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On Some Stochastic Hyperbolic Equations with Symplectic Characteristics

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Abstract
We study the effect of Gaussian perturbations on a class of model hyperbolic partial differential equations with double symplectic characteristics in low spatial dimensions. The coefficients of our partial differential operators contain harmonic oscillators in the space variables, while the noise is additive, white in time and colored in space. We provide sufficient conditions on the spectral measure of the covariance functional describing the noise that allows for the existence of a random field solution for the resulting stochastic partial differential equation. Furthermore, we show how the symplectic structure of the set of multiple points affects the regularity of the noise needed to build a measurable process solution. Our approach is based on some explicit computations for the fundamental solutions of several model partial differential operators together with their explicit Fourier transforms.

Keywords Stochastic partial differential equations · Hyperbolic equations with double characteristics · Gaussian noise · Random field solution

1 Introduction

A number of recent papers, tapping into the powerful techniques presented in the seminal works [12, 13, 24], have extended several classical deterministic results for solutions of hyperbolic operators of various types—linear, semilinear, and whose principal symbols have multiple involutive or symplectic characteristics—to the stochastic framework: for a current review, encompassing also other types of classical PDEs in open domains in $\mathbb{R}^n$ and Riemannian manifolds as well, one can consult, among others, [1–7, 10, 11]. Our goal in this work is to continue the study of possible extensions to the

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stochastic framework by analyzing three model linear stochastic operators, we deem not having been previously examined in the literature. In particular, we would like to understand how the symplectic geometry of the principal symbols and the conditions on the lower order terms influence the construction of the random field solutions via the corresponding colors of the noise, an analysis started in [7].

The general problem is presented here in a compact form:

$$\begin{cases} P_i u_i = \dot{F}_i\,(t,x) \quad t \geq 0, x \in \mathbb{R}^i, i \in \{1,2,3\} \\ u_i\,(0,x) = 0 \\ \partial_t u_i\,(0,x) = 0, \end{cases} \tag{1}$$

where formally

$$F_i(\phi) := \int_{\mathbb{R}^{i+1}} \phi(t,x)\dot{F}_i(t,x)\,dtdx, \quad \phi \in C_0^\infty\left(\mathbb{R}^{i+1}\right),$$

is a mean zero family of normal random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with covariance

$$E(F_i(\phi)F_i(\psi)) = \int_0^\infty dt \int_{\mathbb{R}^i} dx \int_{\mathbb{R}^i} dy\, \phi(t,x) f_i(x-y)\psi(t,y), \tag{2}$$

for $\phi, \psi \in C_0^\infty\left(\mathbb{R}^{i+1}\right)$. The operators P_i, $i \in \{1,2,3\}$ are defined as

$$P_1 := D_t^2 - (D_x^2 + x^2), \tag{3}$$
$$P_2 := D_t^2 - \mu(D_x^2 + x^2 D_y^2) + bD_y, \quad |b| < \mu, \tag{4}$$
$$P_3 := D_t^2 - \mu(D_x^2 + x^2 D_y^2) - aD_z^2 + bD_y, \quad |b| < \mu,\ a > 0. \tag{5}$$

P_1 is a 0-th order perturbation of the wave operator, which is strictly hyperbolic, and we use it mostly as an introduction to some of the techniques used in the latter cases. Regarding the two- and three-dimensional operators P_2 and P_3, their corresponding principal symbols are

$$p_2 := \tau^2 - \mu(\xi^2 + x^2\eta^2),$$
$$p_3 := \tau^2 - \mu(\xi^2 + x^2\eta^2) - a\zeta^2,$$

which are readily seen to be hyperbolic with respect to τ and vanishing at the second order on the C^∞-manifolds

$$\Sigma_2 := \{(t,x,y,\tau,\xi,\eta) \in \dot{T}^*\mathbb{R}^3, x = \tau = \xi = 0\},$$
$$\Sigma_3 := \{(t,x,y,z,\tau,\xi,\eta,\zeta) \in \dot{T}^*\mathbb{R}^4, x = \tau = \xi = \zeta = 0\},$$

where $\dot{T}^*\mathbb{R}^i$ denotes the phase space cotangent bundle minus the zero section. The manifold Σ_2 is purely symplectic in its space variables and covariables, i.e. the two-form $d\xi \wedge dx$ is non degenerate or equivalently the Poisson brackets of the functions defining the manifold do not all vanish on the manifold itself, while Σ_3 presents a mixed involutive-symplectic structure. Some review of the classical symplectic framework for differential equations with multiple characteristics can be found, among others, in [7, 8, 15].

Calling H_i the Hamiltonian field of p_i, the fundamental matrix $F_i(\omega)$ associated to p_i at a point $\omega \in \Sigma_i$ is computed as $F_i(\omega) = \frac{1}{2} d H_i(\omega)$. One can easily check that for all points $\omega_i \in \Sigma_i$ both $F_2(\omega_2)$ and $F_3(\omega_3)$ have just two non-zero complex eigenvalues $\pm i\mu\eta$. One then defines in this case the corresponding operator to be of non-effectively hyperbolic type [15]. The request of well-posedness of the Cauchy problem in the C^∞ and Gevrey classes for this type of operators gives rise to the (strict) Levi type condition $|b| < \mu$, where μ is what is usually known as the positive trace of the associated harmonic oscillator (see e.g. [14, 15]). The (non-strict) Levi condition $|b| \leq \mu$ is, as it is well known, necessary—and in its strict version sufficient as well—in order to reach the well-posedness of the Cauchy problem for P_3 in the C^∞ category, see e.g. [16]. It will be shown below how, even in the stochastic construction, this condition on the lower order terms plays a fundamental role for the explicit existence of the random field solution.

The hyperbolic operator with symbol p_2 has been studied in the deterministic context in a number of papers, see for instance [17, 18, 20], aiming to calculate Poisson formulas for hypoelliptic operators on compact manifolds and studying the propagation of singularities for their solutions. Both these objectives rely on having at one's disposal a rather explicit expression for the fundamental solution; thus in order to develop similar results in a stochastic setup we must necessarily possess a rather sizeable knowledge of such a fundamental solution. That being said, the presence of harmonic oscillators in the symbols p_2 and p_3 together with the symplectic nature of their double manifolds makes the more traditional approaches of extending the classical calculi of pseudo-differential analysis to the stochastic setting much more complicated. This is essentially due to the fact that the simple bicharacteristic curves exhibit a periodic nature, making it very hard to present any fundamental solution as an integral Fourier operator whose phase is supported over those very curves. An explicit procedure is then required and this is what will be done in the present work through suitable Hermite functions expansions and a precise control over the related coefficients.

Finally, we observe that the arguments we utilize in our analysis could very well be used to deal with a more general second order hyperbolic quadratic form with symplectic characteristics. Nevertheless, the simpler nature of our model examples allows us to remove some of the complexities in [20] or [17], where only the codimension 3 was considered, and develop a sharper understanding of the structure of our fundamental solution processes. We observe that the symplectic nature of the multiple manifold is not by itself an a-priori obstacle to obtaining a closed form fundamental solution, as was proven in [7]. What makes the problem more challenging in our case is indeed the presence of the harmonic oscillators.

