eta^q f(x, k, q\eta^{-1}(k - u)D\eta); \end{aligned}$$

thus, by the growth assumption (5.1), we obtain

$$\begin{aligned} (*) & \leq Q \int_{A_{k,t}} (1 - \eta^q) f(x, u, Du) dx \\ & \quad + \int_{A_{k,t}} \mu(x) \left\{ \frac{2^q}{(t-s)^q} (u - k)^q + k^\gamma + 1 \right\} dx. \end{aligned} \quad (5.6)$$

As a consequence, by combining (5.5) - (5.6), and by recalling the properties of η in (5.4), we derive the following estimate

$$\begin{aligned} \int_{A_{k,s}} f(x, u, Du) dx &\leq Q \int_{A_{k,t} \setminus A_{k,s}} f(x, u, Du) dx \\ &+ \frac{c}{(t-s)^q} \int_{A_{k,R}} \mu(x) ((u-k)^q + k^\gamma) dx \\ &+ \|\mu\|_{L^s(B_R(x_0))} |A_{k,R}|^{1/s'}, \end{aligned} \quad (5.7)$$

where $c > 0$ is a constant only depending on q and Q .

With the above estimate (5.7) at hand, we can easily complete the proof of the proposition. Indeed, starting from the cited (5.7) and adding to both sides of this estimate the term

$$Q \int_{A_{k,s}} f(x, u, Du) dx$$

(notice that this integral is finite, since u is a quasi-minimizer of (3.1) and thus $f(x, u, Du)$ is integrable on $A_{k,s} \subseteq B_s(x_0) \Subset \Omega$), we clearly get

$$\begin{aligned} \int_{A_{k,s}} f(x, u, Du) dx &\leq \frac{Q}{Q+1} \int_{A_{k,t}} f(x, u, Du) dx \\ &+ \frac{c}{(t-s)^q} \int_{A_{k,R}} \mu(x) ((u-k)^q + k^\gamma) dx + \|\mu\|_{L^s(B_R(x_0))} |A_{k,R}|^{1/s'}; \end{aligned}$$

then, by applying Lemma 4.6 with the choice

- i) $\tau_0 := \rho$ and $\tau_1 := R$;
- ii) $\phi(r) = \int_{A_{k,r}} f(x, u, Du) dx$;
- iii) $\theta = \frac{Q}{Q+1} \in (0, 1)$;
- iv) $\alpha = q$ and $A = c \int_{A_{k,R}} \mu(x) ((u-k)^q + k^\gamma) dx$;
- v) $B = \|\mu\|_{L^s(B_R(x_0))} |A_{k,R}|^{1/s'}$;

we obtain the desired (5.3), and the proof is complete. $\square$

Remark 5.2. We explicitly notice that the integrability assumption (5.2) ensures that the right-hand side of (5.3) is finite. Indeed, since $\mu \in L_{\text{loc}}^s(\Omega)$, by the classical Hölder inequality we get

$$\begin{aligned} \int_{A_{k,R}} \mu(x) ((u-k)^q + k^\gamma) dx &\leq \int_{A_{k,R}} \mu(x) (|u|^q + k^\gamma) dx \\ &\leq \int_{B_R(x_0)} \mu(x) (|u|^q + k^\gamma) dx \\ &\text{(by Hölder's inequality, since } \mu \in L^s(B_R(x_0))) \\ &\leq \|\mu\|_{L^s(B_R(x_0))} (\|u\|_{L^{qs'}(B_R(x_0))}^q + |B_R(x_0)|^{1/s'} k^\gamma). \end{aligned}$$

On the other hand, when f satisfies the assumption **(H2)** in its full strength (as required in the statement of Theorem 3.2), we know from Proposition 4.3 that any quasi-minimizer of (3.1) satisfies (5.2).

6. PROOF OF THEOREM 3.2

Thanks to all the results established so far, we are finally ready to provide the proof of Theorem 3.2 (of which we inherit all the assumptions and the notation). Throughout what follows, we simply denote by

$$c(\mathbf{data})$$

any constant (possibly different from line to line) which only depends on the *data in assumption (H2)*, that is, the exponents p_i, q, γ and the L^{r_i} and L^s -norms of λ_i^{-1}, μ , respectively, on some fixed compact subset of Ω .

