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A class of reset linear systems: the reset-delayed linear systems and their structural and stability properties

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Abstract: This work introduces a new class of reset linear systems, called *reset-delayed linear systems*, which consists of multivariable dynamical systems featuring a continuous-time state evolution interrupted by discontinuities of some, or even all the state variables at isolated time instants. In particular, these discontinuities abide by an algebraic equation imposing that the state variables involved take the same values they respectively had a certain amount of time before the time instant when the discontinuity is triggered. In this way, the evolutions of the state variables involved turn out to be delayed by the amount of time considered in the reset operation. In the presence of suitably chosen forcing actions, reset-delayed linear systems can effectively model repetitive behaviors which imply a discontinuity at the junction between one cycle and the subsequent one. Herein, some structural properties of this class of multivariable linear dynamical systems are studied and the geometric notions of invariance and controlled invariance are formalized and applied to the solution of the disturbance decoupling problem.

Keywords: Reset-delayed linear systems, time-delay systems, impulsive linear systems, hybrid linear systems with state jumps, structural properties, stability, disturbance decoupling.

1. INTRODUCTION

Reset linear systems made their first appearance in the control literature of the late fifties as devices aimed at overcoming some critical limitations related with the integral action in linear controllers – typically, the phenomenon now well-known as integrator wind-up. In particular, the reset linear system introduced by Clegg (1958), also known as *Clegg integrator*, consisted of a linear integrator whose state was subject to a reset to zero triggered by a zero crossing of the tracking error. A more sophisticated version of the earliest reset linear system was proposed in the mid seventies by Horowitz and Rosenbaum (1975) to decrease the feedback loop transmission bandwidth in feedback systems with large parameters uncertainties. The reset linear system introduced by Horowitz and Rosenbaum was characterized by the presence of the so-called first-order reset element, which would have become the standard in the field for the subsequent decades.

In particular, first-order reset elements attracted special attention between the late nineties and the two-thousands. The benefits of employing control schemes including a first-order reset element, instead of conventional ones, were assessed through specific experimental setups (Bakkeheim et al., 2008). The stability of general reset systems was formally analyzed (Beker et al., 2004; Guo et al., 2009), also in the presence of uncertainties (Guo et al., 2012) — a more comprehensive study of these aspects can be found in the later book (Guo et al., 2016). New classes of first-order reset elements, which differed from the previous ones for the condition triggering the state reset, were proposed in (Nešić et al., 2008) and subsequently used, e.g., in (Nešić et al., 2011).

The latest years have seen the consolidation of the description of reset control systems as *hybrid linear systems with state jumps* (or, *impulsive linear systems*), where the state goes instantaneously to zero either at the occurrence of a specific event or at designated time instants which belong to the set of the reset time instants. In this framework, a method for modeling time-driven multivariable reset control systems in the frequency domain was developed in (Buitenhuis et al., 2023). A stability analysis of event-driven multivariable reset control systems was carried out in (Satoh, 2023) by means of Lyapunov-like functions. A stability analysis of event-driven reset control systems was developed by means of frequency-domain methods, with a focus on single-input single-output control architectures, in (Dastjerdi et al., 2023).

In this work, a new class of reset linear systems is introduced, henceforth called *reset-delayed linear systems*. The systems of this class are multivariable dynamical systems which exhibit a continuous-time state evolution interrupted by discontinuities of some, or even all the state variables, which, at isolated time instants, assume the same values they respectively had a certain amount of time before the time instant at which the discontinuity takes place. This means that, differently from standard

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reset linear systems, the class of reset linear systems introduced herein is such that the state variables subject to discontinuity are not forced to assume the zero value but they are brought back to the respective values they had a certain amount of time before the time instant at which the discontinuity occurs. Thus, according to this rule, the evolutions of the state variables subject to discontinuities are, in fact, delayed by the amount of time considered in the reset operation. Like in standard reset linear systems, state discontinuities may be either event-driven or time-driven. However, only time-driven discontinuities are considered herein and the time at which the discontinuities occur are called jump times.