The function $f_i : \mathbb{R}^i \to \mathbb{R}$ is taken to be continuous except at most at zero, and even. This requirements are necessary to guarantee that the functional expressed in (2) is non-negative definite. As seen in [22], this is also equivalent to the existence of a non-negative tempered measure ν_i whose Fourier transform is f_i, i.e., for all

$\phi \in \mathcal{S}(\mathbb{R}^i)$

$$\int_{\mathbb{R}^i} \phi(x) f_i(x) dx = \int_{\mathbb{R}^i} \mathcal{F}\phi(\xi) \nu_i(d\xi),$$

where, and henceforth, $\mathcal{F}\phi$ denotes the Fourier transform

$$\mathcal{F}\phi(\xi) := \int_{\mathbb{R}^i} e^{-ix\cdot\xi} \phi(x) dx$$

and

$$\mathcal{F}^{-1}\phi(x) := \frac{1}{2\pi} \int_{\mathbb{R}^i} e^{ix\cdot\xi} \phi(\xi) d\xi.$$

Denoting with E_i the fundamental solution of the operator P_i, we will say that $u_i(t, x)$ defined as

$$u_i(t, x) = \int_0^t \int_{\mathbb{R}^i} E_i\,(t-s, x-y)\, \dot{F}_i(s, y) ds\, dy \tag{6}$$

is a *random field solution* to (1) if the stochastic integral is well defined and the map $(t, x) \mapsto u_i(t, x)$ is measurable. For the theory of stochastic integration, we refer to [12], where the author performs an extension and adaptation of Walsh's construction of the martingale measure stochastic integral (see [24]). We will shortly describe it. We denote with $\mathcal{D}\,(\mathbb{R}^p)$ the space of functions $\phi \in C_0^\infty\,(\mathbb{R}^p)$ endowed with the topology described by the following notion of convergence. Given a sequence $(\phi_n)_{n\in\mathbb{N}} \subset \mathcal{D}\,(\mathbb{R}^p)$ and a function $\phi \in \mathcal{D}\,(\mathbb{R}^p)$ we say that ϕ_n converges to ϕ and write $\phi_n \to \phi$ if:

- there exists a compact set $K \subset \mathbb{R}^p$ such that $\text{supp}\,(\phi_n - \phi) \subset K$ for all $n \geq 1$,
- $\lim_{n\to\infty} D^\alpha \phi_n = D^\alpha \phi$ uniformly in K for every multi-index α.

Let $\mathcal{B}_b(\mathbb{R}^i)$ e the σ-field of all bounded Borel sets of $\mathbb{R}^i$. The first step is to extend F_i to a *worthy martingale measure* (see [24]). Using (2) we can verify that F_i is L^2-continuous and, as such, we can extend it to a σ-finite L^2-valued measure by approximating indicator functions of sets in $\mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^i)$ with elements of $\mathcal{D}\left(\mathbb{R}^{i+1}\right)$. We set

$$M_{i,t}(B) = F_i([0, t] \times B), \quad B \in \mathcal{B}_b(\mathbb{R}^i)$$

and

$$\mathcal{F}^0_{i,t} = \sigma\left(M_{i,s}(B), s \leq t, B \in \mathcal{B}_b(\mathbb{R}^i)\right), \quad \mathcal{F}_{i,t} = \mathcal{F}^0_{i,t} \vee \mathcal{N},$$

with $\mathcal{N}$ being the σ-field generated by the $\mathbb{P}$-null sets. We then have that, by construction, $t \to M_{i,t}(B)$ is a continuous martingale and

$$F_i(\phi) = \int_{\mathbb{R}^+} \int_{\mathbb{R}^i} \phi(t,x) M_i(dt,dx), \quad \phi \in \mathcal{D}(\mathbb{R}^{i+1}).$$

We call a function $(s,x,\omega) \mapsto g(s,x,\omega)$ *elementary* if it is of the form

$$g(s,x,\omega) = 1_{(a,b]}(s) 1_A(x) X(\omega) \ , \quad a,b \in \mathbb{R}, \ 0 \le a < b, \ A \in \mathcal{B}_b(\mathbb{R}^i),$$

where X is a $\mathcal{F}_{i,a}$-measurable r.v. We denote by $\mathcal{E}$ the space of all finite linear combinations of elementary functions and call *predictable* the σ-field on $\mathbb{R}_+ \times \mathbb{R}^i \times \Omega$ generated by the elements of $\mathcal{E}$. Moreover, set

$$||g||_+ := E\left(\int_0^t ds \int_{\mathbb{R}^i} dy \int_{\mathbb{R}^i} dx |g(s,x,\cdot)| f_i(x-y) |g(s,y,\cdot)|\right) \tag{7}$$

and

$$||g||_0 := E\left(\int_0^t ds \int_{\mathbb{R}^i} dy \int_{\mathbb{R}^i} dx g(s,x,\cdot) f_i(x-y) g(s,y,\cdot)\right). \tag{8}$$

In [24], Walsh defines the martingale-measure stochastic integral on the complete space $\mathcal{P}_+$ of predictable functions g with $||g||_+ < +\infty$.
On the other hand, Dalang in his work [12] is able to extend such a construction defining the martingale

$$t \mapsto \int_0^t \int_{\mathbb{R}^i} v(s,x,\cdot) M(ds,dx)$$

for all elements v of the completion $\mathcal{P}_0$ of $(\mathcal{E}, ||\cdot||_0)$. Moreover, denoting as $\bar{\mathcal{P}}$ the space of all predictable functions $h(t,x,\omega)$ such that $x \mapsto h(t,x,\omega) \in \mathcal{S}'(\mathbb{R}^i)$ for every $(t,\omega) \in [0,T] \times \Omega$, $\mathcal{F}h(t,\cdot,\omega)(\xi)$ is a function a.s. and

$$||h||_0' := E\left(\int_0^t ds \int_{\mathbb{R}^i} |\mathcal{F}h(t,\cdot,\omega)(\xi)|^2 \nu_i(d\xi)\right)^{1/2} < +\infty, \tag{9}$$

and calling $\mathcal{E}_0$ the subset of $\mathcal{P}_+$ consisting of all functions $g(s,x,\omega)$ such that $x \mapsto g(s,x,\omega) \in \mathcal{S}(\mathbb{R}^i)$, then we can identify $\mathcal{P}_0$ with the set of elements $h(t,x,\omega)$ of $\bar{\mathcal{P}}$ such that it exists a sequence $h_n(t,x,\omega)$ of elements of $\mathcal{E}_0$ that gives

$$\lim_{n\to\infty} ||h_n - h||_0' = 0. \tag{10}$$

Furthermore, we observe that this implies that for elements $h(s,x,\omega) \in \mathcal{P}_0$

$$||h||_0 = ||h||_0'.$$

Finally, we define the physicist's Hermite polynomials and Hermite functions as (see for instance [21])

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2},$$
$$\Psi_n(x) = \frac{1}{(2^n n! \sqrt{\pi})^{1/2}} e^{\frac{-x^2}{2}} H_n(x), \tag{11}$$

and name

$$\rho_n(\eta, \zeta) := \mu|\eta|(2n+1) + a\zeta^2 - b\eta$$
$$\tilde{\rho}_n(\eta) := \rho_n(\eta, 0).$$

We are now ready to state our main result, which contains all the several claims proven below, itemized according to the increasing number of space variables. Recalling that our stochastic PDEs are

$$\begin{cases} P_i u_i = \dot{F}_i(t,x) & t \geq 0, x \in \mathbb{R}^i, i \in \{1,2,3\} \\ u_i(0,x) = 0 \\ \partial_t u_i(0,x) = 0, \end{cases}$$

where the operators P_i, $i \in \{1,2,3\}$ are defined as

$$P_1 := D_t^2 - (D_x^2 + x^2),$$
$$P_2 := D_t^2 - \mu(D_x^2 + x^2 D_y^2) + bD_y, \quad |b| < \mu$$
$$P_3 := D_t^2 - \mu(D_x^2 + x^2 D_y^2) - aD_z^2 + bD_y, \quad |b| < \mu,\ a > 0,$$

we have the following:

Theorem 1.1 *(i) Let $\nu_1(d\xi_0) = d\xi_0$ be the Lebesgue measure. Then the one-dimensional problem has a random field solution given by*

$$u_1(t,x) = \int_0^t \int_{\mathbb{R}} E_1(t-s, x; x_0)\, \dot{F}_1(s, x_0) ds\, dx_0$$

with

$$E_1(t,x,x_0) = -H(t) \sum_{n=0}^{\infty} \frac{\sin(\sqrt{2n+1}\, t)}{\sqrt{2n+1}} \Psi_n(x_0) \Psi_n(x).$$

.

