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A Novel Family of Starlike Functions Involving Quantum Calculus and a Special Function

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Abstract: The intent of quantum calculus, briefly q-calculus, is to find q-analogues of mathematical entities so that the original object is achieved when a certain limit is taken. In the case of q-analogue of the ordinary derivative, the limit is $q \to 1^-$. Also, the study of integral as well as differential operators has remained a significant field of inquiry from the early developments of function theory. In the present article, a subclass $S_{sc}(\mu, q)$ of functions being analytic in $\mathcal{D} = \{z \in \mathbb{C} : |z| < 1\}$ is introduced. The definition of $S_{sc}(\mu, q)$ involves the concepts of subordination, that of q-derivative and q-Ruscheweyh operators. Since coefficient estimates and coefficient functionals provide insights into different geometric properties of analytic functions, for this newly defined subclass, we investigate coefficient estimates up to a_4, in which both bounds for $|a_2|$ and $|a_3|$ are sharp, while that of $|a_4|$ is sharp in one case. We also discuss the sharp Fekete–Szegö functional for the said class. In addition, Toeplitz determinant bounds up to $T_3(2)$ (sharp in some cases) and sufficient condition are obtained. Several consequences derived from our above-mentioned findings are also part of the discussion.

Keywords: Analytic functions; subordination; Toeplitz determinant; q-derivative operator; q-Ruscheweyh operator

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1. Introduction

We denote the open unit disk by $\mathcal{D}$ and by $\mathcal{A}$ and the family of all functions $\mathfrak{F} : \mathcal{D} \to \mathbb{C}$ that are analytic and satisfy the normalization conditions. $\mathfrak{F}(0) = 0$, $\mathfrak{F}'(0) = 1$, that is:

$$\mathfrak{F}(z) = z + \sum_{\mathfrak{k}=2}^{\infty} a_{\mathfrak{k}} z^{\mathfrak{k}}. \tag{1}$$

The subfamily of $\mathcal{A}$, consisting of the so-called functions being univalent, that is, injective in $\mathcal{D}$, is labeled by $\mathcal{S}$. Furthermore, we recall the class $\mathcal{P}$ of Carathéodory functions p (see [1,2]), having their real parts positive in $\mathcal{D}$ and normalized by $p(0) = 1$, i.e.,

$$p(z) = 1 + \sum_{\mathfrak{k}=1}^{\infty} p_{\mathfrak{k}} z^{\mathfrak{k}}. \tag{2}$$

A function $\mathfrak{F} \in \mathcal{A}$ is called starlike if

$$\mathrm{Re}\left(\frac{z\mathfrak{F}'(z)}{\mathfrak{F}(z)}\right) > 0, \qquad (z \in \mathcal{D})$$

and convex if

$$\mathrm{Re}\left(1 + \frac{z\mathfrak{F}''(z)}{\mathfrak{F}'(z)}\right) > 0, \qquad (z \in \mathcal{D}).$$

The respective subclasses of starlike and convex functions are labeled by $\mathcal{S}^*$ and $\mathcal{C}$. These two subclasses can also be defined by the use of the subordination principle.

If for $\mathfrak{F}$ and g being analytic in $\mathcal{D}$, there is an analytic function w in $\mathcal{D}$ satisfying the properties $|w(z)| \leq |z|$, $w(0) = 0$ and $\mathfrak{F}(z) = g(w(z))$, then $\mathfrak{F}$ is subordinate to g, symbolically; $\mathfrak{F} \prec g$. Additionally, $\mathfrak{F} \prec g$ if, and only if, $\mathfrak{F}(\mathcal{U}) \subset g(\mathcal{U})$, provided that g is univalent. Using this concept of subordination, $\mathcal{S}^*$ and $\mathcal{C}$ are defined as

$$\mathcal{S}^* = \left\{\mathfrak{F} \in \mathcal{A} : \frac{z\mathfrak{F}'(z)}{\mathfrak{F}(z)} \prec \frac{1-z}{1+z}, z \in \mathcal{D}\right\},$$

and

$$\mathcal{C} = \left\{\mathfrak{F} \in \mathcal{A} : \frac{(z\mathfrak{F}'(z))'}{\mathfrak{F}'(z)} \prec \frac{1-z}{1+z}, \quad z \in \mathcal{D}\right\}.$$

While various subclasses of $\mathcal{S}^*$ and $\mathcal{C}$, respectively, are defined as

$$\mathcal{S}^*(\phi) = \left\{\mathfrak{F} \in \mathcal{A} : \frac{z\mathfrak{F}'(z)}{\mathfrak{F}(z)} \prec \phi(z) \quad z \in \mathcal{D}\right\},$$

and

$$\mathcal{C}(\phi) = \left\{\mathfrak{F} \in \mathcal{A} : 1 + \frac{z\mathfrak{F}''(z)}{\mathfrak{F}'(z)} \prec \phi(z), \quad z \in \mathcal{D}\right\}.$$

where the specific univalent function having a positive real part denoted by ϕ is known as the Ma and Minda [3] type function. For different choices of ϕ, the researchers have defined numerous subclasses of $\mathcal{S}^*$ and $\mathcal{C}$. For example, in [4], ϕ is chosen as $1 + z/(1 - \alpha z^2)$, $(0 \leq \alpha \leq 1)$, in [5] as $1 + \xi \sinh(z)$, $(0 < \xi < 1/\sinh(1))$, and in [6] as $1 + \tanh(z)$. A few more choices of ϕ can be seen, for example in [7–11].

Furthermore, if $\mathfrak{F}, g \in \mathcal{A}$ with $\mathfrak{F}$ given in (1) and g as

$$g(z) = z + \sum_{\mathfrak{k}=2}^{\infty} b_{\mathfrak{k}} z^{\mathfrak{k}},$$

then their convolution (or Hadamard product) $f * g$ is defined by

$$(\mathfrak{F} * g) = z + \sum_{\mathfrak{k}=2}^{\infty} a_{\mathfrak{k}} b_{\mathfrak{k}} z^{\mathfrak{k}}. \tag{3}$$

Pólya and Schoenberg [12] conjectured that the class $\mathcal{C}$ is closed under convolution; that is, whenever $\mathfrak{F}$ and g are in $\mathcal{C}$, then $\mathfrak{F} * g$ is in $\mathcal{C}$. This conjecture was then proved by Ruscheweyh and Sheil-Small [13,14] in 1973.

Recent years have seen a dramatic growth of research work and their applications in q-calculus-related fields. These developments can be demonstrated, for example, in the finite differences theory, quantum mechanics, analytic function theory, gamma function theory, operator theory, and most currently in geometric theory of univalent holomorphic and univalent harmonic functions. Quantum calculus also concerns graphical interpretations in physics. Visualizing the motion of particles in a q-system and modeling the structure of q-systems in phase space, in quantum field theory, the graphical depiction of particle interactions and in physical systems, and graphically representing the scaling attributes of fractals are just a few examples. Thus, the widespread applications of quantum calculus, briefly denoted as q-calculus, in several research directions in mathematics and physics have sparked considerable interest among researchers. In the recent past, the extensive research work of Aral et al. in [15] shows that q-analysis has gained substantial observance from researchers, especially within the region of function theory, as quantum calculus can be utilized to generate tools to a that have applications in investigating different subclasses of analytic functions. In 1990 Ismail, Merkes, and Styer [16] introduced the notion of q-extension of the subclass $\mathcal{S}^*$ and hence utilized q-calculus in G.F.T for the first time. Additionally, for complex functions, the use of differential and integral operators plays a central role in defining and generalizing new subclasses. In continuation, the q-analogues of such operators are extensively used in G.F.T. We refer, for example, to [17,18] for q-integral operators and to [19–23] for q-differential operators. More related work can be seen in [24–28]. These findings, among several others, highlight the critical demand for meaningful developments in q- and fractional q-calculus within the context of G.F.T. Several have worked to propel this theory forward by defining particular classes using q-calculus. Their collective efforts have broadened the horizons and deepened the understanding of G.F.T, setting the stage for additional inquiries and developments in complex analysis. The esteem of q-calculus as an essential instrument to delineate classes and decipher characteristics underscores its significance in the progressive refinement of geometric function theory. For the latest updates and contributions on this topic, see [29–33].

The initiation of the integral operator is credited to Alexander [34]. The inverse of Jackson's q-integration, known as q-derivative or Jackson derivative, was introduced by Jackson [35]. To gain in-depth and comprehensive insight into the theoretical underpinnings and practical implementations of fractional q-derivative operators in the wider framework of G.F.T, we refer to the informative review paper by Srivastava [36]. The elaboration of operator theory in this scenario has inspired many researchers to work in this direction.

We briefly review the key concepts of the q-calculus to facilitate understanding in the sequel. We start with a definition given in [37].

Definition 1. *The symbol $[\vartheta]_q$ is the q-number if*

$$[\vartheta]_q = \frac{1-q^{\vartheta}}{1-q}, \; 0<q<1 \text{ and } \vartheta \in \mathbb{C}. \tag{4}$$

When ϑ is a natural number, we obtain

$$[\vartheta]_q = \sum_{\mathfrak{k}=0}^{\vartheta-1} q^{\mathfrak{k}},$$

and

$$[\vartheta]_q! = \prod_{\mathfrak{k}=1}^{\vartheta} [\mathfrak{k}]_q \, ,$$

where $[\vartheta]_q!$ is known as ϑth q-factorial. Observe that $[0]_q! = 0$.

Next, we recall a second concept introduced in [37].

Definition 2. *The q-generalized version of Pochhammer symbol is given by*

$$[\vartheta]_{q,\mathfrak{j}} = [\vartheta]_q[\vartheta+1]_q[\vartheta+2]_q \ldots [\vartheta+\mathfrak{j}-1]_q.$$

where ϑ is any complex number and $\mathfrak{j} \in \mathbb{N}$. Also, $[\vartheta]_{q,0} = 1$.