On these bases, reset-delayed linear systems can be profitably employed, for instance, to exploit the reactivity of dynamically unstable linear systems, yet avoiding state divergence in the long term. Moreover, if the jump times are equally spaced and the delay implied by the reset operation is equal to the jump period, with a suitable choice of the control input, reset-delayed linear systems can effectively produce cyclic behaviors which include a state discontinuity at the junction between one cycle and the subsequent one. Otherwise, on the same assumptions on the jump times and the delay implied by the reset operation, in the presence of forcing actions and/or disturbances, reset-delayed linear systems can produce behaviors which consist of a sequence of rounds, where, at the beginning of each round, some, not necessarily zero, initial conditions are restored. Therefore, this class of multivariable dynamical systems can also be effectively used, for instance, to simulate the behavior of teams of autonomous systems engaged in patrolling or reconnaissance, respectively.

Since the mathematical description of reset-delayed systems is akin to that of impulsive linear systems, the study of this class of dynamical systems is approached herein with methodologies similar to those developed for impulsive linear systems, e.g., in (Medina and Lawrence, 2009; Carnevale et al., 2016; Zattoni et al., 2017, 2018): namely, methodologies grounded on the structural methods evolved from the geometric approach originally conceived for linear time-invariant systems (Basile and Marro, 1992; Wonham, 1985). In this work, the notions of invariance and controlled invariance are formulated first, and then applied to the solution of the disturbance decoupling problem. In particular, the solvability of the disturbance decoupling problem is characterized in structural terms by using the classic geometric approach (Basile and Marro, 1992; Wonham, 1985) and its extensions to several other classes of dynamical systems, such as nonlinear systems (Isidori, 1995), 2-D systems (Conte and Perdon, 1988), systems over rings and time delay-systems (Conte and Perdon, 2000), switching systems (Otsuka, 2010; Zattoni et al., 2016), and the same impulsive linear systems (Perdon et al., 2017).

The paper is organized as follows. Section 2 formally introduces the class of the reset-delayed linear systems. Section 3 introduces a subclass of reset-delayed linear systems, the fully reset-delayed linear systems, and shows a sufficient condition for their global asymptotic stability. Section 4 sets forth the geometric definitions of invariant subspace and controlled invariant subspace for reset-delayed linear systems. Section 5 deals with the disturbance decoupling problem stated for reset-delayed linear systems and shows a necessary and sufficient solvability condition (also with the additional requirement of stability) in the case of fully reset-delayed linear systems. Section 6 presents some concluding remarks.

Notation: The symbols $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{N}$ stand for the sets of real numbers, nonnegative real numbers, and natural numbers, respectively. Matrices and linear maps are denoted by slanted capital letters, like A. Sets, vector spaces, and subspaces are denoted by calligraphic capital letters, like $\mathcal{X}$. The image and the kernel of A are denoted by $\operatorname{Im} A$ and $\operatorname{Ker} A$, respectively. The notation T^{-1} stands for the inverse of the nonsingular square matrix T. The notation $A^{-1}\mathcal{V}$, where A is a not necessarily invertible linear map and $\mathcal{V}$ is a subspace, stands for the inverse image of $\mathcal{V}$ through A. The symbol I_n stands for the identity matrix of dimension n.

2. RESET-DELAYED LINEAR SYSTEMS

Reset-delayed linear systems are introduced as a special class of linear impulsive systems which exhibit the following behavior. While the continuous-time dynamics does not show any special feature characterizing the class considered, the jump behavior is such that, at any time at which the jump occurs, say t, some, or even all the state variables instantaneously take the (not necessarily zero) values they respectively had at the time $(t-\delta(t))$, where, for any $t \in \mathbb{R}^+$, $\delta(t) > 0$ is a real value. Thus, each time the jump occurs, the evolution of the state (or, of some of its components) is *delayed* in time by $\delta(t)$.