(ii) Let $\nu_2 = d\xi\, \hat{\nu}_2(d\hat{\eta})$ and assume that

$$\int_{\mathbb{R}} \frac{1}{|\hat{\eta}|^{1/2}} \hat{\nu}_2(d\hat{\eta}) < +\infty; \tag{12}$$

then the two-dimensional problem has a random field solution given by

$$u_2(t, x, y) = \int_0^t \int_{\mathbb{R}^2} E_2\left(t - s, x, y - y_0; x_0\right) \dot{F}_1(s, x_0, y_0) ds\, dx_0\, dy_0$$

with

$$E_2\left(t, x, y; x_0\right) = -\frac{H\left(t\right)}{2\pi} \times \int_{\mathbb{R}} e^{iy\eta} |\eta|^{1/2} \sum_{n=0}^{\infty} \frac{\sin\left(\tilde{\rho}_n\left(\eta\right)^{1/2} t\right)}{\tilde{\rho}_n\left(\eta\right)^{1/2}} \Psi_n\left(x_0|\eta|^{1/2}\right) \Psi_n\left(x|\eta|^{1/2}\right) d\eta.$$

(iii) Let $\nu_3(d\xi, d\hat{\eta}, d\hat{\zeta}) = d\xi\, \hat{\nu}_3(d\hat{\eta}, d\hat{\zeta})$. Assume $\hat{\nu}_3$ is absolutely continuous with respect to the Lebesgue measure and that it admits a density of the form $(\hat{\eta}, \hat{\zeta}) \mapsto w\left(|\hat{\eta}|^2 + |\hat{\zeta}|^2\right)$ for which it exists an $\alpha < 1/3$ such that

$$\int_{\mathbb{R}^2} \frac{1}{\left(|\hat{\eta}|^2 + |\hat{\zeta}|^2\right)^{\alpha}} \hat{\nu}_3(d\hat{\eta}, d\hat{\zeta}) < +\infty. \tag{13}$$

Then the three-dimensional problem has a random field solution given by

$$u_3(t, x, y, z) = \int_0^t \int_{\mathbb{R}^3} E_3\left(t - s, x, y - y_0, z - z_0; x_0\right) \times \dot{F}_3(s, x_0, y_0, z_0) ds\, dx_0\, dy_0\, dz_0$$

with

$$E_3\left(t, x, y, z; x_0\right) = -\frac{H\left(t\right)}{2\pi} \times \int_{\mathbb{R}^2} e^{iy\eta + iz\zeta} |\eta|^{1/2} \sum_{n=0}^{\infty} \frac{\sin\left(\rho_n\left(\eta, \zeta\right)^{1/2} t\right)}{\rho_n\left(\eta, \zeta\right)^{1/2}} \Psi_n\left(x_0|\eta|^{1/2}\right) \Psi_n\left(x|\eta|^{1/2}\right) d\eta\, d\zeta.$$

We would like to highlight again how the color of the noise, observable through the corresponding measures, relates to the geometry of the double manifolds.

The plan of the paper is as follows. In Sect. 2 we organize the proof for the one-dimensional case, parts of which are to be used in the later sections. In particular in Subsect. 2.1 the formal fundamental solution is computed and in Subsect. 2.2 the corresponding random field is constructed. Section 3 contains the main arguments. In Subsect. 3.1 the formal fundamental solutions are explicitly calculated for both dimensions 2 and 3. Subsections 3.2 and 3.3 are devoted to the estimate of the formal solutions obtained, and in the final Subsect. 3.4 the existence of the random field solutions as stated in Theorem 1.1 is eventually proven.

2 The Case of One Spatial Dimension

2.1 Fundamental Solution

We now proceed with the case of one spatial dimension. This, besides proving the related simple case, helps us collecting all the major tools needed for the subsequent parts.

We consider the problem

$$P_1 E_1(t, x; x_0) = \delta(t)\delta(x - x_0).$$

Performing an Hermite series expansion of $x \mapsto E_1(t, x; x_0)$ and using the well-known identities (see [19] or [23])

$$\sum_{n=0}^{\infty} \Psi_n(x)\Psi_n(x_0) = \delta(x - x_0), \tag{14}$$

$$(\partial_x^2 - x^2 + 2n + 1)\Psi_n(x) = 0 \quad \Leftrightarrow \quad (D_x^2 + x^2)\Psi_n(x) = (2n + 1)\Psi_n(x) \tag{15}$$

we get

$$P_1 \sum_{n=0}^{\infty} E_{1,n}(t, x_0)\Psi_n(x) = \sum_{n=0}^{\infty} (P_{1,n} E_{1,n})(t, x_0)\Psi_n(x) = \sum_{n=0}^{\infty} \delta(t)\Psi_n(x)\Psi_n(x_0), \tag{16}$$

where

$$P_{1,n} = D_t^2 - (2n + 1).$$

Equating term by term in (16) we obtain the ODE problem

$$\begin{cases} P_{1,n} E_{1,n}(t, x_0) = \delta(t)\Psi_n(x_0), \\ E_{1,n}(0, x_0) = \partial_t E_{1,n}(0, x_0) = 0, \end{cases}$$

whose solution is

$$\begin{aligned} E_{1,n}(t, x_0) &= -\Psi_n(x_0) \int_0^t \delta(t - \tau) \frac{\sin(\sqrt{2n+1}\,\tau)}{\sqrt{2n+1}}\, d\tau \\ &= -H(t) \frac{\sin(\sqrt{2n+1}\,t)}{\sqrt{2n+1}} \Psi_n(x_0), \end{aligned} \tag{17}$$

where $H(t)$ denotes the Heavyside function,i.e,

$$H(t) = \begin{cases} 1, & t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$E_1(t,x;x_0) = -H(t)\sum_{n=0}^{\infty}\frac{\sin(\sqrt{2n+1}\,t)}{\sqrt{2n+1}}\Psi_n(x_0)\Psi_n(x).$$

Now, the Hermite functions are eigenvectors of the Fourier transform (see e.g. [23]); in particular

$$\mathcal{F}\Psi_n(\eta) = (-i)^n\Psi_n(\eta)$$

and so

$$\begin{aligned}\mathtt{F}_1(t,x;\xi_0) :=& \mathcal{F}E_1(t,x;\cdot)\,(\xi_0)\\ =& -H(t)\sum_{n=0}^{\infty}(-i)^n\frac{\sin\left(t\sqrt{2n+1}\right)}{\sqrt{2n+1}}\Psi_n(\xi_0)\Psi_n(x).\end{aligned} \tag{18}$$

2.2 Random Field Solution

For the reader's convenience, we recall the expression for the candidate random field solution

$$u_1(t,x) = \int_0^t\int_{\mathbb{R}} E_1\,(t-s,x;x_0)\,\dot{F}_1(s,x_0)ds\,dx_0.$$

In order to prove that the expression above is indeed a real-valued process, we need to show that $E_1\,(t-s,x;x_0)$ is an element of the space $\mathcal{P}_0$. To do that, the first step is to show the following.