We also mention (see [35]) the notion of q-difference operator:

Definition 3. *For $\mathfrak{F} \in \mathcal{A}$, the q-difference operator $(0 < q < 1)$; $\mathfrak{D}_q : \mathcal{A} \to \mathcal{A}$ is*

$$\left(\mathfrak{D}_q\mathfrak{F}\right)(z) = \begin{cases} \frac{\mathfrak{F}(z)-\mathfrak{F}(qz)}{z-qz}; & \text{if } z \neq 0, \\ \mathfrak{F}'(0); & \text{if } z = 0. \end{cases} \tag{5}$$

Also, for $\mathfrak{F}$ of the form (1), we have

$$\left(\mathfrak{D}_q\mathfrak{F}\right)(z) = 1 + \sum_{\mathfrak{k}=2}^{\infty} [\mathfrak{k}]_q a_{\mathfrak{k}} z^{\mathfrak{k}-1}. \tag{6}$$

It is worth noting that

$$\lim_{q \to 1^-} [\mathfrak{k}]_q = \mathfrak{k},$$

and

$$\lim_{q \to 1^-} \left(\mathfrak{D}_q\mathfrak{F}\right)(z) = \mathfrak{F}'(z).$$

Recently, using the concepts of subordination and that of q-difference operator $\mathfrak{D}_q$, Matarneh K. et al. in [38] introduced the class $\mathcal{K}(\mathfrak{l}, \mathfrak{m}, q, \zeta)$. They obtained the following sharp coefficient bounds for functions $\mathfrak{F} \in \mathcal{K}(\mathfrak{l}, \mathfrak{m}, q, \zeta)$ as

$$|a_2| \leq \frac{(\mathfrak{l}-\mathfrak{m})\,|\tau|}{2\left((1+\zeta)[2]_q - 1\right)},\ |a_3| \leq \frac{(\mathfrak{l}-\mathfrak{m})|\tau|^2}{4\left([3]_q\left(1+\zeta[2]_q\right)-1\right)}\left(5-\mathfrak{m}+\frac{(\mathfrak{l}-\mathfrak{m})}{(1+\zeta)[2]_q-1}\right),$$

they also investigated the sharp Fekete–Szegö inequality:

$$|a_3 - \mu a_2^2| \leq \frac{(\mathfrak{l}-\mathfrak{m})\,|\tau|}{4\left([3]_q\left(1+\zeta[2]_q\right)-1\right)} \max\{2, |\mathfrak{m}-5+$$

$$\tau\left(\frac{(\mathfrak{l}-\mathfrak{m})}{(1+\zeta)[2]_q-1}\left(\frac{\left([3]_q\left(1+\zeta[2]_q\right)-1\right)\mu}{(1+\zeta)[2]_q-1}-1\right)\right)\Bigg|\Bigg\},$$

where $-1 \leq \mathfrak{m} < \mathfrak{l} \leq 1$, $\tau = \left(1-\sqrt{5}\right)/2$, and $\zeta \geq 0$. Furthermore, necessary and sufficient conditions of certain subclasses of functions that are q-starlike can be seen in [39].

Here we also recall (see [19]) the q-Ruscheweyh differential operator:

Definition 4. *The differential operator $\mathfrak{R}_q^{\mu} : \mathcal{A} \to \mathcal{A}$, $(\mu > -1)$, operates on $\mathfrak{F}$ given in (1) as*

$$\mathfrak{R}_q^{\mu}\mathfrak{F}(z) = \varphi(q, 1+\mu; z) * \mathfrak{F}(z) = z + \sum_{\mathfrak{k}=2}^{\infty} \Psi_{\mathfrak{k},\mu,q} a_{\mathfrak{k}} z^{\mathfrak{k}}, \tag{7}$$

where

$$\varphi(q, 1+\mu; z) = z + \sum_{\mathfrak{k}=2}^{\infty} \Psi_{\mathfrak{k},\mu,q} z^{\mathfrak{k}},\ \textit{with}\ \ \Psi_{\mathfrak{k},\mu,q} = [\mu+1]_{q,\mathfrak{k}-1}/[\mathfrak{k}-1]_q!\,,$$

is the q-Ruscheweyh operator. The following identity holds for the operator $\mathfrak{R}_q^{\mu}$:

$$z\,\mathfrak{D}_q\left(\mathfrak{R}_q^{\mu}\mathfrak{F}(z)\right) = \left(1+[\mu]_q/q^{\mu}\right)\mathfrak{R}_q^{\mu+1}\mathfrak{F}(z) - \left([\mu]_q/q^{\mu}\right)\mathfrak{R}_q^{\mu}\mathfrak{F}(z).$$

Specifically, if $\mu \in \{0,1,2,\ldots,\mathrm{j},\ldots\}$, *then*

$$\mathfrak{R}_q^0\mathfrak{F}(z) = \mathfrak{F}(z),$$

$$\mathfrak{R}_q^1\mathfrak{F}(z) = z\mathfrak{D}_q\mathfrak{F}(z),$$

In general,

$$\mathfrak{R}_q^{\mathrm{j}}\mathfrak{F}(z) = \left(1/[\mathrm{j}]_q!\right)z\mathfrak{D}_q^{\mathrm{j}}\left(z^{\mathrm{j}-1}\mathfrak{F}(z)\right).$$

Also, as

$$\mathfrak{D}_q\left(\mathfrak{R}_q^{\mathrm{j}}\mathfrak{F}(z)\right) = 1 + \sum_{\mathrm{j}=2}^{\infty}\Psi_{\mathrm{j},\mu,q}[\mathrm{j}]_q a_n z^{\mathrm{j}-1},$$

and

$$\lim_{q\to 1^-}\mathfrak{R}_q^{\mu}(\mathfrak{F}(z)) = \mathfrak{R}^{\mu}(\mathfrak{F}(z)),$$

thus, the operator in Definition 4 becomes the operator, defined in [40], when $q \to 1^-$.

In the recent past, Tang et al. [41] used the operator $\mathfrak{R}_q^{\mu}$ and the subordination $z\mathfrak{D}_q\left(\mathfrak{R}_q^{\mu}\mathfrak{F}(z)\right)/\mathfrak{R}_q^{\mu}\mathfrak{F}(z) \prec (1+qz)/(1-q^2z)$, to introduce the class $S(\mu,q)$. They obtained the sufficient condition for this class as

$$\sum_{\mathfrak{k}=2}^{\infty}\left([\mathfrak{k}]_q\left(1+q^2\right) - 1 + q\right)\Psi_{\mathfrak{k},\mu,q}|a_{\mathfrak{k}}| \le q + q^2.$$

In addition to coefficient bounds and Fekete–Szegö inequality, the authors in the same article also investigated the sharp Toeplitz determinant bounds for functions in $S(\mu,q)$ as

$$|T_2(1)| \le 1 + \left(\frac{1+q}{\Psi_{2,\mu,q}}\right)^2, \quad |T_2(2)| \le \left(\frac{1+q}{\Psi_{2,\mu,q}}\right)^2 + \left(\frac{1+q+q^2}{\Psi_{3,\mu,q}}\right)^2$$

and

$$|T_2(3)| \le \frac{\left(1+q+q^2\right)^2}{\Psi_{3,\mu,q}^2} + \frac{(1+q)\left(1+q^2\right)^2}{\Psi_{4,\mu,q}^2}.$$

Taking motivation from these research works related to q-calculus, we employ the q-Ruscheweyh operator $\mathfrak{R}_q^{\mu}$ with the notion of subordination to introduce our class. For the subordination function, we defined the q-analogue of $\phi(z) = (1-\sin z)/\cos z$ in the same way as Tang et al. [41] defined for $(1+z)/(1-z)$.

Now, we give a formal definition of our new subclass $S_{sc}(\mu,q)$.

Definition 5. *The class* $S_{sc}(\mu,q)$ *consists* $\mathfrak{F} \in \mathcal{A}$, *for which the relation*

$$\frac{z\mathfrak{D}_q\left(\mathfrak{R}_q^{\mu}\mathfrak{F}(z)\right)}{\mathfrak{R}_q^{\mu}\mathfrak{F}(z)} \prec 1 + \frac{1+q}{2q}\left(\frac{1-\sin(q^2z)}{\cos(q^2z)} - 1\right), z \in \mathcal{D}$$

holds. That is,

$$S_{sc}(\mu,q) = \left\{\mathfrak{F} \in \mathcal{A} : \frac{z\mathfrak{D}_q\left(\mathfrak{R}_q^{\mu}\mathfrak{F}(z)\right)}{\mathfrak{R}_q^{\mu}f(z)} \prec 1 + \frac{1+q}{2q}\left(\frac{1-\sin(q^2z)}{\cos(q^2z)} - 1\right), z \in \mathcal{D}\right\},$$

where $0 < q \le 1$ *and* $\mu \in \mathbb{N}$.

For this new subclass $S_{sc}(\mu, q)$, we intend to investigate the sufficient condition, coefficient estimates, Fekete–Szegö inequality, and Toeplitz determinant bounds.

2. Preliminary Lemmas

In this section, we recall several statements that have appeared in [3,42–44] and that will be applied in the proofs of our results.

Lemma 1. *If p in $\mathcal{P}$ is as in (2), then $|p_{\mathfrak{j}}| \leq 2$, ($\mathfrak{j}$ = 1, 2, 3, . . .). The sharpness of the inequality follows from $p_1(z) = (1+z)/(1-z)$.*

Proof. See [42]. □

Lemma 2. *If p of the form (2) is in $\mathcal{P}$, then, for any number μ being complex,*

$$\left|p_2 - \mu p_1^2\right| \leq \max\{2, 2\ |2\mu - 1|\}.$$

For this inequality, the functions $p_1(z) = (1+z)/(1-z)$ and $p_2(z) = \left(1+z^2\right)/\left(1-z^2\right)$ provide the sharpness.

Proof. See [3]. □

Lemma 3. *Let $p \in \mathcal{P}$ be as in (2). Then for all $\mathfrak{j},\ \mathfrak{k} \in \mathbb{N}$ with $\mathfrak{k} < \mathfrak{j}$ and any complex number μ*

$$|p_{\mathfrak{j}} - \mu\, p_{\mathfrak{k}}\, p_{\mathfrak{j}-\mathfrak{k}}| \leq \max\{2,\ 2|2\mu - 1|\}.$$

This inequality is bestpossible for $p_1(z)$.