In general, the jump behavior may be time driven or event driven. However, this work considers time-driven jump behaviors only. Hence, the set of the jump time instants is introduced as a finite or countably infinite ordered set

$$\mathcal{S} = \{t_0, t_1, \ldots\} \subseteq \mathbb{R}^+$$

such that

$$\inf_{k \in \mathbb{N}} \{t_k - t_{k-1}\} \geq \alpha$$

for some $\alpha > 0$. It is worth just mentioning that the condition above ensures that the set $\mathcal{S}$ of the jump time instants has no accumulation points, which prevents the arising of Zeno phenomena.

Moreover, it is assumed that $\delta(t)$ is a constant function: i.e.,

$$\delta(t) = \delta > 0 \quad \text{for all } t \in \mathbb{R}^+.$$

With the above premises, a reset-delayed linear system Σ is described by equations of the form

$$\Sigma \equiv \begin{cases} \dot{x}(t) \;=\; A\,x(t) + B\,u(t), & \text{for } t \notin \mathcal{S}, \\ x(t) \;=\; J_\Delta\, x(t^-), & \text{for } t \in \mathcal{S}, \\ y(t) \;=\; C\,x(t), \end{cases} \tag{1}$$

where $t \in \mathbb{R}^+$ is the independent time variable; $x \in \mathcal{X} = \mathbb{R}^n$, $u \in \mathcal{U} = \mathbb{R}^m$, and $y \in \mathcal{Y} = \mathbb{R}^p$ are the state, the input, and the output of Σ, respectively; A, B, and C are real matrices of suitable dimensions; J_Δ is an $n \times n$ matrix of the form

$$J_\Delta = \begin{bmatrix} I_{n_1} & 0 \\ 0 & I_{n_2}\Delta \end{bmatrix}, \tag{2}$$

where Δ is the delay operator defined, for any time function $f : \mathbb{R}^+ \to \mathbb{R}$ and any $t \in \mathbb{R}^+$, by

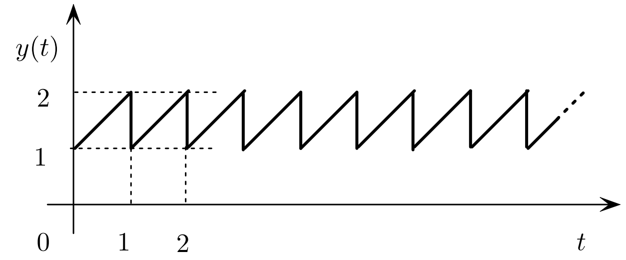

Fig. 1. Example 1: The sawtooth signal generated by the reset-delayed linear system Σ

$$\Delta f(t) = \begin{cases} f(t-\delta), & \text{for } t \geq \delta, \\ f(0), & \text{for } t < \delta, \end{cases}$$

with $0 < \delta \in \mathbb{R}$. Moreover,

$$x(t^-) = \lim_{t \to t^-} x(t)$$

and $\mathcal{S} = \{t_0, t_1, \ldots\} \subseteq \mathbb{R}^+$ is the set of the jump time instants, as previously introduced.

In (1), the first block of (differential) equations defines the *flow dynamics* of Σ, while the second block of (algebraic) equations defines the *jump-delay behavior* of Σ.

Example 1. Let Σ be a system of the form (1), given by

$$\Sigma \equiv \begin{cases} \dot{x}(t) = u(t), & \text{for } t \notin \mathcal{S}, \\ x(t) = \Delta\, x(t^-), & \text{for } t \in \mathcal{S}, \\ y(t) = x(t), \end{cases}$$

where x, u, $y \in \mathbb{R}$ and $\mathcal{S} = \{\delta, 2\delta, \ldots\} \subseteq \mathbb{R}^+$. Taking $x(0) = 1$ and $\delta = 1$, with the constant input $u(t) = 1$, the plot of the output $y(t)$, which coincides with the state $x(t)$, of Σ has the form shown in Fig. 1. Clearly, with a suitable initialization, the system Σ can be used to generate a variety of sawtooth signals in response to constant inputs or, more generally, in response to inputs which periodically repeat on each time interval $[k\,\delta,\ (k+1)\,\delta)$ with $k = 0, 1, \ldots$.