Lemma 2.2.1 *Let* $\mathtt{F}_1(t,x;\xi_0)$ *be defined as in* (18)*. Then*

$$I := \int_0^t\int_{\mathbb{R}} |\mathtt{F}_1\,(t-s,x;\xi_0)\,|^2 d\xi_0\,ds < +\infty.$$

Proof By Parseval identity, we have

$$I = \int_0^t\sum_{n=0}^{+\infty}\frac{\sin^2\left((t-s)\sqrt{2n+1}\right)}{2n+1}\Psi_n^2(x)ds \le t\sum_{n=0}^{+\infty}\frac{1}{2n+1}\Psi_n^2(x).$$

Now, we recall that there exist constants C_1 and C_2 such that, (see e.g. [9])

$$C_1 n^{-1/6} < \max_{x\in\mathbb{R}}\Psi_n^2(x) < C_2 n^{-1/6} \quad n \ge 1. \tag{19}$$

This implies that

$$I \le C_2 t\left(\Psi_0^2(x) + \sum_{n=1}^{+\infty}(2n+1)^{-1}n^{-1/6}\right) \le C_2 t\left(\pi^{-1/4} + \sum_{n=1}^{+\infty}(2n+1)^{-1}n^{-1/6}\right)$$

which is bounded. □

This shows that the fundamental solution $E_1(t-s,x;x_0)$ belongs to the space $\bar{\mathcal{P}}$. However, to prove that $E_1 \in \mathcal{P}_0$ as well we need to find a sequence of functions $\left(E_{1,m}\right)_{m\in\mathbb{N}}$ in $\mathcal{E}_0$ converging to E_1 in $\bar{\mathcal{P}}$.
Unsurprisingly, we set

$$E_{1,m}(t-s,x;x_0) := -H(t)\sum_{n=0}^{m}\frac{\sin\left((t-s)\sqrt{2n+1}\right)}{\sqrt{2n+1}}\Psi_n(x_0)\Psi_n(x).$$

Firstly, we observe that for each $m\in\mathbb{N}$, $x_0 \mapsto E_{1,m}(t,x;x_0)\in\mathcal{S}(\mathbb{R})$. Moreover, setting

$$\mathrm{F}_1^m(t-s,x;\xi_0) := -H(t)\sum_{n=m}^{\infty}(-i)^n\frac{\sin\left((t-s)\sqrt{2n+1}\right)}{\sqrt{2n+1}}\Psi_n(\xi_0)\Psi_n(x),$$

we have

$$\left|\left|E_{1,m}-E_1\right|\right|_0^2 = \int_0^t ds\int_{\mathbb{R}}\left|\mathrm{F}_1^m(t-s,x;\xi_0)\right|^2 d\xi_0.$$

Hence, from the proof of Lemma 2.2.1, we gather

$$\left|\left|E_{1,m}-E_1\right|\right|_0^2 \leq C_2 t\sum_{n=m}^{+\infty}(2n+1)^{-1}n^{-1/6}\xrightarrow{m\to+\infty}0.$$

Now, it remains to prove measurability, and we do that by checking $\mathbb{L}^2(\Omega)$-continuity of the solution process in the separate variables. Denoting

$$S_1(t,x,h) = \sum_{n=0}^{+\infty}\frac{\sin^2\left(t\sqrt{2n+1}\right)}{2n+1}\left(\Psi_n(x+h)-\Psi_n(x)\right)^2,$$

the increment in space gives

$$\begin{aligned}
&\mathbb{E}\left[|u_1(t,x+h)-u_1(t,x)|^2\right]\\
&= \int_0^t ds\int_{\mathbb{R}}|\mathrm{F}_1(t-s,x+h;\xi_0)-\mathrm{F}_1(t-s,x;\xi_0)|^2 d\xi_0\\
&= \int_0^t S_1(t-s,x,h)ds.
\end{aligned}$$

Thanks to (19) we have

$$S_1(t,x,h) \leq 4C_2\left(\pi^{-1/4}+\sum_{n=1}^{+\infty}(2n+1)^{-1}n^{-1/6}\right) < +\infty, \quad (20)$$

granting uniform convergence and consequently the continuity of $S_1(t-s, x, h)$ with respect to his last variable. Therefore, by dominated convergence

$$\lim_{h\to 0} \mathbb{E}\left[|u_1(t, x+h) - u_1(t, x)|^2\right] = 0. \tag{21}$$

For the increment in time we write

$$\begin{aligned}
&\mathbb{E}\left[|u_1(t+h, x) - u_1(t, x)|^2\right] \\
&\leq 2\int_0^t ds \int_{\mathbb{R}} |\mathrm{F}_1\,(t+h-s, x; \xi_0) - \mathrm{F}_1\,(t-s, x; \xi_0)|^2\, d\xi_0 \\
&\quad + 2\int_t^{t+h} ds \int_{\mathbb{R}} |\mathrm{F}_1\,(t+h-s, x; \xi_0)|^2\, d\xi_0 \\
&= 2I_1(h) + 2I_2(h).
\end{aligned}$$

Lemma 2.2.1 gives directly

$$\lim_{h\to 0} I_2(h) = 0.$$

Similarly to the increment in the x variable, we call

$$\tilde{S}_1(t, x, h) := \sum_{n=0}^{+\infty} \frac{\left(\sin\left((t+h)\sqrt{2n+1}\right) - \sin\left(t\sqrt{2n+1}\right)\right)^2}{\sqrt{2n+1}} \Psi_n^2(x)$$

and by Parseval's identity, we obtain

$$I_1(h) = \int_0^t \tilde{S}_1(t-s, x, h)ds.$$

For $\tilde{S}_1(t-s, x, h)$, a bound similar to (20) holds. So, by dominated convergence

$$\lim_{h\to 0} I_1(h) = 0.$$

3 The Case of Two and Three Spatial Dimensions

3.1 Common Computations: Fundamental Solution

Here we explicitly derive the fundamental solution of the operator P_3. Since the structure of the calculation is much the same as in the two-dimensional case, we omit the latter and recover the corresponding fundamental solution at the end of the section.

Hence, our problem becomes now

$$P_3 E_3 = \delta(t)\delta(x - x_0)\delta(y)\delta(z).$$

Taking the Fourier transform with respect to (y, z) we get

$$\widehat{P_3 E_3} = \left(D_t^2 - \mu\left(D_x^2 + x^2\eta^2\right) - a\zeta^2 + b\eta\right)\hat{E}_3\left(t, x, \eta, \zeta; x_0\right) = \delta\left(t\right)\delta\left(x - x_0\right). \tag{22}$$

Putting $\sigma = |\eta|^{1/2}x$ we define

$$W\left(t, \sigma, \eta, \zeta; x_0\right) = \hat{E}_3\left(t, |\eta|^{-1/2}x, \eta, \zeta; x_0\right);$$

thus

$$\partial_\sigma^2 W\left(t, \sigma, \eta, \zeta; x_0\right) = |\eta|^{-1}\partial_x^2\hat{E}_3\left(t, |\eta|^{-1/2}x, \eta, \zeta; x_0\right)$$

and, consequently

$$\left(D_x^2 + x^2\eta^2\right)\hat{E}_3\left(t, |\eta|^{-1/2}x, \eta, \zeta; x_0\right) = |\eta|\left(D_\sigma^2 + \sigma^2\right)W\left(t, \sigma, \eta, \zeta; x_0\right).$$