Proof. See [44]. □

Lemma 4. *Let $p \in \mathcal{P}$ be as given in (2) with $B \in [0,1]$ and $2B^2 - B \leq D \leq B$, then*

$$\left|p_3 - 2Bp_1p_2 + Dp_1^3\right| \leq 2.$$

Proof. See [43]. □

3. Main Results

Let us formulate a sufficiency criterion for a function to be in $S_s(\mu, q)$.

Theorem 1. *If $\mathfrak{F}$ of the form (1) is in $\mathcal{A}$, and is such that*

$$\sum_{\mathfrak{k}=0}^{\infty} \frac{1}{2\mathfrak{k}+1} \left(\frac{q\sum_{\mathfrak{j}=2}^{\infty}\left([\mathfrak{j}]_q - 1\right)\Psi_{\mathfrak{j},\mu,q}|a_{\mathfrak{j}}|}{\left|1 + q - \sum_{\mathfrak{j}=2}^{\infty}\left(q[\mathfrak{j}]_q + 1\right)\Psi_{\mathfrak{j},\mu,q}|a_{\mathfrak{j}}|\right|} \right)^{2\mathfrak{k}+1} \leq \frac{q^2}{2} \tag{8}$$

then $\mathfrak{F} \in S_{sc}(\mu, q)$.

Proof. Let

$$\mathfrak{G}(z) = \frac{z\mathfrak{D}_q\left(\mathfrak{R}_q^{\mu}\mathfrak{F}(z)\right)}{\mathfrak{R}_q^{\mu}\mathfrak{F}(z)},$$

and consider

$$\left|\frac{\frac{\pi}{4}-\tan^{-1}\left(\frac{2q}{1+q}(\mathfrak{G}(z)-1)+1\right)}{\frac{q^2}{2}}\right|.$$

We need to show that

$$\left|\frac{\frac{\pi}{4}-\tan^{-1}\left(\frac{2q}{1+q}(\mathfrak{G}(z)-1)+1\right)}{\frac{q^2}{2}}\right|<1$$

Observe that

$$\begin{aligned}
&\left|\frac{\frac{\pi}{4}-\tan^{-1}\left(\frac{2q}{1+q}(\mathfrak{G}(z)-1)+1\right)}{\frac{q^2}{2}}\right| \\
&=\frac{2}{q^2}\left|\tan^{-1}(1)-\tan^{-1}\left(1+\frac{2q}{1+q}(\mathfrak{G}(z)-1)\right)\right| \\
&=\frac{2}{q^2}\left|\tan^{-1}\left(\frac{q-q\mathfrak{G}(z)}{1+q\mathfrak{G}(z)}\right)\right| \\
&=\frac{2}{q^2}\left|\tan^{-1}\left(\frac{q\mathfrak{R}_q^{\mu}\mathfrak{F}(z)-qz\mathfrak{D}_q\left(\mathfrak{R}_q^{\mu}\mathfrak{F}(z)\right)}{\mathfrak{R}_q^{\mu}\mathfrak{F}(z)+qz\mathfrak{D}_q\left(\mathfrak{R}_q^{\mu}\mathfrak{F}(z)\right)}\right)\right| \\
&=\frac{2}{q^2}\left|\tan^{-1}\left(\frac{\left(qz+\sum_{\mathrm{j}=2}^{\infty}q\Psi_{\mathrm{j},\mu,q}a_{\mathrm{j}}z^{\mathrm{j}}\right)-\left(qz+\sum_{\mathrm{j}=2}^{\infty}q[\mathrm{j}]_q\Psi_{\mathrm{j},\mu,q}a_{\mathrm{j}}z^{\mathrm{j}}\right)}{z+\sum_{\mathrm{j}=2}^{\infty}\Psi_{\mathrm{j},\mu,q}a_{\mathrm{j}}z^{\mathrm{j}}+qz+\sum_{\mathrm{j}=2}^{\infty}q[\mathrm{j}]_q\Psi_{\mathrm{j},\mu,q}a_{\mathrm{j}}z^{\mathrm{j}}}\right)\right| \\
&=\frac{2}{q^2}\left|\tan^{-1}\left(\frac{\sum_{\mathrm{j}=2}^{\infty}q\left(1-[\mathrm{j}]_q\right)\Psi_{\mathrm{j},\mu,q}a_{\mathrm{j}}z^{\mathrm{j}-1}}{1+q+\sum_{\mathrm{j}=2}^{\infty}\left(q[\mathrm{j}]_q+1\right)\Psi_{\mathrm{j},\mu,q}a_{\mathrm{j}}z^{\mathrm{j}-1}}\right)\right| \\
&=\frac{2}{q^2}\left|\sum_{\mathfrak{k}=0}^{\infty}\frac{(-1)^{\mathfrak{k}}}{2\mathfrak{k}+1}\left(\frac{q\sum_{\mathrm{j}=2}^{\infty}\left(1-[\mathrm{j}]_q\right)\Psi_{\mathrm{j},\mu,q}a_{\mathrm{j}}z^{\mathrm{j}-1}}{(1+q)+\sum_{\mathrm{j}=2}^{\infty}\left(q[\mathrm{j}]_q+1\right)\Psi_{\mathrm{j},\mu,q}a_{\mathrm{j}}z^{\mathrm{j}-1}}\right)^{2\mathfrak{k}+1}\right| \\
&\leq\frac{2}{q^2}\sum_{\mathfrak{k}=0}^{\infty}\frac{1}{2\mathfrak{k}+1}\left(\frac{q\sum_{\mathrm{j}=2}^{\infty}\left([\mathrm{j}]_q-1\right)\Psi_{\mathrm{j},\mu,q}|a_{\mathrm{j}}||z|^{\mathrm{j}-1}}{\left|1+q-\sum_{\mathrm{j}=2}^{\infty}\left(q[\mathrm{j}]_q+1\right)\Psi_{\mathrm{j},\mu,q}|a_{\mathrm{j}}||z|^{\mathrm{j}-1}\right|}\right)^{2\mathfrak{k}+1} \\
&\leq\frac{2}{q^2}\sum_{\mathfrak{k}=0}^{\infty}\frac{1}{2\mathfrak{k}+1}\left(\frac{q\sum_{\mathrm{j}=2}^{\infty}\left([\mathrm{j}]_q-1\right)\Psi_{\mathrm{j},\mu,q}|a_{\mathrm{j}}|}{\left|1+q-\sum_{\mathrm{j}=2}^{\infty}\left(q[\mathrm{j}]_q+1\right)\Psi_{\mathrm{j},\mu,q}|a_{\mathrm{j}}|\right|}\right)^{2\mathfrak{k}+1},
\end{aligned}$$

thus,

$$\begin{aligned}
&\left|\frac{\frac{\pi}{4}-\tan^{-1}\left(\frac{2q}{1+q}(\mathfrak{G}(z)-1)+1\right)}{\frac{q^2}{2}}\right|<1 \\
&\Leftrightarrow\frac{2}{q^2}\sum_{\mathfrak{k}=0}^{\infty}\frac{1}{2\mathfrak{k}+1}\left(\frac{q\sum_{\mathrm{j}=2}^{\infty}\left([\mathrm{j}]_q-1\right)\Psi_{\mathrm{j},\mu,q}|a_{\mathrm{j}}|}{\left|1+q-\sum_{\mathrm{j}=2}^{\infty}\left(q[\mathrm{j}]_q+1\right)\Psi_{\mathrm{j},\mu,q}|a_{\mathrm{j}}|\right|}\right)^{2\mathfrak{k}+1}<1 \\
&\Leftrightarrow\sum_{\mathfrak{k}=0}^{\infty}\frac{1}{2\mathfrak{k}+1}\left(\frac{q\sum_{n=\mathrm{j}}^{\infty}\left([\mathrm{j}]_q-1\right)\Psi_{\mathrm{j},\mu,q}|a_{\mathrm{j}}|}{\left|1+q-\sum_{\mathrm{j}=2}^{\infty}\left(q[\mathrm{j}]_q+1\right)\Psi_{\mathrm{j},\mu,q}|a_{\mathrm{j}}|\right|}\right)^{2\mathfrak{k}+1}<\frac{q^2}{2},
\end{aligned}$$

which is the inequality (8). Hence, there is some Schwarz function $w(z)$:

$$\frac{\frac{\pi}{4}-\tan^{-1}\left(\frac{2q}{1+q}(\mathfrak{G}(z)-1)+1\right)}{\frac{q^2}{2}}=w(z),$$

solving the above equation for $\mathfrak{G}(z)$,

$$\mathfrak{G}(z)=1+\frac{1+q}{2q}\left(\frac{1-\sin\left(q^2w(z)\right)}{\cos\left(q^2w(z)\right)}-1\right),$$

thus,

$$\mathfrak{G}(z)\prec 1+\frac{1+q}{2q}\left(\frac{1-\sin\left(q^2z\right)}{\cos\left(q^2z\right)}-1\right),$$

or

$$\frac{z\mathfrak{D}_q\left(\mathfrak{N}_q^{\mu}\mathfrak{F}(z)\right)}{\mathfrak{N}_q^{\mu}\mathfrak{F}(z)}\prec 1+\frac{1+q}{2q}\left(\frac{1-\sin\left(q^2z\right)}{\cos\left(q^2z\right)}-1\right),$$

which implies that $\mathfrak{F}\in S_{sc}(\mu,q)$. □

Remark 1. *From the above Theorem 1, we have*

$$\left|\frac{\frac{\pi}{4}-\tan^{-1}\left(\frac{2q}{1+q}(\mathfrak{G}(z)-1)+1\right)}{\frac{q^2}{2}}\right|$$

$$\leq\frac{2}{q^2}\sum_{\mathfrak{k}=0}^{\infty}\frac{1}{2\mathfrak{k}+1}\left(\frac{\sum_{\mathrm{j}=2}^{\infty}\left(q[\mathrm{j}]_q-q\right)\Psi_{\mathrm{j},\mu,q}|a_{\mathrm{j}}|}{\left|1+q-\sum_{\mathrm{j}=2}^{\infty}\left(q[\mathrm{j}]_q+1\right)\Psi_{\mathrm{j},\mu,q}|a_{\mathrm{j}}|\right|}\right)^{2\mathfrak{k}+1}. \tag{9}$$