Proof (of Theorem 3.2). Let $u \in W_{\text{loc}}^{1,1}(\Omega)$ be a quasi-minimizer of (3.1), and let $x_0 \in \Omega$, $R_0 \leq 1$ be such that $B_0 = B_{R_0}(x_0) \Subset \Omega$.

Owing to Proposition 4.3, we know that

$$u \in L_{\text{loc}}^{qs'}(\Omega); \quad (6.1)$$

hence, for every fixed $0 < R \leq R_0$ we can define

$$J_h := \int_{A_{k_h, \rho_h}} (u - k_h)^{qs'} dx \quad (h \geq 0),$$

where $(\rho_h)_h$ and $(k_h)_h$ are given, respectively, by

$$\begin{aligned} (*) \quad \rho_h &:= \frac{R}{2} + \frac{R}{2^{h+1}} = \frac{R}{2} \left(1 + \frac{1}{2^h}\right); \\ (*) \quad k_h &:= d \left(1 - \frac{1}{2^{h+1}}\right) \end{aligned}$$

(for some $d \geq 2$ to be chosen later on). Moreover, we also define

$$\bar{\rho}_h := \frac{\rho_h + \rho_{h+1}}{2} = \frac{R}{2} \left(1 + \frac{3}{4 \cdot 2^h}\right).$$

We explicitly observe that, since $(\rho_h)_h$ is decreasing and $(k_h)_h$ is increasing, the sequence $(J_h)_h$ is decreasing: in fact, we have

$$\begin{aligned} J_{h+1} &= \int_{A_{k_{h+1}, \rho_{h+1}}} (u - k_{h+1})^{qs'} dx \leq \int_{A_{k_{h+1}, \rho_h}} (u - k_{h+1})^{qs'} dx \\ &\leq \int_{A_{k_{h+1}, \rho_h}} (u - k_h)^{qs'} dx \leq \int_{A_{k_h, \rho_h}} (u - k_h)^{qs'} dx = J_h. \end{aligned} \quad (6.2)$$

Furthermore, since $\rho_h \leq R \leq R_0$ we have

$$\begin{aligned} J_h &= \int_{A_{k_h, \rho_h}} (u - k_h)^{qs'} dx \leq \int_{A_{k_h, \rho_h}} |u|^{qs'} dx \\ &\leq \int_{B_R(x_0)} |u|^{qs'} dx = \|u\|_{L^{qs'}(B_R(x_0))}^{qs'} < +\infty. \end{aligned} \quad (6.3)$$

All that being said, we now turn to prove the following claim, which is the real core of our argument: *there exists a constant $\mathbf{C} > 0$, depending on the data appearing in assumption (H2), such that*

$$J_{h+1} \leq \mathbf{C} [1 + \|u\|_{L^{\bar{\sigma}^*}(B_R(x_0))}^{(\gamma-q)s'}]^\frac{q}{p} \cdot \frac{1}{d\delta_1 R^{\delta_2}} \cdot \lambda^h J_h^{1+\alpha}, \quad \text{for all } h \geq 0, \quad (6.4)$$

for suitable constants $\delta_1, \delta_2, \alpha > 0$ and $\lambda > 1$.

To begin with, we choose a sequence $(\zeta_h)_h$ of cut-off functions such that

$$\begin{aligned} \text{i) } & \zeta_h \in C_c^\infty(B_{\bar{\rho}_h}(x_0)), \\ \text{ii) } & \zeta_h \equiv 1 \text{ in } B_{\rho_{h+1}}(x_0), \\ \text{iii) } & |D\zeta_h| \leq 2^{h+4}/R. \end{aligned} \quad (6.5)$$

Moreover, to keep the notation as simple as possible, we set

$$u_h = (u - k_h)_+ = \max\{u - k_h, 0\}.$$

Now, starting from the very definition of J_h , and applying the Hölder inequality with exponent $\bar{\sigma}^*/(qs') > 1$ (see assumption (3.4)-(ii)), we get

$$\begin{aligned} J_{h+1} & \leq |A_{k_{h+1}, \rho_{h+1}}|^{1-\frac{qs'}{\bar{\sigma}^*}} \left(\int_{A_{k_{h+1}, \bar{\rho}_h}} ((u - k_{h+1})\zeta_h)^{\bar{\sigma}^*} dx \right)^{\frac{qs'}{\bar{\sigma}^*}} \\ & = |A_{k_{h+1}, \rho_{h+1}}|^{1-\frac{qs'}{\bar{\sigma}^*}} \left(\int_{B_{\bar{\rho}_h}(x_0)} (u_{h+1}\zeta_h)^{\bar{\sigma}^*} dx \right)^{\frac{qs'}{\bar{\sigma}^*}}, \end{aligned}$$

where we have also used the properties of ζ_h in (6.5).