3. FULLY RESET-DELAYED LINEAR SYSTEMS AND THEIR GLOBAL ASYMPTOTIC STABILITY

A special class of reset-delayed linear systems, henceforth referred to as *fully reset-delayed linear systems*, is characterized by a jump-delayed behavior such that, at each jump time instant, *all* the state variables take the values they respectively had a given amount of time before the considered jump time instant.

With reference to the mathematical description of reset-delayed linear systems given by (1), where J_Δ is defined by (2), fully reset-delayed linear systems admit the same mathematical description, where (2) particularizes as

$$J_\Delta = I_n\, \Delta. \tag{3}$$

This means that, at each jump time $t_k \in \mathcal{S}$, all the components of the state take the values they respectively had at the time $t_k - \delta$. Hence, at each jump time, the state evolution of Σ is *time-delayed* by δ.

A sufficient condition for the global asymptotic stability of fully reset-delayed linear systems is given by the following proposition.

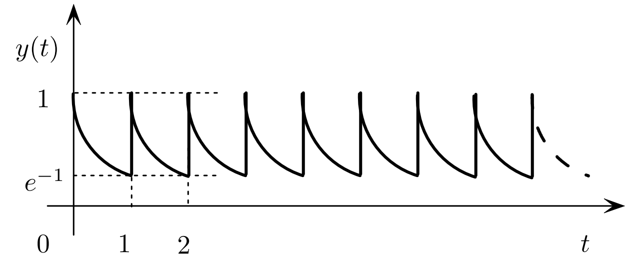

Fig. 2. Example 2: The sawtooth signal generated by the fully reset-delayed linear system Σ

Proposition 1. Let Σ be a fully reset-delayed linear system – i.e., a reset-delayed linear system described by (1), where J_Δ is defined by (3). Let the positive real constant δ satisfy the condition

$$0 < \delta < \min\{t_0, \inf_{k \in \mathbb{N}} \{t_k - t_{k-1}\}\}. \tag{4}$$

Then, Σ is globally asymptotically stable if its flow dynamics is asymptotically stable.

Proof. The proof hinges on the fact that the state evolution of Σ on the time interval $[t_k, t_{k+1})$ coincides with the state evolution of its flow dynamics on $[t_k - k\delta, t_{k+1} - k\delta)$. Let $\bar{x}(t)$ denote an auxiliary variable whose dynamics coincides with the free flow dynamics of Σ: i.e.,

$$\dot{\bar{x}}(t) = A\bar{x}(t),$$

where A is the same as in (1). Since the flow dynamics of Σ is asymptotically stable by assumption, for any $\varepsilon > 0$, there exists $\tau > 0$ such that $\|\bar{x}(t)\| < \varepsilon$ for all $t > \tau$. By (4), there exists a (sufficiently large) k such that $t_k > \tau + k\delta$. Note that, again by (4), this implies $t_{k+1} > \tau + (k+1)\delta$, and so forth. Let t be any time instant in $[t_k, t_{k+1})$. Note that the special choice of t_k implies $t - k\delta > \tau$. Consequently, $\|x(t)\| = \|\bar{x}(t - k\delta)\| < \varepsilon$ and this is true for any $t \in [t_k, t_{k+1})$. By iterating the same reasoning on the subsequent time interval, $[t_{k+1}, t_{k+2})$, one gets that $\|x(t)\| < \varepsilon$ for any $t \in [t_{k+1}, t_{k+2})$ and so forth for all the subsequent time intervals, which proves the statement. □

Remark 1. Concerning Proposition 1, it is worth noting that the asymptotic stability of the flow dynamics of the fully reset-delayed linear system Σ does not imply the global asymptotic stability of Σ if condition (4) is not satisfied. This fact is illustrated by the following example.