Now, if for $\mathfrak{F}\in\mathcal{A}$ of the form (1), we have

$$\sum_{\mathfrak{k}=0}^{\infty}\frac{1}{2\mathfrak{k}+1}\left(\frac{\sum_{\mathrm{j}=2}^{\infty}\left(q[\mathrm{j}]_q-q\right)\Psi_{\mathrm{j},\mu,q}|a_{\mathrm{j}}|}{\left|1+q-\sum_{\mathrm{j}=2}^{\infty}\left(q[\mathrm{j}]_q+1\right)\Psi_{\mathrm{j},\mu,q}|a_{\mathrm{j}}|\right|}\right)^{2\mathfrak{k}+1}>\frac{q^2}{2},$$

then from (9), either

$$\left|\frac{\frac{\pi}{4}-\tan^{-1}\left(\frac{2q}{1+q}(\mathfrak{G}(z)-1)+1\right)}{\frac{q^2}{2}}\right|<1,$$

(which leads us to the conclusion that $\mathfrak{F}\in S_{sc}(\mu,q)$), or

$$\left|\frac{\frac{\pi}{4}-\tan^{-1}\left(\frac{2q}{1+q}(\mathfrak{G}(z)-1)+1\right)}{\frac{q^2}{2}}\right|\geq 1,$$

which shows that $\mathfrak{F}$ cannot be a member of $S_{sc}(\mu,q)$. Thus, Theorem 1 guarantees for $\mathfrak{F}\in\mathcal{A}$ to be in $S_{sc}(\mu,q)$ if it satisfies (8). On the other hand, if $\mathfrak{F}$ does not satisfy the inequality (8), the theorem remains calm because in this case, $\mathfrak{F}$ may or may not belong to the class $S_{sc}(\mu,q)$.

Remark 2. *If for $\mathfrak{F}$, $1+q$ is greater than $\sum_{\mathrm{j}=2}^{\infty}\left(q[\mathrm{j}]_q+1\right)\Psi_{\mathrm{j},\mu,q}|a_{\mathrm{j}}|$, then from inequality (8), we get*

$$|a_{\mathrm{j}}|\leq\frac{(1+q)M}{\sum_{\mathrm{j}=2}^{\infty}\left\{(1+M)q[\mathrm{j}]_q-(q-M)\right\}\Psi_{\mathrm{j},\mu,q}},$$

where $M = \left((2\mathfrak{k}+1)q^2/2\right)^{1/(2\mathfrak{k}+1)}$ *and* $(\mathfrak{k} = 0, 1, 2, \ldots)$*, which implies that* $\mathfrak{F} \in S_{sc}(\mu, q)$*.*

4. Coefficient Estimates

G.F.T concerns geometric attributes of analytic functions. Also, as coefficient bounds of analytic functions provide a view into the image domains of these functions, it is natural to study the coefficient bounds in this field of study.

Theorem 2. *Let* $\mathfrak{F} \in S_{sc}(\mu, q)$ *and be as in (1). Then*

$$|a_2| \leq \frac{1+q}{2\Psi_{2,\mu,q}}, \tag{10}$$

$$|a_3| \leq \begin{cases} \frac{1}{2\Psi_{3,\mu,q}}; & \text{if} \quad 0 < q \leq \frac{\sqrt{5}-1}{2} \\ \frac{1+q+q^2}{4\Psi_{3,\mu,q}}; & \text{if} \quad \frac{\sqrt{5}-1}{2} < q \leq 1, \end{cases} \tag{11}$$

and,

$$|a_4| \leq \begin{cases} \frac{1+q}{2(1+q+q^2)\Psi_{4,\mu,q}}; & \text{if} \quad 0 < q \leq q_0 \simeq 0.7771997339, \\ \frac{(1+q)\left(10+q+2q^2\right)}{8(1+q+q^2)\Psi_{4,\mu,q}}; & \text{if} \quad q_0 < q \leq \frac{\sqrt{17}-1}{4}, \\ \frac{(1+q)\left(4+q+2q^2\right)}{4(1+q+q^2)\Psi_{4,\mu,q}}; & \text{if} \quad \frac{\sqrt{17}-1}{4} < q \leq q_1 \simeq 0.78412415795, \\ \frac{(1+q)\left(4q^4+3q^3+18q^2+9q+15\right)}{24(1+q+q^2)\Psi_{4,\mu,q}}; & \text{if} \quad q_1 < q \leq 1. \end{cases} \tag{12}$$

The inequalities for $|a_2|$ *and* $|a_3|$ *are sharp; however, only the first bound of* $|a_4|$ *is sharp.*

Proof. For $\mathfrak{F} \in S_{sc}(\mu, q)$, we have

$$\frac{z\,\mathfrak{D}_q\left(\mathfrak{R}_q^\mu \mathfrak{F}(z)\right)}{\mathfrak{R}_q^\mu \mathfrak{F}(z)} \prec 1 + \frac{1+q}{2q}\left(\frac{1-\sin\left(q^2 z\right)}{\cos\left(q^2 z\right)} - 1\right),$$

and hence, some $w(z) = \sum_{k=1}^{\infty} \mathfrak{w}_k z^k$, $z \in \mathcal{D}$ exists that satisfies $|w\ | < 1$, $w(0) = 0$ and

$$\frac{z\,\mathfrak{D}_q\left(\mathfrak{R}_q^\mu \mathfrak{F}(z)\right)}{\mathfrak{R}_q^\mu \mathfrak{F}(z)} = 1 + \frac{1+q}{2q}\left(\frac{1-\sin\left(q^2 w(z)\right)}{\cos\left(q^2 w(z)\right)} - 1\right). \tag{13}$$

Now, let

$$\frac{z\mathfrak{D}_q\left(\mathfrak{R}_q^\mu \mathfrak{F}(z)\right)}{\mathfrak{R}_q^\mu \mathfrak{F}(z)} = 1 + \sum_{\mathfrak{k}=1}^{\infty} \mathfrak{c}_\mathfrak{k} z^\mathfrak{k}. \tag{14}$$

Using Equations (6) and (7), Equation (14) takes the form

$$\begin{aligned} z + \sum_{\mathfrak{k}=2}^{\infty} [\mathfrak{k}]_q \Psi_{\mathfrak{k},\mu,q} a_\mathfrak{k} z^\mathfrak{k} &= z + \sum_{\mathfrak{k}=2}^{\infty} \Psi_{\mathfrak{k},\mu,q} a_\mathfrak{k} z^\mathfrak{k} + \sum_{\mathfrak{k}=1}^{\infty} \mathfrak{c}_\mathfrak{k} z^{\mathfrak{k}+1} \\ &+ \sum_{\mathfrak{k}=1}^{\infty} \mathfrak{c}_\mathfrak{k} z^\mathfrak{k} \times \sum_{\mathfrak{k}=2}^{\infty} \Psi_{\mathfrak{k},\mu,q} a_\mathfrak{k} z^\mathfrak{k}. \end{aligned}$$

Applying Cauchy product formula [45], after some simplification, we have

$$\sum_{\mathfrak{k}=2}^{\infty} \left(\left([\mathfrak{k}]_q - 1\right)\Psi_{\mathfrak{k},\mu,q} a_\mathfrak{k} - \mathfrak{c}_{\mathfrak{k}-1}\right) z^\mathfrak{k} = \sum_{\mathfrak{k}=3}^{\infty} \left(\sum_{m=3}^{\mathfrak{k}} \mathfrak{c}_{\mathfrak{k}-m+1} \Psi_{m-1,\mu,q} a_{m-1}\right) z^\mathfrak{k}.$$

Comparing coefficients of $z^2,\ z^3$,and z^4 and simplifying, we have

$$\begin{aligned}
\mathfrak{c}_1 &= \left([2]_q - 1\right)\Psi_{2,\mu,q}a_2,\\
\mathfrak{c}_2 &= \left([3]_q - 1\right)\Psi_{3,\mu,q}a_3 - t_2\Psi^2_{2,\mu,q}a_2^2,\\
\mathfrak{c}_3 &= \left([4]_q - 1\right)\Psi_{4,\mu,q}a_4 - t_3\Psi_{2,\mu,q}\Psi_{3,\mu,q}a_2a_3 + t_2\Psi^3_{2,\mu,q}a_2^3,
\end{aligned}$$

where $t_2 = [2]_q - 1$ and $t_3 = [3]_q + [2]_q - 2$.

Using Equation (4), after some simplification, the above three equations take the form

$$\mathfrak{c}_1 = q\Psi_{2,\mu,q}a_2, \tag{15}$$

$$\mathfrak{c}_2 = q(1+q)\Psi_{3,\mu,q}a_3 - q\Psi^2_{2,\mu,q}a_2^2, \tag{16}$$

$$\mathfrak{c}_3 = q\left(1+q+q^2\right)\Psi_{4,\mu,q}a_4 - q(2+q)\Psi_{2,\mu,q}\Psi_{3,\mu,q}a_2a_3 + q\Psi^3_{2,\mu,q}a_2^3. \tag{17}$$

Using Equations (15)–(17) in Equation (14), we obtain

$$\begin{aligned}
\frac{z\,\mathfrak{D}_q\left(\mathfrak{R}^{\mu}_{q}\mathfrak{F}(z)\right)}{\mathfrak{R}^{\mu}_{q}\mathfrak{F}(z)} &= 1 + q\Psi_{2,\mu,q}a_2 z + \left(\left(q+q^2\right)\Psi_{3,\mu,q}a_3 - q\Psi^2_{2,\mu,q}a_2^2\right)z^2\\
&+\left(q\left(1+q+q^2\right)\Psi_{4,\mu,q}a_4 - q(2+q)\Psi_{2,\mu,q}\Psi_{3,\mu,q}a_2a_3 + q\Psi^3_{2,\mu,q}a_2^3\right)z^3 + \dots
\end{aligned}\tag{18}$$

Also, for the Schwarz function $w(z) = \sum_{\mathfrak{k}=1}^{\infty}\mathfrak{w}_{\mathfrak{k}}z^{\mathfrak{k}}$, we have

$$\begin{aligned}
&1 + \frac{1+q}{2q}\left(\frac{1-\sin\left(q^2w(z)\right)}{\cos\left(q^2w(z)\right)} - 1\right)\\
&= 1 - \frac{q(1+q)}{2}\left\{\mathfrak{w}_1 z + \left(\mathfrak{w}_2 - \frac{q^2}{2}\mathfrak{w}_1^2\right)z^2 + \left(\mathfrak{w}_3 - q^2\mathfrak{w}_1\mathfrak{w}_2 + \frac{q^4}{3}\mathfrak{w}_1^3\right)z^3 + \dots\right\}
\end{aligned}\tag{19}$$