On the other hand, since we know that $u \in W_{\text{loc}}^{1,\sigma}(\Omega) \cap L_{\text{loc}}^{\bar{\sigma}^*}(\Omega)$ (see Propositions 4.1 and 4.3, respectively), it is readily seen that

$$v := u_{h+1}\zeta_h \in W_0^{1,\sigma}(B_{\bar{\rho}_h}(x_0))$$

(also recall that $\sigma_i \leq \bar{\sigma}^*$, see (4.5)); as a consequence, we are entitled to apply Proposition 4.4 (with $\Omega' = B_{\bar{\rho}_h}(x_0) \Subset \Omega$), obtaining

$$\begin{aligned} J_{h+1} & \leq c |A_{k_{h+1}, \rho_{h+1}}|^{1-\frac{qs'}{\bar{\sigma}^*}} \times \\ & \quad \times \left\{ \prod_{i=1}^n \left[\|\lambda_i^{-1}\|_{L^{r_i}(B_{R_0}(x_0))} \int_{B_{\bar{\rho}_h}(x_0)} \lambda_i |v_{x_i}|^{p_i} dx \right]^{\frac{1}{p_i}} \right\}^{\frac{qs'}{n}} \\ & \leq c(\mathbf{data}) |A_{k_{h+1}, \rho_{h+1}}|^{1-\frac{qs'}{\bar{\sigma}^*}} \times \\ & \quad \times \prod_{i=1}^n \left[\int_{B_{\bar{\rho}_h}(x_0)} \lambda_i(x) |(u_{h+1}\zeta_h)_{x_i}|^{p_i} dx \right]^{\frac{qs'}{p_i n}}. \end{aligned} \quad (6.6)$$

To proceed further, we set

$$\mathfrak{J}_i = \int_{B_{\bar{\rho}_h}(x_0)} \lambda_i(x) |(u_{h+1}\zeta_h)_{x_i}|^{p_i} dx \quad (1 \leq i \leq n),$$

and we observe that, by combining the growth condition in assumption **(H2)** with the Caccioppoli-type inequality in Proposition 5.1, we have

$$\begin{aligned}
\mathfrak{J}_i &\leq c(\mathbf{data}) \int_{B_{\bar{\rho}_h}(x_0)} \lambda_i(x) \{ |(\zeta_h)_{x_i}|^{p_i} u_{h+1}^{p_i} + \zeta_h^{p_i} |(u_{h+1})_{x_i}|^{p_i} \} dx \\
&\quad (\text{by properties of } \zeta_h \text{ in (6.5)}) \\
&\leq c(\mathbf{data}) \left(\frac{2^{h+4}}{R} \right)^{p_i} \int_{A_{k_{h+1}, \bar{\rho}_h}} \lambda_i(x) (u - k_{h+1})^{p_i} dx \\
&\quad + c(\mathbf{data}) \int_{A_{k_{h+1}, \bar{\rho}_h}} \lambda_i(x) |u_{x_i}|^{p_i} dx \\
&\quad (\text{here we use assumption (H2)}) \\
&\leq c(\mathbf{data}) \left(\frac{2^{h+4}}{R} \right)^{p_i} \int_{A_{k_{h+1}, \bar{\rho}_h}} \lambda_i(x) (u - k_{h+1})^{p_i} dx \\
&\quad + c(\mathbf{data}) \int_{A_{k_{h+1}, \bar{\rho}_h}} f(x, u, Du) dx \\
&\quad (\text{by the Caccioppoli-type inequality in Proposition 5.1}) \\
&\leq c(\mathbf{data}) \left(\frac{2^{h+4}}{R} \right)^{p_i} \int_{A_{k_{h+1}, \bar{\rho}_h}} \lambda_i(x) (u - k_{h+1})^{p_i} dx \\
&\quad + c(\mathbf{data}) \left(\frac{2^{h+3}}{R} \right)^q \int_{A_{k_{h+1}, \rho_h}} \mu(x) ((u - k_{h+1})^q + k_{h+1}^\gamma) dx \\
&\quad + c(\mathbf{data}) \|\mu\|_{L^s(B_R(x_0))} |A_{k_{h+1}, \rho_h}|^{1/s'}.
\end{aligned}$$