Example 2. Let Σ be a system of the form (1), where J_Δ is defined by (3), given by

$$\Sigma \equiv \begin{cases} \dot{x}(t) = -x(t), & \text{for } t \notin \mathcal{S}, \\ x(t) = \Delta\, x(t^-), & \text{for } t \in \mathcal{S}, \\ y(t) = x(t), \end{cases} \tag{5}$$

where x, u, $y \in \mathbb{R}$ and $\mathcal{S} = \{\delta, 2\delta, \ldots\} \subseteq \mathbb{R}^+$. The flow dynamics is asymptotically stable, since its only pole is at -1. However, since $x(k\delta) = x(0)$ for all $k \in \mathbb{N}$, Σ is not globally asymptotically stable. For instance, taking $x(0) = 1$ and $\delta = 1$, the plot of the output $y(t)$, which coincides with the state $x(t)$, of Σ has the sawtooth form shown in Fig. 2.

In more complex cases, in which the value of some, but not of all, the state variables is affected by the jump-delay, stability is more difficult to characterize.

4. INVARIANCE AND CONTROLLED INVARIANCE FOR RESET-DELAYED LINEAR SYSTEMS

In order to analyze the geometric structure of reset-delayed linear systems, it is convenient to introduce the following notions.

Definition 1. Let Σ be a reset-delayed linear system of the form (1), where J_Δ is defined by (2). A subspace $\mathcal{V} \subseteq \mathcal{X}$ is said to be:

- an *invariant subspace* for Σ if, for any $x_0 \in \mathcal{V}$, the free state evolution $x(t)$ with $x(0) = x_0$ belongs to $\mathcal{V}$: i.e., $x(t) \in \mathcal{V}$ for all $t \in \mathbb{R}^+$;
- a *controlled invariant subspace* for Σ if, for any $x_0 \in \mathcal{V}$, there exists an input $u(t)$, with $u(t) \in \mathcal{U}$ for all $t \in \mathbb{R}^+$, such that the corresponding forced state evolution $x(t)$ with $x(0) = x_0$ belongs to $\mathcal{V}$: i.e., $x(t) \in \mathcal{V}$ for all $t \in \mathbb{R}^+$.

In order to give an algebraic characterization of the above properties, the following technical lemma is needed.

Lemma 1. Let $\mathcal{V} \subseteq \mathbb{R}^n$ be a subspace and let $n = n_1 + n_2$. Then,

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \mathcal{V} \subseteq \mathcal{V} \quad \text{if and only if} \quad \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix} \mathcal{V} \subseteq \mathcal{V}.$$

Proof. Any $x \in \mathcal{V}$ is of the form

$$x = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix} x.$$

Therefore, if any one of the two addends on the right-hand side of the previous equality belongs to $\mathcal{V}$, also the other belongs to $\mathcal{V}$. □

Hence, the following results, concerning the notions introduced in Definition 1, are given.

Proposition 2. Let Σ be a reset-delayed linear system of the form (1), with J_Δ defined by (2). A subspace $\mathcal{V} \subseteq \mathcal{X}$ is an invariant subspace for Σ if the following conditions hold:

$$A\mathcal{V} \subseteq \mathcal{V}, \tag{6}$$

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \mathcal{V} \subseteq \mathcal{V} \;\text{ or, equivalently, }\; \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix} \mathcal{V} \subseteq \mathcal{V}. \tag{7}$$

Proof. By (6), $\mathcal{V}$ is an invariant subspace for the flow dynamics of Σ. In particular, this means that, for any $x_0 \in \mathcal{V}$, the free state evolution $x(t)$ with $x(0) = x_0$ belongs to $\mathcal{V}$ for all t, with $0 \le t < t_0$, where $t_0 \in \mathcal{S}$ is the first jump time instant. Moreover, since $\mathcal{V}$ is closed, also $x(t_0^-)$ belongs to $\mathcal{V}$. Then, since

$$x(t_0) = J_\Delta\, x(t_0^-) = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} x(t_0^-) + \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix} \Delta\, x(t_0^-), \tag{8}$$

where, as $x(t_0^-)$ is defined as a limit,

$$\Delta x(t_0^-) = \begin{cases} x(t_0 - \delta), & \text{if } t_0 \ge \delta, \\ x(0), & \text{if } t_0 < \delta, \end{cases} \tag{9}$$

from (7) it follows that $x(t_0)$ belongs to $\mathcal{V}$. By iterating the same argument on the subsequent time intervals $[t_k, t_{k+1})$, with $k \in \mathbb{N}$, one gets that the free state evolution $x(t)$, with $x(0) = x_0 \in \mathcal{V}$, belongs to $\mathcal{V}$ for all $t \in \mathbb{R}^+$. □