Furthermore, as for each Schwarz function $w(z)$, there is some function $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \dots$ in $\mathcal{P}$, for which

$$\frac{1+w}{1-w} = p,$$

simplification and comparison of coefficients of $z,\ z^2$, and z^3 yield the results

$$\begin{aligned}
\mathfrak{w}_1 &= \frac{1}{2}p_1,\\
\mathfrak{w}_2 &= \frac{1}{2}\left(p_2 - \frac{1}{2}p_1^2\right),\\
\mathfrak{w}_3 &= \frac{1}{2}\left(p_3 - p_1p_2 + \frac{1}{4}p_1^3\right).
\end{aligned}$$

Now, substituting these values of $\mathfrak{w}_1$, $\mathfrak{w}_2$, and $\mathfrak{w}_3$ into Equation (19), we have

$$\begin{aligned}
1 + \frac{1+q}{2q}\left(\frac{1-\sin\left(q^2w(z)\right)}{\cos\left(q^2w(z)\right)} - 1\right) &= 1 - \frac{q(1+q)}{4}p_1z - \frac{q(1+q)}{4}\left(p_2 - \frac{2+q^2}{4}p_1^2\right)z^2\\
&-\frac{q(1+q)}{4}\left(p_3 - \left(\frac{2+q^2}{2}\right)p_1p_2 + \frac{1}{12}\left(q^4+3q^2+3\right)p_1^3\right)z^3 + \dots
\end{aligned}\tag{20}$$

Utilizing (18) and (20) in (13) and equating coefficients of

z, z^2, and z^3 ,

$$a_2 = -\frac{1+q}{4\Psi_{2,\mu,q}}p_1, \tag{21}$$

$$a_3 = -\frac{1}{4\Psi_{3,\mu,q}}\left(p_2 - \frac{q^2+q+3}{4}p_1^2\right), \tag{22}$$

and

$$a_4 = -\frac{1+q}{4(1+q+q^2)\Psi_{4,\mu,q}}\left(p_3 - \frac{2q^2+q+6}{4}p_1p_2 + \frac{4q^4+3q^3+18q^2+9q+27}{48}p_1^3\right). \tag{23}$$

Applying Lemma 1 to (21), we obtain

$$|a_2| \leq \frac{1+q}{2\Psi_{2,\mu,q}}.$$

The equality is achieved for $\mathfrak{F}$, which satisfies the Equation (13) for $w(z) = z$, given by

$$\mathfrak{F}(z) = z - \frac{1+q}{2\Psi_{2,\mu,q}}z^2 + \frac{1+q+q^2}{4\Psi_{3,\mu,q}}z^3 - \frac{(1+q)(3+3q+6q^2+3q^3+4q^4)}{24(1+q+q^2)\Psi_{4,\mu,q}}z^4 + \dots . \tag{24}$$

Now, taking the modulus of Equation (22), we have

$$|a_3| = \frac{1}{4\Psi_{3,\mu,q}}\left|p_2 - \mu p_1^2\right|,$$

where $\mu = \frac{q^2+q+3}{4}$. Thus, by Lemma 2, we have

$$|a_3| \leq \frac{1}{4\Psi_{3,\mu,q}}2\max\{1, \ |2\mu - 1|\}, \tag{25}$$

where

$$|2\mu - 1| = \frac{1+q+q^2}{2}.$$

Case I: For $0 < q \leq \frac{\sqrt{5}-1}{2}$: $|2\mu - 1| \leq 1$; therefore, the inequality (25) takes the form

$$|a_3| \leq \frac{1}{2\Psi_{3,\mu,q}}.$$

Sharpness is provided by the function $\mathfrak{F}$, which satisfies the Equation (13) for $w(z) = z^2$, given as

$$\mathfrak{F}(z) = z - \frac{1}{2\Psi_{3,\mu,q}}z^3 + \dots \quad . \tag{26}$$

Case II: $\frac{\sqrt{5}-1}{2} < q \leq 1$: $|2\mu - 1| > 1$; therefore, the inequality (25) yields the following:

$$|a_3| \leq \frac{|2\mu - 1|}{2\Psi_{3,\mu,q}} = \frac{1+q+q^2}{4\Psi_{3,\mu,q}}.$$

Sharpness follows for $\mathfrak{F}$ given in (24).

Finally, we discuss the upper bound of $|a_4|$.

Case I: For $0 < q \leq q_0$, where $q_0 \simeq 0.7771997339$, from Equation (23), we have

$$|a_4| = \frac{1+q}{4(1+q+q^2)\Psi_{4,\mu,q}}\left|p_3 - 2Bp_1p_2 + Dp_1^3\right|, \tag{27}$$

with

$$B = \frac{2q^2 + q + 6}{8} \text{ and } D = \frac{4q^4 + 3q^3 + 18q^2 + 9q + 27}{48}.$$

Since both the requirements of Lemma 4 are fulfilled, from Equation (27) we obtain

$$|a_4| \leq \frac{1+q}{2(1+q+q^2)\Psi_{4,\mu,q}},$$

and a function $\mathfrak{F}$ that satisfies the Equation (13) for $w(z) = z^3$, i.e.,

$$\mathfrak{F}(z) = z - \frac{1+q}{2(1+q+q^2)\Psi_{4,\mu,q}} z^4 + \dots \ .$$

proves its sharpness.

Furthermore, we rearrange Equation (23) as

$$\begin{aligned} |a_4| &= \frac{1+q}{4(1+q+q^2)\Psi_{4,\mu,q}} \left| \begin{array}{c} \left(p_3 - \frac{2q^2+q+6}{8} p_1 p_2\right) \\ -\frac{2q^2+q+6}{8} p_1 \left(p_2 - \frac{4q^4+3q^3+18q^2+9q+27}{6(2q^2+q+6)} p_1^2\right) \end{array} \right| \\ &\leq \frac{1+q}{4(1+q+q^2)\Psi_{4,\mu,q}} \left[\begin{array}{c} \left|p_3 - \frac{2q^2+q+6}{8} p_1 p_2\right| \\ +\frac{2q^2+q+6}{8} |p_1| \left|p_2 - \frac{4q^4+3q^3+18q^2+9q+27}{6(2q^2+q+6)} p_1^2\right| \end{array} \right] \\ &= \frac{1+q}{4(1+q+q^2)\Psi_{4,\mu,q}} \left[|p_3 - \mu_1 p_1 p_2| + \frac{2q^2+q+6}{8} |p_1| \left|p_2 - \mu_2 p_1^2\right| \right] \end{aligned} \tag{28}$$

with

$$\mu_1 = \frac{2q^2+q+6}{8} \text{ and } \mu_2 = \frac{4q^4 + 3q^3 + 18q^2 + 9q + 27}{6(2q^2+q+6)}.$$

Applying Lemma 1 to Lemma 3 in (28), we have

$$|a_4| \leq \frac{1+q}{4(1+q+q^2)\Psi_{4,\mu,q}} \left[\max\{2,\ 2|2\mu_1 - 1|\} + \frac{2q^2+q+6}{2} \max\{1,\ |2\mu_2 - 1|\} \right], \tag{29}$$

where $|2\mu_1 - 1| = \frac{2q^2+q+2}{4}$ and $|2\mu_2 - 1| = \frac{4q^4+3q^3+12q^2+6q+9}{3(2q^2+q+6)}$.

Case II: For $q_0 < q \leq \frac{\sqrt{17}-1}{4}$:
$|2\mu_1 - 1| \leq 1$ and $|2\mu_2 - 1| \leq 1$; therefore, from (29), we have

$$|a_4| \leq \frac{(1+q)\left(10 + q + 2q^2\right)}{8(1+q+q^2)\Psi_{4,\mu,q}}.$$

Case III: For $\frac{\sqrt{17}-1}{4} < q \leq q_1$, where $q_1 \simeq 0.78412415795$: $|2\mu_1 - 1| \geq 1$ and $|2\mu_2 - 1| \leq 1$; therefore, from (29), we have

$$|a_4| \leq \frac{(1+q)\left(2q^2 + q + 4\right)}{4(1+q+q^2)\Psi_{4,\mu,q}}.$$

Case IV: For $q_1 < q \leq 1 : |2\mu_1 - 1| > 1$ and $|2\mu_2 - 1| > 1$; therefore, from (29), we have

$$\begin{aligned} |a_4| &\leq \frac{1+q}{4(1+q+q^2)\Psi_{4,\mu,q}} \left[\frac{2q^2+q+2}{2} + \frac{2q^2+q+6}{2} \left(\frac{4q^4 + 3q^3 + 12q^2 + 6q + 9}{3(2q^2+q+6)} \right) \right] \\ &= \frac{(1+q)\left(4q^4 + 3q^3 + 18q^2 + 9q + 15\right)}{24(1+q+q^2)\Psi_{4,\mu,q}}. \end{aligned}$$

□

5. Fekete–Szegö Inequality

The *mth* Hankel determinant denoted as $H_m(\mathrm{j})$ with integers $m \geq 1,\ \mathrm{j} \geq 1$, introduced by Noonan and Thomas [46], for functions $\mathfrak{F}$ of form (1) is

$$H_m(\mathrm{j}) = \det \begin{bmatrix} a_{\mathrm{j}} & a_{\mathrm{j}+1} & \cdots & a_{\mathrm{j}+m-1} \\ a_{\mathrm{j}+1} & a_{\mathrm{j}+2} & \cdots & a_{\mathrm{j}+m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{\mathrm{j}+m-1} & a_{\mathrm{j}+m} & \cdots & a_{\mathrm{j}+2m-2} \end{bmatrix},$$

where $a_1 = 1$. Furthermore, for $m = 2,\ \mathrm{j} = 1$. $H_2(1) = a_3 - a_2^2$ is the particular form of the renowned Fekete–Szegö functional: $a_3 - \mu\, a_2^2$, for $\mu = 1$. The upper bound for $\left|a_3 - \mu a_2^2\right|$, where μ is any real or complex scalar, was first obtained in [47] for the class $\mathcal{S}$ of univalent functions.