From this, since $\lambda_i \leq 2\mu$ a.e. in Ω (see Lemma 4.5), and taking into account that $p_1, \dots, p_n \leq q$ (by assumption **(H2)**), we obtain

$$\begin{aligned}
\mathfrak{J}_i &\leq c(\mathbf{data}) \left(\frac{2^{h+4}}{R} \right)^q \int_{A_{k_{h+1}, \rho_h}} \mu(x) [1 + (u - k_{h+1})^q] dx \\
&\quad + c(\mathbf{data}) \left(\frac{2^{h+3}}{R} \right)^q \int_{A_{k_{h+1}, \rho_h}} \mu(x) ((u - k_{h+1})^q + k_{h+1}^\gamma) dx \\
&\quad + c(\mathbf{data}) \|\mu\|_{L^s(B_R(x_0))} |A_{k_{h+1}, \rho_h}|^{1/s'} \\
&\leq c(\mathbf{data}) \left(\frac{2^h}{R} \right)^q \times \\
&\quad \times \left\{ \int_{A_{k_{h+1}, \rho_h}} \mu(x) ((u - k_{h+1})^q + k_{h+1}^\gamma) dx + |A_{k_{h+1}, \rho_h}|^{1/s'} \right\},
\end{aligned}$$

where we have also used the fact that $\bar{\rho}_h \leq \rho_h$ and $k_{h+1} \geq d/2 \geq 1$.

Summing up, by combining this last estimate with (6.6), and by exploiting Hölder's inequality with exponent s (see (6.1)), we conclude that

$$\begin{aligned}
J_{h+1} &\leq c(\mathbf{data}) |A_{k_{h+1}, \rho_{h+1}}|^{1 - \frac{qs'}{\bar{\sigma}^*}} \cdot \prod_{i=1}^n \mathfrak{I}_i^{\frac{qs'}{np_i}} \\
&\leq c(\mathbf{data}) |A_{k_{h+1}, \rho_{h+1}}|^{1 - \frac{qs'}{\bar{\sigma}^*}} \left(\frac{2^h}{R} \right)^{\frac{q^2 s'}{\bar{p}}} \times \\
&\quad \times \left\{ \int_{A_{k_{h+1}, \rho_h}} \mu(x) ((u - k_{h+1})^q + k_{h+1}^\gamma) dx + |A_{k_{h+1}, \rho_h}|^{1/s'} \right\}^{\frac{qs'}{\bar{p}}} \\
&\leq c(\mathbf{data}) |A_{k_{h+1}, \rho_{h+1}}|^{1 - \frac{qs'}{\bar{\sigma}^*}} \left(\frac{2^h}{R} \right)^{\frac{q^2 s'}{\bar{p}}} \mathcal{R}^{\frac{q}{\bar{p}}},
\end{aligned} \tag{6.7}$$

where we have introduced the shorthand notation

$$\mathcal{R} := \int_{A_{k_{h+1}, \rho_h}} ((u - k_{h+1})^q + k_{h+1}^\gamma)^{s'} dx + |A_{k_{h+1}, \rho_h}|.$$

With estimate (6.7) at hand, the final step towards the proof of the claimed (6.4) consists of estimating the above term $\mathcal{R}$.

To this end we first observe that, by definition J_h , we have

$$\begin{aligned}
J_h &= \int_{A_{k_h, \rho_h}} (u - k_h)^{qs'} dx \geq \int_{A_{k_{h+1}, \rho_h}} (u - k_h)^{qs'} dx \\
&\geq (k_{h+1} - k_h)^{qs'} |A_{k_{h+1}, \rho_h}| = \left(\frac{d}{2^{h+2}} \right)^{qs'} |A_{k_{h+1}, \rho_h}| \\
&\geq \left(\frac{d}{2^{h+2}} \right)^{qs'} |A_{k_{h+1}, \rho_{h+1}}|.
\end{aligned} \tag{6.8}$$