Proposition 3. Let Σ be a reset-delayed linear system of the form (1), with J_Δ defined by (2). A subspace $\mathcal{V} \subseteq \mathcal{X}$ is a controlled invariant subspace for Σ if the condition

$$A\mathcal{V} \subseteq \mathcal{V} + \operatorname{Im} B \tag{10}$$

holds together with (7).

Proof. By (10), $\mathcal{V}$ is a controlled invariant subspace for the flow dynamics of Σ. In particular, this means that, for any $x_0 \in \mathcal{V}$, there exists an input $u(t)$, with $u(t) \in \mathcal{U}$ for all $t \le t_0$, such that the corresponding forced state evolution $x(t)$ with $x(0) = x_0$ belongs to $\mathcal{V}$ for all t, with $0 \le t < t_0$. Moreover, since $\mathcal{V}$ is closed in $\mathcal{X}$, also $x(t_0^-)$ belongs to $\mathcal{V}$. Then, since $x(t_0)$ is given by (8), where $\Delta\, x(t_0^-)$ is defined by (9), from (7) it follows that, for the same input $u(t)$, $x(t_0)$ belongs to $\mathcal{V}$. By iterating the same argument on the subsequent time intervals $[t_k, t_{k+1})$, with $k \in \mathbb{N}$, one gets that there exists an input $u(t)$, with $u(t) \in \mathcal{U}$ for all $t \in \mathbb{R}^+$, possibly discontinuous at the jump time instants, such that the corresponding forced state evolution $x(t)$ with $x(0) = x_0 \in \mathcal{V}$ belongs to $\mathcal{V}$ for all $t \in \mathbb{R}^+$. □

Proposition 4. Let Σ be a reset-delayed linear system of the form (1), with J_Δ defined by (2). Let $\mathcal{V}_1 \subseteq \mathcal{X}$ and $\mathcal{V}_2 \subseteq \mathcal{X}$ be two controlled invariant subspaces for Σ. Then, the sum $\mathcal{V}_1 + \mathcal{V}_2$ is a controlled invariant subspace for Σ.

Proof. Since both $\mathcal{V}_1$ and $\mathcal{V}_2$ satisfy (7) and (10), it is easily seen that the same holds for $\mathcal{V}_1 + \mathcal{V}_2$. □

On the basis of Proposition 4, it can be shown, by standard algebraic arguments, that the set of all controlled invariant subspaces contained in a subspace $\mathcal{W} \subseteq \mathcal{X}$ is an upper semilattice with respect to the sum of subspaces, and, thus, it has a maximal element that will be denoted by $\mathcal{V}^*(\mathcal{W})$.

Algorithm 1. Let Σ be a reset-delayed linear system of the form (1), with J_Δ defined by (2). Let $\mathcal{W}$ be a subspace of $\mathcal{X}$. Then, the sequence of subspaces $\{V_k\}_{k \in \mathbb{N}}$, recursively defined by

$$\mathcal{V}_0 = \mathcal{W}, \tag{11}$$

$$\mathcal{V}_{k+1} = \mathcal{V}_k \cap A^{-1}(\mathcal{V}_k + \operatorname{Im} B) \cap \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}^{-1} \mathcal{V}_k, \quad \text{for } k = 1, 2, \dots, \tag{12}$$

and, likewise, the sequence of subspaces $\{V_k'\}_{k \in \mathbb{N}}$, recursively defined by

$$\mathcal{V}_0' = \mathcal{W}, \tag{13}$$

$$\mathcal{V}_{k+1}' = \mathcal{V}_k' \cap A^{-1}(\mathcal{V}_k' + \operatorname{Im} B) \cap \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix}^{-1} \mathcal{V}_k', \quad \text{for } k = 1, 2, \dots, \tag{14}$$

converge to $\mathcal{V}^*(\mathcal{W})$ in n steps at most.