Theorem 3. *Let $\mathfrak{F}$ of the form (1) be in $S_{sc}(\mu, q)$. Then, for any $\nu \in \mathbb{C}$,*

$$\left|a_3 - \nu a_2^2\right| \leq \frac{1}{2\Psi_{3,\mu,q}} \max\left\{1,\ \left|\frac{1+q+q^2}{2} - \frac{v(1+q)^2\Psi_{3,\mu,q}}{2\Psi_{2,\mu,q}^2}\right|\right\} \tag{30}$$

This inequality is sharp.

Proof. From the two Equations (21) and (22),

$$\begin{aligned} \left|a_3 - v a_2^2\right| &= \frac{1}{16\Psi_{2,\mu,q}^2\Psi_{3,\mu,q}}\left|4\Psi_{2,\mu,q}^2 p_2 - \left\{\left(3+q+q^2\right)\Psi_{2,\mu,q}^2 - \nu(1+q)^2\Psi_{3,\mu,q}\right\}p_1^2\right| \\ &= \frac{1}{4\Psi_{3,\mu,q}}\left|p_2 - \frac{1}{4}\left(3+q+q^2 - \frac{v(1+q)^2\Psi_{3,\mu,q}}{\Psi_{2,\mu,q}^2}\right)p_1^2\right|. \end{aligned}$$

Applying Lemma 2 to the last equation and simplifying, we obtain the required result (30). Furthermore, the inequality (30) when

$$\left|\frac{1+q+q^2}{2} - \frac{v(1+q)^2\Psi_{3,\mu,q}}{2\Psi_{2,\mu,q}^2}\right| \leq 1,$$

is sharp for $\mathfrak{F}$ in (26). On the other hand, if

$$\frac{1}{2}\left|\left(1+q+q^2\right) - \frac{v(1+q)^2\Psi_{3,\mu,q}}{\Psi_{2,\mu,q}^2}\right| > 1,$$

then sharpness follows for $\mathfrak{F}$ in (24), which satisfies the Equation (13) for $w(z) = z$. □

Corollary 1. *For $\mathfrak{F} \in S_{sc}(\mu, q)$ and any $\nu \in \mathbb{C}$,*

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \frac{1}{4\Psi_{3,\mu,q}}\left(\left(1+q+q^2\right) - \frac{\nu(1+q)^2\Psi_{3,\mu,q}}{\Psi_{2,\mu,q}^2}\right); & \text{if } \ v < -v_1, \\ \frac{1}{2\Psi_{3,\mu,q}}; & \text{if } \ -v_1 \leq v \leq v_2, \\ \frac{1}{4\Psi_{3,\mu,q}}\left(\frac{v(1+q)^2\Psi_{3,\mu,q}}{\Psi_{2,\mu,q}^2} - \left(1+q+q^2\right)\right); & \text{if } \ \nu > v_2. \end{cases}$$

where $v_1 = \frac{\left(1-q-q^2\right)\Psi^2_{2,\mu,q}}{(1+q)^2\Psi_{3,\mu,q}}$, $v_2 = \frac{\left(3+q+q^2\right)\Psi^2_{2,\mu,q}}{(1+q)^2\Psi_{3,\mu,q}}$. *This result is sharp.*

Proof. Since v is real,

$$\frac{1}{2}\left|\left(1+q+q^2\right) - \frac{v(1+q)\Psi_{3,\mu,q}}{\Psi^2_{2,\mu,q}}\right| \leq 1,$$

if, and only if,

$$-1 \leq \frac{1}{2}\left(\left(1+q+q^2\right) - \frac{v(1+q)\Psi_{3,\mu,q}}{\Psi^2_{2,\mu,q}}\right) \leq 1,$$

or

$$-\frac{1-q-q^2}{(1+q)^2}\frac{\Psi^2_{2,\mu,q}}{\Psi_{3,\mu,q}} \leq v \leq \frac{3+q+q^2}{(1+q)^2}\frac{\Psi^2_{2,\mu,q}}{\Psi_{3,\mu,q}},$$

On the other hand,

$$\left|\frac{1+q+q^2}{2} - \frac{v(1+q)\Psi_{3,\mu,q}}{2\Psi^2_{2,\mu,q}}\right| > 1,$$

if, and only if,

$$\frac{1}{2}\left(1+q+q^2 - \frac{\nu(1+q)\Psi_{3,\mu,q}}{\Psi^2_{2,\mu,q}}\right) > 1$$

or

$$\frac{1}{2}\left(1+q+q^2 - \frac{v(1+q)\Psi_{3,\mu,q}}{\Psi^2_{2,\mu,q}}\right) < -1,$$

i.e.,

$$\nu < -\frac{1-q-q^2}{(1+q)^2}\frac{\Psi^2_{2,\mu,q}}{\Psi_{3,\mu,q}} \quad \text{or } \nu > \frac{3+q+q^2}{(1+q)^2}\frac{\Psi^2_{2,\mu,q}}{\Psi_{3,\mu,q}}.$$

Hence the result. □

Remark 3. *For* $\mu = 1$, *the second and third coefficients of a function* $\mathfrak{F}$ *satisfying*

$$\frac{z\mathfrak{D}_q\left(\mathfrak{R}^{\mu}_q\mathfrak{F}(z)\right)}{\mathfrak{R}^{\mu}_q\mathfrak{F}(z)} = \frac{z\mathfrak{D}_q\left(z\mathfrak{D}_q\mathfrak{F}(z)\right)}{z\mathfrak{D}_q\mathfrak{F}(z)} = 1 + \frac{1+q}{2q}\left(\frac{1-\sin(q^2z)}{\cos(q^2z)} - 1\right),$$

for any q *with* $0 < q < 1$, *i.e.,* $\mathfrak{F} \in S_{sc}(1,q)$ *are*

$$a_2 = -\frac{1}{2}, \textit{ and } a_3 = \frac{1}{4}.$$

Also, as for $\mu = 1$, $\Psi_{\jmath,\mu,q} = [\jmath]_q$. *Therefore, the result of the Corollary 1 takes the form*

$$\left|a_3 - va_2^2\right| \leq \begin{cases} \frac{1-\nu}{4}; & \textit{if} \quad v < 1 - \frac{2}{[3]_q}, \\ \frac{1}{2[3]_q}; & \textit{if} \quad 1 - \frac{2}{[3]_q} \leq v \leq 1 + \frac{2}{[3]_q}, \\ \frac{\nu-1}{4}; & \textit{if} \quad \nu > 1 + \frac{2}{[3]_q}. \end{cases} \tag{31}$$

For $a_2 = -\frac{1}{2}$ *and* $a_3 = \frac{1}{4}$: $\left|a_3 - va_2^2\right| = \frac{1}{4}|1-\nu|$. *Now, if* $\nu = 1 - \frac{2}{[4]_q}$, *then*

$$\left|a_3 - va_2^2\right| = \frac{1}{4}\left|1 - \left(1 - \frac{2}{[4]_q}\right)\right| = \frac{1}{4}(1-\nu),$$

if $\nu = 1$, *then*

$$\left|a_3 - \nu a_2^2\right| = \frac{1}{4}|1-1| = 0 < \frac{1}{2[3]_q},$$

if $\nu = 1 + \frac{2}{[4]_q}$, *then*

$$\left|a_3 - \nu a_2^2\right| = \frac{1}{4}\left|1 - \left(1 + \frac{2}{[4]_q}\right)\right| = \frac{1}{4}(\nu - 1).$$

Hence, for three different values of ν *lying in three different intervals, as specified in the above* (31)*, the three different values of* $\left|a_3 - \nu a_2^2\right|$ *are confirmed.*

6. Toeplitz Determinant

The Symmetric Toeplitz determinant for functions $\mathfrak{F}$ of the form (1) introduced in [48], denoted by $T_m(\mathrm{j})$ with integers $m \geq 1,\ \mathrm{j} \geq 1$, is

$$T_m(\mathrm{j}) = \det\begin{bmatrix} a_{\mathrm{j}} & a_{\mathrm{j}+1} & \cdots & a_{\mathrm{j}+m-1} \\ a_{\mathrm{j}+1} & a_{\mathrm{j}} & \cdots & a_{\mathrm{j}+m-2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{\mathrm{j}+m-1} & a_{\mathrm{j}+m-2} & \cdots & a_{\mathrm{j}} \end{bmatrix}, \tag{32}$$

where $a_1 = 1$.

Theorem 4. *Let* $\mathfrak{F} \in S_{sc}(\mu, q)$ *and be as in* (1)*. Then*

$$|T_2(1)| \leq 1 + \left(\frac{1+q}{2\Psi_{2,\mu,q}}\right)^2,$$

$$|T_2(2)| \leq \begin{cases} \left(\frac{1+q}{2\Psi_{2,\mu,q}}\right)^2 + \left(\frac{1}{2\Psi_{3,\mu,q}}\right)^2; & \mathit{if}\ \ 0 < q \leq \frac{\sqrt{5}-1}{2}, \\ \left(\frac{1+q}{2\Psi_{2,\mu,q}}\right)^2 + \left(\frac{1+q+q^2}{4\Psi_{3,\mu,q}}\right)^2; & \mathit{if}\ \ \frac{\sqrt{5}-1}{2} < q \leq 1, \end{cases}$$