Therefore, since Dirac's delta is homogeneous of degree -1, naming $\Lambda := D_t^2 - \mu|\eta|\left(D_\sigma^2 + \sigma^2\right) - a\zeta^2 + b\eta$, (22) finally becomes

$$\Lambda W\left(t, \sigma, \eta, \zeta; x_0\right) = |\eta|^{1/2}\delta\left(t\right)\delta\left(\sigma - x_0|\eta|^{1/2}\right). \tag{23}$$

We expand $\sigma \mapsto W(t, \sigma, \eta, \zeta; x_0)$ in Hermite functions

$$W(t, \sigma, \eta, \zeta; x_0) = \sum_{n=0}^{\infty} W_n\left(t, \eta, \zeta; x_0\right)\Psi_n\left(\sigma\right)$$

and define

$$\Lambda_n = D_t^2 - \mu|\eta|\left(2n + 1\right) - a\zeta^2 + b\eta.$$

It follows by (14) and (15) that (23) transforms into

$$\begin{aligned}\Lambda W(t, \sigma, \eta, \zeta; x_0) &= \Lambda\sum_{n=0}^{\infty} W_n\left(t, \eta, \zeta; x_0\right)\Psi_n\left(\sigma\right)\\ &= \sum_{n=0}^{\infty}\Lambda_n W_n\left(t, \eta, \zeta; x_0\right)\Psi_n\left(\sigma\right)\\ &= \delta\left(t\right)\sum_{n=0}^{\infty}\Psi_n\left(x_0|\eta|^{1/2}\right)\Psi_n\left(\sigma\right),\end{aligned}$$

and we split the sum term by term, equating the coefficients of the expansion

$$\left(\Lambda_n W_n\right)\left(t, \eta, \zeta; x_0\right) = \delta\left(t\right)\Psi_n\left(x_0|\eta|^{1/2}\right)|\eta|^{1/2},$$

so that we may focus on the ODE problem

$$\begin{cases}(\partial_t+\rho_n(\eta,\zeta))\,W_n(t,\eta,\zeta;x_0)=-\delta(t)\,\Psi_n\left(x_0|\eta|^{1/2}\right)|\eta|^{1/2},\\ W_n(0,\eta,\zeta;x_0)=\partial_t W_n(0,\eta,\zeta;x_0)=0,\end{cases}\tag{24}$$

which is solved by

$$\begin{aligned}W_n(t,\eta,\zeta;x_0)&=-\,\Psi_n\left(x_0|\eta|^{1/2}\right)|\eta|^{1/2}\cdot\int_0^t\delta(t-\tau)\,\frac{\sin\left(\rho_n(\eta,\zeta)^{1/2}\,\tau\right)}{\rho_n(\eta,\zeta)^{1/2}}d\tau\\&=-\,H(t)\,\Psi_n\left(x_0|\eta|^{1/2}\right)|\eta|^{1/2}\frac{\sin\left(\rho_n(\eta,\zeta)^{1/2}\,t\right)}{\rho_n(\eta,\zeta)^{1/2}}.\end{aligned}$$

Therefore, we find

$$W(t,\sigma,\eta,\zeta;x_0)=-H(t)\,|\eta|^{1/2}\sum_{n=0}^{\infty}\frac{\sin\left(\rho_n(\eta,\zeta)^{1/2}\,t\right)}{\rho_n(\eta,\zeta)^{1/2}}\Psi_n\left(x_0|\eta|^{1/2}\right)\Psi_n(\sigma),$$

which readily gives

$$\hat{E}_3(t,x,\eta,\zeta;x_0)=-H(t)\,|\eta|^{1/2}\sum_{n=0}^{\infty}\frac{\sin\left(\rho_n(\eta,\zeta)^{1/2}\,t\right)}{\rho_n(\eta,\zeta)^{1/2}}\Psi_n\left(x_0|\eta|^{1/2}\right)\Psi_n\left(|\eta|^{1/2}x\right)$$

so that

$$\begin{aligned}&E_3(t,x,y,z;x_0)=-\frac{H(t)}{2\pi}\\&\times\int_{\mathbb{R}^2}e^{iy\eta+iz\zeta}\,|\eta|^{1/2}\sum_{n=0}^{\infty}\frac{\sin\left(\rho_n(\eta,\zeta)^{1/2}\,t\right)}{\rho_n(\eta,\zeta)^{1/2}}\Psi_n\left(x_0|\eta|^{1/2}\right)\Psi_n\left(x|\eta|^{1/2}\right)d\eta\,d\zeta.\end{aligned}$$

Now, we are interested in

$$\mathrm{F}_3\left(t-s,x,y,z;\xi_0,\hat{\eta},\hat{\zeta}\right):=\mathcal{F}E_3(t-s,x,y-\cdot,z-\cdot,\cdot)\left(\xi_0,\hat{\eta},\hat{\zeta}\right);$$

by translation and dilation properties of the Fourier transform, we see

$$\mathrm{F}_3\left(t-s,x,y,z;\xi_0,\hat{\eta},\hat{\zeta}\right)=e^{iy\hat{\eta}+iz\hat{\zeta}}\int_{-\infty}^{\infty}e^{-i\xi_0x_0}\,\hat{E}_3\left(t-s,x,-\hat{\eta},-\hat{\zeta};x_0\right)dx_0.$$

Thus, we finally get

$$\mathtt{F}_3\left(t-s,x,y,z;\xi_0,\hat{\eta},\hat{\zeta}\right)=-\frac{H\left(t-s\right)}{2\pi}e^{iy\hat{\eta}+iz\hat{\zeta}}$$
$$\times\sum_{n=0}^{\infty}(-i)^n\frac{\sin\left(\rho_n\left(-\hat{\eta},\hat{\zeta}\right)^{1/2}(t-s)\right)}{\rho_n\left(-\hat{\eta},\hat{\zeta}\right)^{1/2}}\Psi_n\left(\frac{\xi_0}{|\hat{\eta}|^{1/2}}\right)\Psi_n\left(|\hat{\eta}|^{1/2}x\right). \tag{25}$$

Following the above procedure along, one can also solve

$$P_2E_2=\delta(t)\delta(x-x_0)\delta(y)$$

and obtain

$$E_2\left(t,x,y;x_0\right)=-\frac{H\left(t\right)}{2\pi}$$
$$\times\int_{\mathbb{R}}e^{iy\eta}|\eta|^{1/2}\sum_{n=0}^{\infty}\frac{\sin\left(\tilde{\rho}_n\left(\eta\right)^{1/2}t\right)}{\tilde{\rho}_n\left(\eta\right)^{1/2}}\Psi_n\left(x_0|\eta|^{1/2}\right)\Psi_n\left(x|\eta|^{1/2}\right)d\eta,$$

where $\tilde{\rho}_n(\eta):=\rho_n(\eta,0)$. And thus

$$\mathtt{F}_2\left(t-s,x,y;\xi_0,\hat{\eta}\right)=-\frac{H\left(t-s\right)}{2\pi}e^{iy\hat{\eta}}$$
$$\times\sum_{n=0}^{\infty}(-i)^n\frac{\sin\left(\tilde{\rho}_n\left(-\hat{\eta}\right)\right)^{1/2}(t-s)\Big)}{\tilde{\rho}_n\left(-\hat{\eta}\right)^{1/2}}\Psi_n\left(\frac{\xi_0}{|\hat{\eta}|^{1/2}}\right)\Psi_n\left(|\hat{\eta}|^{1/2}x\right). \tag{26}$$

3.2 Two Dimensions: Integral Bound

Again we rewrite here the expression for the two-dimensional candidate random field solution

$$u_2(t,x,y)=\int_0^t\int_{\mathbb{R}^2}E_2\left(t-s,x,y-y_0;x_0\right)\dot{F}_1(s,x_0,y_0)ds\,dx_0\,dy_0.$$