Moreover, taking into account that

$$k_{h+1}^{\bar{\sigma}^*} |A_{k_{h+1}, \rho_h}| \leq \int_{A_{k_{h+1}, \rho_h}} |u|^{\bar{\sigma}^*} dx \leq \|u\|_{L^{\bar{\sigma}^*}(B_R(x_0))}^{\bar{\sigma}^*},$$

we immediately derive (remind that $k_h \leq d$ for any $h \geq 0$)

$$\begin{aligned}
\int_{A_{k_{h+1}, \rho_h}} k_{h+1}^{\gamma s'} dx &= \left(k_{h+1}^{\bar{\sigma}^*} |A_{k_{h+1}, \rho_h}| \right)^{\frac{(\gamma-q)s'}{\bar{\sigma}^*}} k_{h+1}^{qs'} |A_{k_{h+1}, \rho_h}|^{1 - \frac{(\gamma-q)s'}{\bar{\sigma}^*}} \\
&\leq \|u\|_{L^{\bar{\sigma}^*}(B_R(x_0))}^{(\gamma-q)s'} d^{qs'} |A_{k_{h+1}, \rho_h}|^{1 - \frac{(\gamma-q)s'}{\bar{\sigma}^*}}.
\end{aligned}$$

As a consequence, reminding (6.2), we obtain

$$\begin{aligned}
&\int_{A_{k_{h+1}, \rho_h}} ((u - k_{h+1})^q + k_{h+1}^\gamma)^{s'} dx \\
&\leq c(\mathbf{data}) \left(J_h + \|u\|_{L^{\bar{\sigma}^*}(B_R(x_0))}^{(\gamma-q)s'} d^{qs'} |A_{k_{h+1}, \rho_h}|^\theta \right),
\end{aligned} \tag{6.9}$$

where we have introduced the shorthand notation

$$\theta = 1 - \frac{(\gamma-q)s'}{\bar{\sigma}^*} \in (0, 1).$$

Summing up, by combining (6.8) - (6.9), and by taking into account (6.3), we then deduce the following estimate for $\mathcal{R}$:

$$\begin{aligned}
\mathcal{R} &\leq c(\mathbf{data}) [1 + \|u\|_{L^{\bar{\sigma}^*}(B_R(x_0))}^{(\gamma-q)s'}] \cdot 2^{hqs'} d^{\frac{q(\gamma-q)(s')^2}{\bar{\sigma}^*}} \{J_h + J_h^\theta\} \\
&\leq c(\mathbf{data}) [1 + \|u\|_{L^{\bar{\sigma}^*}(B_R(x_0))}^{(\gamma-q)s'}] \cdot 2^{hqs'} d^{\frac{q(\gamma-q)(s')^2}{\bar{\sigma}^*}} \times \\
&\quad \times (1 + \|u\|_{L^{qs'}(B_R(x_0))}^{qs'(1-\theta)}) J_h^\theta \\
&\quad \text{(by Hölder's inequality with exponent } \bar{\sigma}^*/(qs') > 1) \\
&\leq c(\mathbf{data}) [1 + \|u\|_{L^{\bar{\sigma}^*}(B_R(x_0))}^{(\gamma-q)s' (1 + \frac{qs'}{\bar{\sigma}^*})}] \times \\
&\quad \times 2^{hqs'} d^{\frac{q(\gamma-q)(s')^2}{\bar{\sigma}^*}} J_h^\theta,
\end{aligned} \tag{6.10}$$

where we have also used the fact that $R_0 \leq 1$.

Thanks to estimate (6.10), we can now complete the proof of (6.4). Indeed, gathering (6.8) - (6.10), from (6.7) we obtain

$$\begin{aligned}
J_{h+1} &\leq c(\mathbf{data}) \left(\frac{2^h}{d}\right)^{qs'(1 - \frac{qs'}{\bar{\sigma}^*})} J_h^{1 - \frac{qs'}{\bar{\sigma}^*}} \cdot \left(\frac{2^h}{R}\right)^{\frac{q^2 s'}{\bar{p}}} \times \\
&\quad \times \left\{ [1 + \|u\|_{L^{\bar{\sigma}^*}(B_R(x_0))}^{(\gamma-q)s' (1 + \frac{qs'}{\bar{\sigma}^*})}] \cdot 2^{hqs'} d^{\frac{q(\gamma-q)(s')^2}{\bar{\sigma}^*}} J_h^\theta \right\}^{\frac{q}{\bar{p}}} \\
&\leq c(\mathbf{data}) [1 + \|u\|_{L^{\bar{\sigma}^*}(B_R(x_0))}^{\frac{q}{\bar{p}}(\gamma-q)s' (1 + \frac{qs'}{\bar{\sigma}^*})}] \cdot \frac{1}{d^{\delta_1} R^{\delta_2}} \cdot \lambda^h J_h^{1+\alpha},
\end{aligned}$$