Whenever $\mathcal{W} = \operatorname{Ker} C$, the subspace $\mathcal{V}^*(\mathcal{W}) = \mathcal{V}^*(\operatorname{Ker} C)$ will be simply denoted by $\mathcal{V}^*$.

Remark 2. If Σ is a fully reset-delayed linear system – i.e., a reset-delayed linear system of the form (1), with J_Δ defined by (3) – by Proposition 3, a subspace $\mathcal{V} \subseteq \mathcal{X}$ is a controlled invariant subspace for Σ if and only it is a controlled invariant subspace for its flow dynamics. In that case, the algorithm defined by (11)–(12), or, equivalently, by (13)–(14), reduces to the classical controlled invariant subspace algorithm for linear time-invariant systems (Basile and Marro, 1992; Wonham, 1985).

If Σ is a fully reset-delayed linear system – i.e., a reset-delayed linear system of the form (1), with J_Δ defined by (3) – given a controlled invariant subspace $\mathcal{V} \subseteq \mathcal{X}$ for Σ, it is possible to find a feedback of the form

$$u(t) = F\,x(t) + v(t),$$

where F is a matrix of suitable dimensions such that

$$(A + B\,F)\,\mathcal{V} \subseteq \mathcal{V},$$

or, in other words, such that $\mathcal{V}$ is an invariant subspace for the compensated system

$$\Sigma^F \equiv \left\{ \begin{array}{lcll} \dot{x}(t) & = & (A + B\,F)\,x(t) + B\,v(t), & \text{for } t \notin \mathcal{S}, \\ x(t) & = & J_\Delta\,x(t^-), & \text{for } t \in \mathcal{S}, \\ y(t) & = & C\,x(t), & \end{array} \right.$$

where J_Δ is defined by (3). Any feedback matrix F with such property is said to be a *friend* of $\mathcal{V}$ (Basile and Marro, 1992; Wonham, 1985).

Let

$$T = [\,T_1 \;\, T_2\,]$$

be a nonsingular $n \times n$ matrix such that the columns of T_1 form a basis of $\mathcal{V}$. Then, with the change of basis defined by $x = Tz$, the system Σ^F takes the form

$$\Sigma^F \equiv \left\{ \begin{array}{lcll} \dot{z}(t) & = & A'\,z(t) + B'\,v(t), & \text{for } t \notin \mathcal{S}, \\ z(t) & = & J_\Delta\,z(t^-), & \text{for } t \in \mathcal{S}, \\ y(t) & = & C'\,z(t) & \end{array} \right.$$

where

$$A' = T^{-1}(A + B\,F)\,T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

with $\dim(A_{11}) = \dim(\mathcal{V})$, $B' = T^{-1}B$, and $C' = C\,T$. It is worth noting that the change of basis has no effect on the form of the jump matrix J_Δ, since J_Δ is defined by (3).

5. THE DISTURBANCE DECOUPLING PROBLEM FOR RESET-DELAYED LINEAR SYSTEMS

The notions and results of the previous sections can be profitably used to deal with a classical control problem stated herein for reset-delayed linear systems: the disturbance decoupling problem (DDP) and the disturbance decoupling problem with stability (DDPS). Given a reset-delayed linear system affected by an unknown disturbance input, the DDP consists in finding, if possible, a state feedback that decouples the output of the compensated system from the disturbance. The additional requirement of global asymptotic stability of the resulting compensated system gives rise to the DDPS. Both problems are formally stated below.