In the present section, we proceed with the study of the fundamental solutions E_2 to argue that the formal expression above is well defined as a real-valued process. We start proving

Lemma 3.2.1 *Let* $\mathtt{F}_2\left(t-s,x,y;\xi_0,\hat{\eta}\right)$ *be defined as in* (26) *and the assumptions of Theorem* 1.1 *(ii) hold. Then*

$$I_2:=\int_0^t\int_{\mathbb{R}^2}\left|\mathtt{F}_2\left(t-s,x,y;\xi_0,\hat{\eta}\right)\right|^2\nu_2\left(d\xi_0,d\hat{\eta}\right)\,ds<+\infty.$$

Proof Let us call

$$J\left(t-s,x,y;\hat{\eta}\right):=\int_{\mathbb{R}}\left|\mathrm{F}_2\left(t-s,x,y;\xi_0,\hat{\eta}\right)\right|^2\,d\xi_0$$

and change variable in the following way: $\lambda=|\hat{\eta}|^{-1/2}\xi_0$. Then

$$J\left(t-s,x,y;\hat{\eta}\right):=\int_{\mathbb{R}}\left|\mathrm{F}_2\left(t-s,x,y;|\hat{\eta}|^{1/2}\lambda,\hat{\eta}\right)\right|^2|\hat{\eta}|^{1/2}\,d\lambda.$$

Now, if we denote

$$\begin{aligned}g_n\left(t-s,x,y;\hat{\eta}\right):=&\frac{H\left(t-s\right)}{2\pi}e^{iy\hat{\eta}}\\&\times(-i)^n\,\frac{\sin\left(\tilde{\rho}_n\left(-\hat{\eta}\right)^{1/2}(t-s)\right)}{\tilde{\rho}_n\left(-\hat{\eta}\right)^{1/2}}\Psi_n\left(|\hat{\eta}|^{1/2}x\right),\end{aligned}\tag{27}$$

we get

$$\mathrm{F}_2\left(t-s,x,y;|\hat{\eta}|^{1/2}\lambda,\hat{\eta}\right)=\sum_{n=0}^{\infty}g_n\left(t-s,x,y;\hat{\eta}\right)\Psi_n(\lambda)$$

and hence, by Parseval's identity,

$$\int_{\mathbb{R}}\left|\mathrm{F}_2\left(t-s,x,y;|\hat{\eta}|^{1/2}\lambda,\hat{\eta}\right)\right|^2\,d\lambda=\sum_{n=0}^{\infty}g_n\left(t-s,x,y;\hat{\eta}\right)^2.\tag{28}$$

It follows that

$$\begin{aligned}I_2&=\int_0^t\int_{\mathbb{R}}|\hat{\eta}|^{1/2}\sum_{n=0}^{\infty}g_n\left(t-s,x,y;\hat{\eta}\right)^2\hat{\nu}_2\left(d\hat{\eta}\right)\,ds\\&=\frac{1}{4\pi^2}\int_0^t\int_{\mathbb{R}}\sum_{n=0}^{\infty}\frac{\sin^2\left(\tilde{\rho}_n^{1/2}(-\hat{\eta})\,(t-s)\right)}{\tilde{\rho}_n(-\hat{\eta})}|\hat{\eta}|^{1/2}\Psi_n^2\left(|\hat{\eta}|^{1/2}x\right)\hat{\nu}_2\left(d\hat{\eta}\right)\,ds.\end{aligned}$$

Recalling (19) and denoting

$$T_0:=\int_0^t\int_{\mathbb{R}}g_0\left(t-s,x,y;\hat{\eta}\right)^2|\hat{\eta}|^{1/2}\hat{\nu}_2\left(d\hat{\eta}\right)\,ds,$$

we get

$$I_2 \leq \frac{C_2}{4\pi^2} \int_0^t \int_{\mathbb{R}} \sum_{n=1}^{\infty} \frac{|\hat{\eta}|^{1/2}}{n^{1/6}\left((2n+1)\mu\left|\hat{\eta}\right| + b\hat{\eta}\right)} \hat{\nu}_2\left(d\hat{\eta}\right) ds + T_0$$
$$\leq \frac{C_2\, t}{4\pi^2 n^{1/6}} \sum_{n=1}^{\infty} \int_{\mathbb{R}} \frac{|\hat{\eta}|^{1/2}}{\left((2n+1)\mu\left|\hat{\eta}\right| + b\hat{\eta}\right)} \hat{\nu}_2\left(d\hat{\eta}\right) + T_0. \tag{29}$$

Now, we have

$$\int_{\mathbb{R}} \frac{|\hat{\eta}|^{1/2}}{\left((2n+1)\mu\left|\hat{\eta}\right| + b\hat{\eta}\right)} \hat{\nu}_2\left(d\hat{\eta}\right) \leq \frac{1}{n\mu} \int_{\mathbb{R}} \left|\hat{\eta}\right|^{-1/2} \hat{\nu}_2(d\hat{\eta}) = \frac{C}{\mu n}$$

where the last inequality comes from the fact that the condition $|b| < \mu$ gives $\mu + \operatorname{sgn}(\hat{\eta})b > 0$. Finally, convergence of the integral T_0 under this measure is easily checked. Indeed, we can find an $\epsilon > 0$ such that $\mu + \operatorname{sgn}(\hat{\eta})b > \epsilon$ for all $\hat{\eta} \in \mathbb{R}$ and observe that $\Psi_0^2\left(|\hat{\eta}|^{1/2}x\right) = \pi^{-1/4}e^{-|\hat{\eta}|x^2/2} \leq \pi^{-1/4}$. Therefore

$$T_0 \leq \frac{t}{4\pi^{9/4}\epsilon} \int_{\mathbb{R}} \left|\hat{\eta}\right|^{-1/2} \hat{\nu}_2\left(d\hat{\eta}\right) < +\infty.$$

□

Now, we want to find a sequence $E_{2,m}(t, x, y; x_0) \in \mathcal{E}_0$ so that $E_{2,m} \to E_2$ in $\bar{\mathcal{P}}$. We set

$$E_{2,m}(t, x, y; x_0) := -\frac{H(t)}{2\pi} \times \int_{\mathbb{R}} e^{iy\eta} |\eta|^{1/2} \sum_{n=0}^{m} \frac{\sin\left(\tilde{\rho}_n(\eta)^{1/2} t\right)}{\tilde{\rho}_n(\eta)^{1/2}} \Psi_n\left(x_0|\eta|^{1/2}\right) \Psi_n\left(x|\eta|^{1/2}\right) d\eta.$$

Again for each $m \in \mathbb{N}$, $(y, x_0) \mapsto E_{2,m}(t, x, y; x_0) \in \mathcal{S}\left(\mathbb{R}^2\right)$, and so calling

$$\mathrm{F}_2^m\left(t-s, x, y; \xi_0, \hat{\eta}\right) := \frac{H(t-s)}{2\pi} e^{iy\hat{\eta}} \times \sum_{n=m}^{+\infty} (-i)^n \frac{\sin\left(\tilde{\rho}_n(-\hat{\eta})^{1/2}(t-s)\right)}{\tilde{\rho}_n(-\hat{\eta})^{1/2}} \Psi_n\left(\frac{\xi_0}{|\hat{\eta}|^{1/2}}\right) \Psi_n\left(|\hat{\eta}|^{1/2}x\right),$$

we have

$$\begin{aligned}\left|\left|E_{2,m}-E_2\right|\right|_0^2 &= \int_0^t ds \int_{\mathbb{R}^2} \left|\mathrm{F}_2^m\left(t-s,x,y;\xi_0,\hat{\eta}\right)\right|^2 \nu_2\left(d\xi_0,d\hat{\eta}\right)\\ &\leq \frac{C_2\,t}{4\pi^2}\sum_{n=m}^{+\infty}\int_{\mathbb{R}} \frac{|\hat{\eta}|^{1/2}}{n^{1/6}\left((2n+1)\,\mu\left|\hat{\eta}\right|+b\hat{\eta}\right)}\hat{\nu}_2(d\hat{\eta})\\ &\leq \tilde{C}_2\,t\sum_{n=m}^{+\infty} n^{-7/6} \xrightarrow{m\to+\infty} 0.\end{aligned}$$