where the constants $\lambda, \delta_1, \delta_2$ and α are explicitly given by

$$\begin{aligned}
(*) \quad \lambda &= 8^{\frac{q^2 s'}{\bar{p}}}; \\
(*) \quad \delta_1 &= qs' \left(1 - \frac{qs'}{\bar{\sigma}^*}\right) - \frac{(\gamma-q)(qs')^2}{\bar{p} \cdot \bar{\sigma}^*} = qs' \left\{ 1 - \frac{qs'}{\bar{\sigma}^*} + \frac{q^2 s'}{\bar{p} \cdot \bar{\sigma}^*} - \frac{\gamma qs'}{\bar{p} \cdot \bar{\sigma}^*} \right\}; \\
(*) \quad \delta_2 &= \frac{q^2 s'}{\bar{p}}; \\
(*) \quad \alpha &= \frac{\theta q}{\bar{p}} - \frac{qs'}{\bar{\sigma}^*} = \frac{q}{\bar{p}} \left\{ 1 + (q - \bar{p}) \frac{s'}{\bar{\sigma}^*} - \frac{\gamma s'}{\bar{\sigma}^*} \right\}.
\end{aligned}$$

We finally observe that, since $s' \geq 1$ and since $q \geq \bar{p}$ (see Lemma 4.5), we clearly have that $\delta_2 > 0$ and $\lambda > 1$; moreover, by using assumption (3.4) it is not difficult to recognize that $\delta_1, \alpha > 0$.

Indeed, as regards δ_1 we have

$$\begin{aligned}
\delta_1 > 0 &\iff \frac{\gamma qs'}{\bar{p} \cdot \bar{\sigma}^*} < 1 - \frac{qs'}{\bar{\sigma}^*} + \frac{q^2 s'}{\bar{p} \cdot \bar{\sigma}^*} \\
&\iff \gamma < \bar{p} \left(\frac{\bar{\sigma}^*}{qs'} - 1 \right) + q \\
&\iff \gamma < \frac{\bar{p}}{q} \cdot \frac{\bar{\sigma}^*}{s'} + q - \bar{p},
\end{aligned} \tag{6.11}$$

and this last inequality is *precisely* the third inequality in (3.4). As regards the number α , instead, we notice that

$$\alpha > 0 \iff \frac{\gamma s'}{\bar{\sigma}^*} < 1 + (q - \bar{p}) \frac{s'}{\bar{\sigma}^*} \iff \gamma < \frac{\bar{\sigma}^*}{s'} + q - \bar{p}$$

and this condition is implied by (6.11) since $\frac{\bar{p}}{q} \leq 1$. It is worth highlighting that, if $p_i = p = \gamma = q$ and $r_i = s = \infty$, then

$$\delta_1 = p(1 - \frac{p}{p^*}) = \frac{p^2}{n} \quad \text{and} \quad \alpha = 1 - \frac{p}{p^*} = \frac{p}{n}.$$

Summing up, the claimed (6.4) is fully established.

Thanks to the estimated (6.4) just established, we are finally in a position to complete the demonstration of the theorem. Indeed, setting

$$A = \mathbf{C} [1 + \|u\|_{L^{\bar{\sigma}^*}(B_R(x_0))}]^{\frac{q}{\bar{p}}(\gamma-q)s'(1+\frac{qs'}{\bar{\sigma}^*})} \cdot \frac{1}{d^{\delta_1} R^{\delta_2}},$$

we can clearly rewrite the cited estimate (6.4) as follows

$$J_{h+1} \leq A \lambda^h J_h^{1+\alpha}.$$

Moreover, taking into account (6.3) (and recalling that $\delta_1 > 0$), we easily see that we can choose $d \geq 2$ in such a way that

$$\begin{aligned} A^{-\frac{1}{\alpha}} \lambda^{-\frac{1}{\alpha^2}} &= \left(\frac{d^{\delta_1} R^{\delta_2}}{\mathbf{C} [1 + \|u\|_{L^{\bar{\sigma}^*}(B_R(x_0))}]^{\frac{q}{\bar{p}}(\gamma-q)s'(1+\frac{qs'}{\bar{\sigma}^*})}} \right)^{1/\alpha} \lambda^{-1/\alpha^2} \\ &\geq \|u\|_{L^{qs'}(B_R(x_0))}^{qs'} \geq J_0. \end{aligned}$$