and

$$|T_2(3)| \leq \begin{cases} \left(\frac{1}{2\Psi_{3,\mu,q}}\right)^2 + \left(\frac{1+q}{2(1+q+q^2)\Psi_{4,\mu,q}}\right)^2; & \mathit{if}\ \ 0 < q \leq \frac{5-1}{2}, \\ \left(\frac{\big(1+q+q^2}{4\Psi_{3,\mu,q}}\right)^2 + \left(\frac{1+q}{2(1+q+q^2)\Psi_{4,\mu,q}}\right)^2; & \mathit{if}\ \ \frac{\sqrt{5}-1}{2} < q \leq q_0, \\ \left(\frac{\big(1+q+q^2}{4\Psi_{3,\mu,q}}\right)^2 + \left(\frac{(1+q)\big(10+q+2q^2\big)}{8(1+q+q^2)\Psi_{4,\mu,q}}\right)^2; & \mathit{if}\ \ q_0 < q \leq \frac{\sqrt{17}-1}{4}, \\ \left(\frac{\big(1+q+q^2}{4\Psi_{3,\mu,q}}\right)^2 + \left(\frac{(1+q)\big(4+q+2q^2\big)}{4(1+q+q^2)\Psi_{4,\mu,q}}\right)^2; & \frac{\sqrt{17}-1}{4} < q \leq q_1, \\ \left(\frac{\big(1+q+q^2}{4\Psi_{3,\mu,q}}\right)^2 + \left(\frac{(1+q)\big(4q^4+3q^3+18q^2+9q+15\big)}{24(1+q+q^2)\Psi_{4,\mu,q}}\right)^2; & \mathit{if}\ \ q_1 < q \leq 1. \end{cases}$$

where $q_0 \simeq 0.7771997339$ *and* $q_1 \simeq 0.78412415795$*. The bound for* $|T_2(1)|$ *and the bound in the second inequality for* $|T_2(2)|$ *are sharp for* $\mathfrak{F}$ *as in* (1)*, where* $\mathfrak{F}$ *is a solution to* (13) *for* $w(z) = iz$*. That is,*

$$\mathfrak{F}(z) = z - \frac{1+q}{2\Psi_{2,\mu,q}}iz^2 - \frac{1+q+q^2}{4\Psi_{3,\mu,q}}z^3 + \frac{(1+q)\left(3+3q+6q^2+3q^3+4q^4\right)}{4(1+q+q^2)\Psi_{4,\mu,q}}iz^4 + \ldots. \tag{33}$$

Proof. For $\mathfrak{F} \in S_{sc}(\mu, q)$ and being of the form (1), we have

$$T_2(\mathrm{j}) = \begin{vmatrix} a_{\mathrm{j}} & a_{\mathrm{j}+1} \\ a_{\mathrm{j}+1} & a_{\mathrm{j}} \end{vmatrix} = a_{\mathrm{j}}^2 - a_{\mathrm{j}+1}^2.$$

By taking modulus and applying triangle inequality, we obtain

$$|T_2(\mathrm{j})| \leq |a_{\mathrm{j}}|^2 + |a_{\mathrm{j}+1}|^2 \tag{34}$$

For $\mathrm{j} = 1$ in (34) and using (10), we have

$$|T_2(1)| \leq 1 + \frac{(1+q)^2}{4\Psi^2_{2,\mu,q}}.$$

Now, for $\mathrm{j} = 2$ in (34) and using (11), we proceed as if $0 < q \leq \frac{\sqrt{5}-1}{2}$:

$$|T_2(2)| \leq |a_2|^2 + |a_3|^2 \leq \frac{(1+q)^2}{4\Psi^2_{2,\mu,q}} + \frac{1}{4\Psi^2_{3,\mu,q}};$$

and if $\frac{\sqrt{5}-1}{2} < q \leq 1$,

$$|T_2(2)| \leq \frac{(1+q)^2}{4\Psi^2_{2,\mu,q}} + \frac{\left(1+q+q^2\right)^2}{4\Psi^2_{3,\mu,q}},$$

Similarly, for n= 3 in (34) and using (11) and (12), we obtain the required inequalities.
□

Remark 4. *For $\mu = 0$ and $q \to 1^-$, we have*

$$|T_2(1)| \leq 2,$$
$$|T_2(2)| \leq \frac{25}{16},$$

and

$$|T_2(3)| \leq \frac{1565}{648}.$$

The inequalities $|T_2(1)| \leq 2$ and $|T_2(2)| \leq \frac{25}{16}$ are sharp for $\mathfrak{F}(z)$, given in (33) with $\mu = 0$ and $q \to 1^-$.

Remark 5. *For $\mu = 1$ and $q \to 1^-$, we have*

$$|T_2(1)| \leq \frac{5}{4},$$
$$|T_2(2)| \leq \frac{5}{16},$$

and

$$|T_2(3)| \leq \frac{3697}{20736}.$$

The inequalities $|T_2(1)| \leq \frac{5}{4}$ and $|T_2(2)| \leq \frac{5}{16}$ are sharp for the same function in (33) with $\mu = 1$ and $q \to 1^-$.

Theorem 5. *Let $\mathfrak{F} \in S_{sc}(\mu, q)$ and be of the form (1). Then*

$$|T_3(1)| \leq \begin{cases} A; & \text{if } \ 0 < q \leq \frac{\sqrt{5}-1}{2} \text{ with } \mu_0 \leq \mu < \infty, \\ B; & \text{if } \ \frac{\sqrt{5}-1}{2} < q \leq q_2 \text{ with } \mu_0 \leq \mu < \infty, \\ C; & \text{if } \ q_2 \leq q < 1 \text{ with } \ -1 < \mu \leq \mu_0. \end{cases}$$

where $A = 1 + \frac{(1+q)^2}{2\Psi^2_{2,\mu,q}} + \frac{1}{4\Psi^2_{3,\mu,q}}$, $B = 1 + \frac{(1+q)^2}{2\Psi^2_{2,\mu,q}} + \frac{1+q+q^2}{8\Psi^2_{3,\mu,q}}$ *and* $C = 1 + \frac{(1+q)^2}{2\Psi^2_{2,\mu,q}} + \frac{(1+q+q^2)}{16\Psi^2_{3,\mu,q}}\left(\frac{2(1+q)^2\Psi_{3,\mu,q}}{\Psi^2_{2,\mu,q}} - 1 - q - q^2\right)$.

The inequality third is sharp.

Proof. For any $\mathfrak{F} \in S_{sc}(\mu, q)$ and of the form (1), we have from (32) that

$$|T_3(1)| = \left|1 - 2a_2^2 + 2a_2^2 a_3 - a_3^2\right| \leq 1 + 2|a_2|^2 + |a_3|\left|a_3 - 2a_2^2\right|.$$

From (10), we have $|a_2| \leq \frac{1+q}{2\Psi_{2,\mu,q}}$ for all q with $0 < q < 1$. Therefore,

$$|T_3(1)| \leq 1 + \frac{(1+q)^2}{2\Psi^2_{2,\mu,q}} + |a_3|\left|a_3 - 2a_2^2\right| \tag{35}$$

since, when $0 < q \leq q_2$ *with* $\mu_0 \leq \mu < \infty$,

$$\frac{1-q-q^2}{(1+q)^2}\frac{\Psi^2_{2,\mu,q}}{\Psi_{3,\mu,q}} < 2 \leq \frac{3+q+q^2}{(1+q)^2}\frac{\Psi^2_{2,\mu,q}}{\Psi_{3,\mu,q}}, \tag{36}$$

and when $q_2 \leq q < 1$ *with* $-1 \leq \mu \leq \mu_0$,

$$2 \geq \frac{(3+q+q^2)\Psi^2_{2,\mu,q}}{(1+q)^2\Psi_{3,\mu,q}}, \tag{37}$$

where $q_2 \simeq 0.7239319265$ and $\mu_0 \simeq 1.42592518828794$. Thus, in view of (36) and (37), from Corollary 1 for $\nu = 2$, we have

$$\left|a_3 - 2a_2^2\right| \leq \begin{cases} D; & \text{if } \ 0 < q \leq q_2 \text{ with } \mu_0 \leq \mu < \infty, \\ E; & \text{if } \ q_2 \leq q < 1 \text{ with } \ -1 \leq \mu \leq \mu_0. \end{cases} \tag{38}$$

where $D = \frac{1}{2\Psi_{3,\mu,q}}$ and $E = \frac{1}{4\Psi_{3,\mu,q}}\left(\frac{2(1+q)^2\Psi_{3,\mu,q}}{\Psi^2_{2,\mu,q}} - (1+q+q^2)\right)$. We use (11) and (38) into (35), which combined may be discussed as follows:

Case I: $0 < q \leq \frac{\sqrt{5}-1}{2}$ with $\mu_0 \leq \mu < \infty$: from (11), we have $|a_3| \leq \frac{1}{2\Psi_{3,\mu,q}}$ and for $0 < q \leq q_2$ with $\mu_0 \leq \mu < \infty$, from (38), we have $\left|a_3 - 2a_2^2\right| \leq \frac{1}{2\Psi_{3,\mu,q}}$. Both these results are valid for $0 < q \leq \frac{\sqrt{5}-1}{2}$ with $\mu_0 \leq \mu < \infty$. Upon substituting these values in (35), we obtain the first required bound for $|T_3(1)|$.

Case II: $\frac{\sqrt{5}-1}{2} < q \leq q_2$ with $\mu_0 \leq \mu < \infty$

For $\frac{\sqrt{5}-1}{2} < q \leq 1$, from (11), we have $|a_3| \leq \frac{1+q+q^2}{4\Psi_{3,\mu,q}}$ and for $0 < q \leq q_2$ with $\mu_0 \leq \mu < \infty$, from (38), we have $\left|a_3 - 2a_2^2\right| \leq \frac{1}{2\Psi_{3,\mu,q}}$. Both these results are valid for $\frac{\sqrt{5}-1}{2} < q \leq q_2$ with $\mu_0 \leq \mu < \infty$. On substituting these values in (35) we obtain the second required bound for $|T_3(1)|$.