Problem 1. (Disturbance Decoupling). Let Σ be a reset-delayed linear system of the form

$$\Sigma \equiv \left\{ \begin{array}{lcll} \dot{x}(t) & = & A\,x(t) + B\,u(t) + D\,d(t), & \text{for } t \notin \mathcal{S}, \\ x(t) & = & J_\Delta\,x(t^-), & \text{for } t \in \mathcal{S}, \\ y(t) & = & C\,x(t), & \end{array} \right. \quad (15)$$

where, in addition to the mathematical description given by (1), with J_Δ defined by (2), the variable $d \in \mathcal{D} = \mathbb{R}^q$ denotes an unknown disturbance input and the matrix D, of suitable dimensions, denotes the corresponding input-distribution matrix. The *Disturbance Decoupling Problem* consists in finding a state feedback

$$u(t) = F\,x(t) \quad (16)$$

such that the output $y(t)$ of the compensated system

$$\Sigma^F \equiv \left\{ \begin{array}{lcll} \dot{x}(t) & = & (A + BF)\,x(t) + D\,d(t), & \text{for } t \notin \mathcal{S}, \\ x(t) & = & J_\Delta\,x(t^-), & \text{for } t \in \mathcal{S}, \\ y(t) & = & C\,x(t), & \end{array} \right. \quad (17)$$

is decoupled from $d(t)$.

Problem 2. (Disturbance Decoupling with Stability). Let Σ be a reset-delayed linear system of the form (15). The *Disturbance Decoupling Problem with Stability* consists in finding a state feedback (16) that solves the DDP and that, in addition, makes the compensated system Σ^F, defined by (17), globally asymptotically stable.

In relation to the above problems, the following results can be stated.

Proposition 5. Let Σ be a fully reset-delayed linear system of the form (15) – i.e., a reset-delayed linear system of the form (15), with J_Δ defined by (3). The DDP stated for Σ is solvable if and only if the following condition holds:

$$\operatorname{Im} D \subseteq \mathcal{V}^*. \quad (18)$$

Proof. Let F be a friend of $\mathcal{V}^*$. With the change of basis defined by $x = T\,z$, where $T = [\,T_1 \;\, T_2\,]$ is a nonsingular $n \times n$ matrix such that the columns of T_1 form a basis of $\mathcal{V}^*$, the compensated system Σ^F takes the form

$$\Sigma^F \equiv$$

$$\left\{ \begin{array}{lcll} \begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} & = & \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} D_1 \\ 0 \end{bmatrix} d(t), & \text{for } t \notin \mathcal{S}, \\ \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} & = & I_n \Delta \begin{bmatrix} z_1(t^-) \\ z_2(t^-) \end{bmatrix}, & \text{for } t \in \mathcal{S}, \\ y(t) & = & [\,0 \;\, C_2\,] \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}, & \end{array} \right.$$

with $\dim z_1 = \dim \mathcal{V}^*$, and the conclusion follows. □

Proposition 6. Let Σ be a fully reset-delayed linear system of the form (15). The DDPS stated for Σ is solvable if and only if condition (18) holds and there exists a friend F of $\mathcal{V}^*$ such that $(A + B\,F)$ is a Hurwitz matrix.

Proof. It follows from Proposition 5 and Proposition 1. □

The algorithmic construction of $\mathcal{V}^*$ makes it possible to implement a practical procedure to test the solvability of the DDP. Moreover, in the case of fully reset-delayed linear systems, the conditions for the existence of a friend F of $\mathcal{V}^*$ such that $(A + B\,F)$ is Hurwitz coincide with those found in the classical linear time-invariant case (Basile and Marro, 1992; Wonham, 1985).

6. CONCLUSIONS

A new class of dynamical systems whose state evolution is (partially) delayed by a reset action has been introduced and mathematically described. Global asymptotic stability has been characterized for a subclass of systems of that kind and some structural notions have been defined and characterized. As a first application, the solvability of the disturbance decoupling problem, also with the additional requirement of global asymptotic stability of the compensated system, has been characterized in structural terms. Further applications to nonintereacting control problems and to model matching problems are feasible and will be the object of future investigations.

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