3.3 Three Dimensions: Integral Bound

For the reader's convenience, we recall the formal expression for the candidate three-dimensional random field solution

$$\begin{aligned}u_3(t,x,y,z) = \int_0^t \int_{\mathbb{R}^3} & E_3\left(t-s,x,y-y_0,z-z_0;x_0\right)\\ &\times \dot{F}_3(s,x_0,y_0,z_0)ds\,dx_0\,dy_0\,dz_0.\end{aligned}$$

In the present section, we would like to understand whether the above epxression is a real-valued process and to do that we need to check if E_3 belongs to $\mathcal{P}_0$. As always, we start with showing that $E_3 \in \bar{\mathcal{P}}$.

Lemma 3.3.1 *Let* $\mathrm{F}_3\left(t-s,x,y,z;\xi_0,\hat{\eta},\hat{\zeta}\right)$ *be defined as in* (25) *and the assumptions of Theorem* 1.1 *(iii) hold. Then*

$$I_3 := \int_0^t \int_{\mathbb{R}^3} \left|\mathrm{F}_3\left(t-s,x,y,z;\xi_0,\hat{\eta},\hat{\zeta}\right)\right|^2 \nu_3\left(d\xi_0,d\hat{\eta},d\hat{\zeta}\right) ds < \infty.$$

Proof The proof is very similar to the one of Proposition 3.2.1. Indeed, calling

$$\begin{aligned}\hat{g}_n\left(t-s,x,y;\hat{\eta},\hat{\zeta}\right) := &\frac{H\,(t-s)}{2\pi}e^{iy\hat{\eta}+iz\hat{\zeta}}\\ &\times(-i)^n\,\frac{\sin\left(\rho_n\left(-\hat{\eta},\hat{\zeta}\right)^{1/2}(t-s)\right)}{\rho_n\left(-\hat{\eta},\hat{\zeta}\right)^{1/2}}\Psi_n\left(|\hat{\eta}|^{1/2}x\right),\end{aligned}$$

and

$$\begin{aligned}\hat{T}_0 &:= \int_0^t \int_{\mathbb{R}^2} \hat{g}_0\left(t-s,x,y,z;\hat{\eta},\hat{\zeta}\right)^2 |\hat{\eta}|^{1/2}\hat{\nu}_3\left(d\hat{\eta},d\hat{\zeta}\right) ds,\\ \hat{q}_n &:= \int_{\mathbb{R}^2} \frac{|\hat{\eta}|^{1/2}}{\left((2n+1)\,\mu\left|\hat{\eta}\right|+a\hat{\zeta}^2+b\hat{\eta}\right)}\hat{\nu}_3\left(d\hat{\eta},d\hat{\zeta}\right),\end{aligned}$$

with the same strategy employed in the previous section, we obtain

$$I_3 \leq \frac{C_2\, t}{4\pi^2} \sum_{n=1}^{\infty} n^{-1/6} \hat{q}_n + \hat{T}_0. \tag{30}$$

Now, since $|b| < \mu$, we have $\mu + \mathrm{sgn}(\hat{\eta})b > 0$ and consequently

$$\hat{q}_n \leq \int_{\mathbb{R}^2} \frac{|\hat{\eta}|^{1/2}}{n\mu\,|\hat{\eta}| + a\hat{\xi}^2} w(\hat{\eta}^2 + \hat{\xi}^2)\, d\hat{\eta} d\hat{\xi} =: J(n).$$

We can then switch to polar coordinates and, recalling Young's inequality

$$\frac{x^p}{p} + \frac{y^q}{q} \geq xy, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

obtain

$$\begin{aligned} J(n) &= 4\int_0^{+\infty} d\rho\; \rho\, w(\rho^2) \int_0^{\pi/2} \frac{(\rho\cos(\theta))^{1/2}}{n\mu\rho\cos(\theta) + a\rho^2\sin^2(\theta)} d\theta \\ &= 4\int_0^{+\infty} d\rho\; \rho^{1/2} w(\rho^2) \int_0^{\pi/2} \frac{(\cos(\theta))^{1/2}}{n\mu\cos(\theta) + a\rho\sin^2(\theta)} d\theta \\ &\leq \frac{4(1/p)^{1/p}(1/q)^{1/q}}{\mu^{1/p}a^{1/q}} n^{-1/p} \int_0^{+\infty} d\rho\; \rho^{1/2-1/q} w(\rho^2) \int_0^{\pi/2} \cos(\theta)^{1/2-1/p} \sin\theta^{-2/q} d\theta. \end{aligned}$$

We have

$$\int_0^{\pi/2} \cos(\theta)^{1/2-1/p} \sin\theta^{-2/q} d\theta = \frac{\Gamma\,(3/4 - 1/2p)\,\Gamma\,(1/2 - 1/q)}{2\Gamma\,(5/4 - 1/2p - 1/q)} < +\infty,$$

with q selected as below. In fact, condition (13) guarantees that we can indeed select q so that

$$\int_0^{+\infty} \rho^{1/2-1/q} w(\rho^2) d\rho = \int_0^{+\infty} \rho^{1-2\alpha} w(\rho^2) d\rho < +\infty.$$

From $\frac{1}{q} = 2\alpha - \frac{1}{2}$ we have

$$\frac{1}{p} = 1 - \frac{1}{q} = \frac{3}{2} - 2\alpha > \frac{5}{6}.$$

Therefore, we get $J(n) \leq \hat{C} n^{-1/p},\ 1/p > 5/6$ which is sufficent for the convergence of the series in (30).
Finally, taking $\epsilon > 0$ such that $\epsilon < \mu + b\,\mathrm{sgn}(\hat{\eta})$ we write

$$T_0 \leq \frac{t}{4\pi^{9/4}} \int_{\mathbb{R}^2} \frac{|\hat{\eta}|^{1/2}}{a\hat{\xi}^2 + \epsilon\,|\hat{\eta}|} w(\hat{\eta}^2 + \hat{\xi}^2)\, d\hat{\eta} d\hat{\xi},$$

which is finite, by the same arguments utilized above. □

Having proved that $E_3 \in \bar{\mathcal{P}}$, in order to show that $E_3 \in \mathcal{P}_0$, we need to find a sequence of functions $E_{3,m} \in \mathcal{E}_0$ such that $E_{3,m} \to E_3$ in $\bar{\mathcal{P}}$. We set

$$E_{3,m}(t,x,y,z;x_0) := -\frac{H(t)}{2\pi} \times \int_{\mathbb{R}^2} e^{iy\eta+iz\zeta} |\eta|^{1/2} \sum_{n=0}^{m} \frac{\sin\left(\rho_n(\eta,\zeta)^{1/2} t\right)}{\rho_n(\eta,\zeta)^{1/2}} \Psi_n\left(x_0|\eta|^{1/2}\right) \Psi_n\left(x|\eta|^{1/2}\right) d\eta\, d\zeta.$$

It is readily seen that $(x_0, y, z) \mapsto E_{3,m}(t,x,y,z;x_0) \in \mathcal{S}\left(\mathbb{R}^3\right)$ for all $m \in \mathbb{N}$, and

$$\left|\left|E_{3,m} - E_3\right|\right|_0^2 \leq \frac{C_2\, t}{4\pi^2} \sum_{n=m}^{+\infty} n^{-1/6} \hat{q}_n \xrightarrow{m\to+\infty} 0.$$

3.4 Common Computations: Measurability

To prove our claim, it is left to show that the maps $(t,x,y) \mapsto u_2(t,x,y)$ and $(t,x,y,z) \mapsto u_3(t,x,y,z)$ are measurable. Our strategy in this regard is to show $L_2(\Omega)$-continuity of the solution processes. We only tackle the three-dimensional case, as the two-dimensional case is pretty much analogous.