Indeed, by Hölder's inequality (and since $R \leq R_0 \leq 1$) we have

$$\|u\|_{L^{qs'}(B_R(x_0))}^{qs'} \leq \|u\|_{L^{\bar{\sigma}^*}(B_R(x_0))}^{qs'} |B_R|^{1-\frac{qs'}{\bar{\sigma}^*}} \leq c_0 [1 + \|u\|_{L^{\bar{\sigma}^*}(B_R(x_0))}]^{qs'}$$

where $c_0 > 0$ is a positive constant only depending on n, q, s, σ ; as a consequence of this estimate, it then suffices to choose

$$d = \left\{ \frac{\mathbf{C} c_0^\alpha \lambda^{1/\alpha}}{R^{\delta_2}} [1 + \|u\|_{L^{\bar{\sigma}^*}(B_R(x_0))}]^{\frac{q}{\bar{p}}(\gamma-q)s'(1+\frac{qs'}{\bar{\sigma}^*})+qs'\alpha} \right\}^{1/\delta_1} \geq 2 \quad (6.12)$$

(by possibly enlarging $\mathbf{C}$ if needed, and since $R_0 \leq 1 < \lambda$).

Thus, by exploiting Lemma 4.7 *with this choice of d* , we get

$$\lim_{h \rightarrow +\infty} J_h = \int_{A_{d,R/2}} |u - d|^{qs'} dx = 0,$$

from which we derive that $|A_{d,R/2}| = 0$, that is,

$$u \leq d \text{ a.e. in } B_{R/2}(x_0).$$

To prove that u is also bounded from below by $-d$ it suffices to observe that, setting $v := -u$, then $v \in W_{\text{loc}}^{1,1}(\Omega)$ is a quasi-minimizer of the functional

$$G(v) := \int_{\Omega} g(x, v, Dv) dx,$$

where $g : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is the function defined by

$$g(x, w, \xi) := f(x, -w, -\xi).$$

As a consequence, since g clearly satisfies assumptions **(H1)** - **(H2)** (with the *same data*), by the above argument we deduce that

$$-u \leq d \text{ for a.e. } x \in B_{R/2}(x_0).$$

Summing up, we have proved that, if $d \geq 2$ is as in (6.12), one has

$$\|u\|_{L^\infty(B_{R/2}(x_0))} \leq d. \quad (6.13)$$

We finally observe that, owing to the explicit expression of α , we have

$$\begin{aligned} & \frac{q}{\bar{p}}(\gamma - q)s' \left(1 + \frac{qs'}{\bar{\sigma}^*}\right) + qs'\alpha \\ &= \frac{qs'}{\bar{p}} \left\{ (\gamma - q) \left(1 + \frac{qs'}{\bar{\sigma}^*}\right) + q \left[1 + (q - \bar{p}) \frac{s'}{\bar{\sigma}^*} - \frac{\gamma s'}{\bar{\sigma}^*}\right] \right\} \\ &= \frac{qs'}{\bar{p}} \left(\gamma - \frac{qs'\bar{p}}{\bar{\sigma}^*} \right); \end{aligned}$$

as a consequence, by definition of δ_1, δ_2 we obtain

$$d \leq c(\mathbf{data}) \frac{1}{R \frac{q\bar{\sigma}^*}{\bar{p}\bar{\sigma}^* - qs'(\gamma - q + \bar{p})}} \left[1 + \|u\|_{L^{\bar{\sigma}^*}(B_R(x_0))}\right]^{\frac{\bar{\sigma}^*\gamma - qs'\bar{p}}{\bar{p}\bar{\sigma}^* - qs'(\gamma - q + \bar{p})}}.$$

This, together with (6.13), gives the desired (3.5). $\square$

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