Case III: $q_2 \leq q < 1$ with $-1 < \mu \leq \mu_0$:

For $\frac{\sqrt{5}-1}{2} < q \leq 1$, from (11), we have $|a_3| \leq \frac{1+q+q^2}{4\Psi_{3,\mu,q}}$ and for $q_2 \leq q < 1$ with $-1 < \mu \leq \mu_0$, from (38), we have

$$\left|a_3 - 2a_2^2\right| \leq \frac{1}{4\Psi_{3,\mu,q}}\left(\frac{2(1+q)^2\Psi_{3,\mu,q}}{\Psi_{2,\mu,q}^2} - \left(1+q+q^2\right)\right).$$

Both these results are valid for $q_2 \leq q < 1$ with $-1 < \mu \leq \mu_0$. Upon substituting these values in (35), we obtain the third required bound for $|T_3(1)|$, and $\mathfrak{F}$ in (33) shows the sharpness. □

Remark 6. *For $\mu = 0$ and $q \to 1^-$, we have*

$$|T_3(1)| \leq \frac{63}{16},$$

and for $\mu = 1$ and $q \to 1^-$, we have

$$|T_3(1)| \leq \frac{25}{16}.$$

Theorem 6. *If $\mathfrak{F}$ as in (1) belongs to $S_{sc}(\mu, q)$, then*

$$|T_3(2)| \leq \begin{cases} 4\left(\frac{1+q}{2\Psi_{2,\mu,q}} + \frac{1+q}{2(1+q+q^2)\Psi_{4,\mu,q}}\right)\Delta; & \text{if } \ 0 < q \leq q_0, \\ 4\left(\frac{1+q}{2\Psi_{2,\mu,q}} + \frac{(1+q)\left(10+q+2q^2\right)}{8(1+q+q^2)\Psi_{4,\mu,q}}\right)\Delta; & \text{if } \ q_0 < q \leq \frac{\sqrt{17}-1}{4}, \\ 4\left(\frac{1+q}{2\Psi_{2,\mu,q}} + \frac{(1+q)\left(4+q+2q^2\right)}{4(1+q+q^2)\Psi_{4,\mu,q}}\right)\Delta; & \text{if } \ \frac{\sqrt{17}-1}{4} < q \leq q_1, \\ 4\left(\frac{1+q}{2\Psi_{2,\mu,q}} + \frac{(1+q)\left(4q^4+3q^3+18q^2+9q+15\right)}{24(1+q+q^2)\Psi_{4,\mu,q}}\right)\Delta; & \text{if } \ q_1 < q \leq 1. \end{cases}$$

where $q_0 \simeq 0.7771997339$, $q_1 \simeq 0.78412415795$ and

$$\Delta = \lambda_1 + 4|\lambda_2| + |\lambda_3| + \lambda_4|2\lambda_5 - 1|$$

with

$$\begin{aligned} \lambda_1 &= \left(\frac{1+q}{4\Psi_{2,\mu,q}}\right)^2, \\ \lambda_2 &= \frac{1}{128}\left(\frac{(1+q)^2(4q^4+3q^3+18q^2+9q+27)}{6(1+q+q^2)\Psi_{2,\mu,q}\Psi_{4,\mu,q}} - \frac{(3+q+q^2)^2}{\Psi_{3,\mu,q}^2}\right), \\ \lambda_3 &= -\frac{1}{8\Psi_{3,\mu,q}^2}, \\ \lambda_4 &= \frac{(1+q)^2}{16(1+q+q^2)\Psi_{2,\mu,q}\Psi_{4,\mu,q}}, \\ \lambda_5 &= \frac{1}{4}\left(6+q+2q^2\right) - \frac{(1+q+q^2)(3+q+q^2)\Psi_{2,\mu,q}\Psi_{4,\mu,q}}{(1+q)^2\Psi_{3,\mu,q}^2}. \end{aligned}$$

Proof. If $\mathfrak{F} \in S_{sc}(\mu, q)$ and are of the form (1), then from (32) we have

$$T_3(2) = (a_2 - a_4)\left(a_2^2 - 2a_3^2 + a_2a_4\right).$$

Taking modulus and applying triangle inequality, we have

$$|T_3(2)| \leq (|a_2| + |a_4|)\left|a_2^2 - 2a_3^2 + a_2 a_4\right|.$$

Using values of $|a_2|$ and $|a_4|$ from (10) and (12), respectively, we have

$$|T_3(2)| \leq \begin{cases} F\left|a_2^2 - 2a_3^2 + a_2 a_4\right|; & \text{if } \; 0 < q \leq q_0, \\ G\left|a_2^2 - 2a_3^2 + a_2 a_4\right|; & \text{if } \; q_0 < q \leq \frac{\sqrt{17}-1}{4}, \\ H\left|a_2^2 - 2a_3^2 + a_2 a_4\right|; & \text{if } \; \frac{\sqrt{17}-1}{4} < q \leq q_1, \\ L\left|a_2^2 - 2a_3^2 + a_2 a_4\right|; & \text{if } \; q_1 < q \leq 1. \end{cases} \tag{39}$$

where

$$\begin{aligned} F &= \frac{1+q}{2\Psi_{2,\mu,q}} + \frac{1+q}{2(1+q+q^2)\Psi_{4,\mu,q}}, \\ G &= \frac{1+q}{2\Psi_{2,\mu,q}} + \frac{(1+q)\left(10+q+2q^2\right)}{8(1+q+q^2)\Psi_{4,\mu,q}}, \\ H &= \frac{1+q}{2\Psi_{2,\mu,q}} + \frac{(1+q)\left(4+q+2q^2\right)}{4(1+q+q^2)\Psi_{4,\mu,q}}, \\ L &= \frac{1+q}{2\Psi_{2,\mu,q}} + \frac{(1+q)\left(4q^4+3q^3+18q^2+9q+15\right)}{24(1+q+q^2)\Psi_{4,\mu,q}}, \end{aligned}$$

$q_0 \simeq 0.7771997339$ and $q_1 \simeq 0.78412415795$.

Now, using (21)–(23), we have

$$\begin{aligned} & a_2^2 - 2a_3^2 + a_2 a_4 \\ & = \left(-\frac{1+q}{4\Psi_{2,\mu,q}} p_1\right)^2 - 2\left(-\frac{1}{4\Psi_{3,\mu,q}}\left(p_2 - \frac{q^2+q+3}{4} p_1^2\right)\right)^2 + \left(-\frac{1+q}{4\Psi_{2,\mu,q}} p_1\right) \end{aligned}$$

$$\times\left(-\frac{1+q}{4(1+q+q^2)\Psi_{4,\mu,q}}\left(p_3 - \frac{2q^2+q+6}{4} p_1 p_2 + \frac{4q^4+3q^3+18q^2+9q+27}{48} p_1^3\right)\right).$$

After some simplification, we obtained

$$a_2^2 - 2a_3^2 + a_2 a_4$$

$$\begin{aligned} = & \frac{(1+q)^2}{16\Psi_{2,\mu,q}^2} p_1^2 + \frac{1}{128}\left(\frac{(1+q)^2\left(4q^4+3q^3+18q^2+9q+27\right)}{6(1+q+q^2)\Psi_{2,\mu,q}\Psi_{4,\mu,q}} - \frac{\left(3+q+q^2\right)^2}{\Psi_{3,\mu,q}^2}\right) p_1^4 \\ & -\frac{1}{8\Psi_{3,\mu,q}^2} p_2^2 + \frac{(1+q)^2}{16(1+q+q^2)\Psi_{2,\mu,q}\Psi_{4,\mu,q}} p_1 \\ & \times\left(p_3 - \left(\frac{1}{4}\left(6+q+2q^2\right) - \frac{(1+q+q^2)\left(3+q+q^2\right)\Psi_{2,\mu,q}\Psi_{4,\mu,q}}{(1+q)^2\Psi_{3,\mu,q}^2}\right) p_1 p_2\right) \\ = & \lambda_1 p_1^2 + \lambda_2 p_1^4 - \lambda_3 p_2^2 + \lambda_4 p_1\left(p_3 - \lambda_5 p_1 p_2\right), \end{aligned}$$

where λ_i's are that given in the statement. Taking modulus, we have

$$\left|a_2^2 - 2a_3^2 + a_2 a_4 \quad\right| \leq \lambda_1 |p_1|^2 + |\lambda_2||p_1|^4 + |\lambda_3||p_2|^2 + \lambda_4 |p_1||p_3 - \lambda_5 p_1 p_2|,$$

by using Lemma 1 and Lemma 3, the above inequality takes the form

$$\left|a_2^2 - 2a_3^2 + a_2 a_4 \quad\right| \leq 4(\lambda_1 + 4|\lambda_2| + |\lambda_3| + \lambda_4 \max\{1, \; |2\lambda_5 - 1|\}) \tag{40}$$

since

$$2\lambda_5 - 1 = \frac{4+q+2q^2}{2} - \frac{(1+q+q^2)(3+q+q^2)\Psi_{2,\mu,q}\Psi_{4,\mu,q}}{(1+q)^2\Psi^2_{3,\mu,q}}$$

is such that

$$1.5 < |2\lambda_5 - 1| < 4, \text{ for } 0 < q < 1 \text{ and } \mu > -1.$$

Therefore, the inequality (40) takes the form

$$\left|a_2^2 - 2a_3^2 + a_2a_4 \quad \right| \le 4(\lambda_1 + 4|\lambda_2| + |\lambda_3| + \lambda_4|2\lambda_5 - 1|) = 4\Delta \text{ (say)}.$$

On subtitution of this bound of $\left|a_2^2 - 2a_3^2 + a_2a_4\right|$ into (39), the result is achieved. □

Remark 7. *For* $\mu = 0,\ q \to 1^-$;

$$|T_3(2)| \le \frac{26095}{2592},$$

and for $\mu = 1$ *and* $q \to 1^-$, *we have*

$$|T_3(2)| \le \frac{6655}{13824}.$$

7. Conclusions

In this study, we have used the concept of q-derivative along with the subordination principle for analytic functions to obtain a new subclass linked to the q-Ruschewey operator. The subordination function we used is, in fact, the q-analogue of the Ma and Minda type function $\phi(z) = (1 - \sin z)/\cos z$. We have explored sufficiency criteria for functions of our newly defined subclass. Since G.F.T is the field in which geometric properties of analytic functions are studied, and as coefficient estimates and coefficient functionals give in-depth insight into its geometric attributes, it seems natural to study the mentioned bounds of an analytic function. We obtained sharp initial coefficient bounds of a_2 and a_3, and also in one case of a_4, and sharp Fekete–Szegö functional for our particular subclass. Furthermore, we have established Toeplitz determinants up to $T_3(2)$, which are sharp in certain cases. We have also discussed various implications that emerge from our results shown above.

Looking at the subordination function $\phi_q(z) = 1 + \frac{1+q}{2q}\left(\frac{1-\sin(q^2z)}{\cos(q^2z)} - 1\right)$, one can easily see the potential for further work in this direction by defining the q-analogues of other Ma and Minda type ϕ functions.

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