We first consider the variable y. We call

$$\bar{F}_3(t-s,x,z;\xi_0,\hat{\eta},\hat{\zeta}) := \frac{H(t-s)}{2\pi} e^{iz\hat{\zeta}} \sum_{n=0}^{+\infty} (-i)^n \frac{\sin\left(\rho_n\left(-\hat{\eta},\hat{\zeta}\right)^{1/2}(t-s)\right)}{\rho_n\left(-\hat{\eta},\hat{\zeta}\right)^{1/2}} \Psi_n \times \left(\frac{\xi_0}{|\hat{\eta}|^{1/2}}\right) \Psi_n\left(|\hat{\eta}|^{1/2} x\right),$$

and get

$$\begin{aligned} &\mathbb{E}\left[|u_3(t,x,y+h,z) - u_3(t,x,y,z)|^2\right] \\ &= \int_0^t ds \int_{\mathbb{R}^3} \left|\mathrm{F}_3\left(t-s,x,y+h,z;\xi_0,\hat{\eta},\hat{\zeta}\right) - \mathrm{F}_3\left(t-s,x,y,z;\xi_0,\hat{\eta},\hat{\zeta}\right)\right|^2 \nu_3(d\xi_0,d\hat{\eta},d\hat{\zeta}) \\ &= \int_0^t ds \int_{\mathbb{R}^3} \left|e^{i\hat{\eta}(y+h)} - e^{i\hat{\eta}y}\right|^2 \left|\bar{F}_3(t-s,x,z;\xi_0,\hat{\eta},\hat{\zeta})\right|^2 \nu_3(d\xi_0,d\hat{\eta},d\hat{\zeta}). \end{aligned}$$

Now, $\bar{F}_3(t-s,x,z;\xi_0,\hat{\eta},\hat{\zeta})$ is integrable and, hence, by dominated convergence, we have

$$\lim_{h\to 0} \mathbb{E}\left[|u_3(t,x,y+h,z)-u_3(t,x,y,z)|^2\right]=0.$$

The z variable is treated the same. For the increment in the x variable, we denote

$$\begin{aligned} S_3(t-s,\hat{\eta},\hat{\zeta},x,h) \; &:= \sum_{n=0}^{+\infty} \frac{\sin^2\left(\rho_n\left(-\hat{\eta},\hat{\zeta}\right)^{1/2}(t-s)\right)}{\rho_n\left(-\hat{\eta},\hat{\zeta}\right)} \\ &\quad \times \left(\Psi_n(|\eta|^{1/2}(x+h))-\Psi_n(|\eta|^{1/2}x)\right)^2, \end{aligned}$$

and write

$$\begin{aligned} &\mathbb{E}\left[|u_3(t,x+h,y,z)-u_3(t,x,y,z)|^2\right] \\ &= \int_0^t ds \int_{\mathbb{R}^3} \left|\mathrm{F}_3\left(t-s,x+h,y,z;\xi_0,\hat{\eta},\hat{\zeta}\right)-\mathrm{F}_3\left(t-s,x,y,z;\xi_0,\hat{\eta},\hat{\zeta}\right)\right|^2 \nu_3(d\xi_0,d\hat{\eta},d\hat{\zeta}) \\ &= \int_0^t ds \int_{\mathbb{R}^2} \frac{|\hat{\eta}|^{1/2}}{4\pi^2} S_3(t-s,\hat{\eta},\hat{\zeta},x,h)\left(1+|\hat{\eta}|^2+|\hat{\zeta}|^2\right)^{-\alpha} d\hat{\eta}d\hat{\zeta}. \end{aligned}$$

Moreover thanks to (19), we have

$$S_3(t-s,\hat{\eta},\hat{\zeta},x,h) \le \sum_{n=0}^{+\infty} \frac{4C_2}{n^{1/6}\rho_n\left(-\hat{\eta},\hat{\zeta}\right)} < +\infty \tag{31}$$

granting uniform convergence and consequently the continuity of $S_3(t-s,\hat{\eta},\hat{\zeta},x,h)$ with respect to his last variable. Furthermore, in the proof of Lemma 3.3.1 we have seen that

$$\int_0^t ds \int_{\mathbb{R}^3} |\eta|^{1/2} \sum_{n=0}^{+\infty} \frac{1}{n^{1/6}\rho_n\left(-\hat{\eta},\hat{\zeta}\right)} \nu_3(d\xi_0,d\hat{\eta},d\hat{\zeta}) < +\infty.$$

Therefore, by dominated convergence we get

$$\lim_{h\to 0} \mathbb{E}\left[|u_3(t,x+h,y,z)-u_3(t,x,y,z)|^2\right]=0. \tag{32}$$

Lastly, the increment in time yields

$$\begin{aligned} &\mathbb{E}\left[|u_3(t+h,x,y,z)-u_3(t,x,y,z)|^2\right] \\ &\le 2\int_0^t ds \int_{\mathbb{R}^3} \left|\mathrm{F}_3\left(t+h-s,x,y,z;\xi_0,\hat{\eta},\hat{\zeta}\right)-\mathrm{F}_3\left(t-s,x,y,z;\xi_0,\hat{\eta},\hat{\zeta}\right)\right|^2 \nu_3(d\xi_0,d\hat{\eta},d\hat{\zeta}) \\ &\quad + 2\int_t^{t+h} ds \int_{\mathbb{R}^3} \left|\mathrm{F}_3\left(t+h-s,x,y,z;\xi_0,\hat{\eta},\hat{\zeta}\right)\right|^2 \nu_3(d\xi_0,d\hat{\eta},d\hat{\zeta}) \\ &= 2J_1(h)+2J_2(h). \end{aligned}$$

Lemma 3.3.1 immediately gives

$$\lim_{h\to 0} J_2(h) = 0.$$

Denoting

$$\tilde{S}_3(t-s,\hat{\eta},\hat{\zeta},x,h) := \sum_{n=0}^{+\infty} \frac{\left(\sin\left(\rho_n^{1/2}(\hat{\eta},\hat{\zeta})(t+h-s)\right) - \sin\left(\rho_n^{1/2}(\hat{\eta},\hat{\zeta})(t-s)\right)\right)^2}{\rho_n(\hat{\eta},\hat{\zeta})} \Psi_n^2(|\eta|^{1/2}x),$$

by Parseval's identity, we obtain

$$J_1(h) = \int_0^t ds \int_{\mathbb{R}^2} \frac{|\hat{\eta}|^{1/2}}{4\pi^2} \tilde{S}_3(t-s,\hat{\eta},\hat{\zeta},x,h) \left(1+|\hat{\eta}|^2+|\hat{\zeta}|^2\right)^{-\alpha} d\hat{\eta} d\hat{\zeta}.$$

For $\tilde{S}_3(t-s,\hat{\eta},\hat{\zeta},x,h)$, the same bound as in (31) holds. So, by dominated convergence, again we have

$$\lim_{h\to 0} J_1(h) = 0.$$

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