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ON A CHEEGER–KOHLER-JOBIN INEQUALITY

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Abstract. We discuss the minimization of a Kohler-Jobin type scale-invariant functional among open, convex set, namely

$$\min\left\{T_2(\Omega)^{\frac{1}{N+2}}h_1(\Omega) : \Omega\subset\mathbb{R}^N,\ \text{convex}\right\}$$

where T_2 denotes the torsional rigidity and h_1 the Cheeger constant. We prove the existence of a minimizer and we conjecture that the ball is the unique minimizer. We provide a sufficient condition for the validity of the conjecture. We also show lack of existence among several class of sets. As a side result we discuss the equivalence of the several definitions of Cheeger constants present in the literature and show a quite general class of sets for which those are equivalent.

Keywords: Cheeger constant, Kohler-Jobin inequality, quantitative estimates, Poincaré-Sobolev constants

MSC 2010: 35P10, 39B62, 49Q10, 49R05

1. Introduction

Let Ω be an open, bounded set in $\mathbb{R}^N$. For $1\le r<N$ and $1\le q<Nr/(N-r)$, or for $r\ge N$ and $1\le q<+\infty$, we define

$$\lambda_{r,q}(\Omega) := \inf\left\{\frac{\int_\Omega|\nabla u|^r\mathrm{d}x}{\left(\int_\Omega|u|^q\mathrm{d}x\right)^{\frac{r}{q}}} : u\in W_0^{1,r}(\Omega)\setminus\{0\}\right\} = \inf\left\{\frac{\int_\Omega|\nabla u|^r\mathrm{d}x}{\left(\int_\Omega|u|^q\mathrm{d}x\right)^{\frac{r}{q}}} : u\in C_c^\infty(\Omega)\setminus\{0\}\right\}, \tag{1.1}$$ def-lpq

which can be interpreted as the principal frequency for the nonlinear eigenvalue problem

$$-\Delta_r u = \lambda\|u\|_{L^q(\Omega)}^{r-q}|u|^{q-2}u \qquad \text{in } \Omega, \qquad\qquad u=0 \qquad \text{on } \partial\Omega\,,$$

where $\Delta_r u = \mathrm{div}(|\nabla u|^{r-2}\nabla u)$ denotes the r-Laplacian of u. Alternatively, $\lambda_{r,q}(\Omega)$ can be defined as the optimal constant C for the Poincaré inequality

$$C\left(\int_\Omega|u|^q\mathrm{d}x\right)^{\frac{r}{q}} \le \int_\Omega|\nabla u|^r\mathrm{d}x\,,$$

in the Sobolev space $W_0^{1,r}(\Omega)$, that is the closur of $C_c^\infty(\Omega)$ under the seminorm

$$u\mapsto\left(\int|\nabla u|^r\right)^{1/r}.$$

Notice that on $C_c^\infty(\Omega)$ the previous is actually a norm. Some of the functionals defined above have been intensively studied in the literature: it is the case of the *p-torsional rigidity* (or just torsion)

$$\lambda_{p,1}^{-1} = T_p(\Omega) = -\frac{p}{(p-1)}\min\left\{\frac{1}{p}\int_\Omega|\nabla u|^p\mathrm{d}x - \int_\Omega u\,\mathrm{d}x \ :\ u\in W_0^{1,p}(\Omega)\right\}.$$

The equivalence of this definition and that in (1.1) can be shown by homogeneity. Notice also that $\lambda_{2,2}(\cdot)$ is the classical first eigenvalue of the Dirichlet-Laplacian and that, if Ω has regular enough boundary, $\lambda_{1,1}(\cdot)$ is *Cheeger constant* of Ω,

$$h_1(\Omega) := \inf\left\{\frac{P(E)}{|E|} : E\subset\overline{\Omega}\right\},$$

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where $P(\cdot)$ denotes the Caccioppoli-De Giorgi's perimeter, and $|\cdot|$ is the N-dimensional Lebesgue measure (see Section 2). The Pólya-Szegö inequality, or to the isoperimetric inequality if $r = q = 1$, ensures that balls minimize $\lambda_{r,q}(\cdot)$ among sets of prescribed measure, that is

$$\lambda_{r,q}(\Omega)|\Omega|^{\frac{r}{N}+\frac{r}{q}-1} \geq \lambda_{r,q}(B)|B|^{\frac{r}{N}+\frac{r}{q}-1}\,, \tag{1.2}$$ quantitative

for any ball B. Moreover, equality holds if and only if Ω is a ball, up to a set of null $r-$capacity. Of course this entails that balls maximize the $p-$torsional rigidity under measure constraint, that is

$$T_p(\Omega)|\Omega|^{-\frac{p+N(p-1)}{N}} \leq T_p(B)|B|^{-\frac{p+N(p-1)}{N}}\,,$$

for any ball B. The latter is known as *Saint-Venant inequality.*

In this paper we will deal mostly with the functionals T_2 and $h_1 = \lambda_{1,1}$. We stress that many results hold for more general cases. Nevertheless we shall need some results which hold for $\lambda_{1,p}$, $p > 1$.

Pólya and Szegö in [22] conjectured that the product of the torsional rigidity (raised to a suitable power) and the first eigenvalue of the Dirichlet-Laplacian was minimized by balls. Intuitively, this tells that the minimality of balls for the eigenvalue is somehow more stable compared to their maximality for the torsion. The conjecture was proved to be true by Kohler-Jobin, who showed, in [17, 18], that

$$T_2(\Omega)^{\frac{2}{N+2}}\lambda_{2,2}(\Omega) \geq T_2(B)^{\frac{2}{N+2}}\lambda_{2,2}(B)\,, \tag{1.3}$$ original-KJ

with equality if and only if Ω is a ball, up to a set of 2-capacity zero. The exponent $2/(N+2)$ is chosen so that the functional is scale invariant. The Pólya and Szegö's conjecture can be naturally extended to the more general family of functionals $T_p^\theta(\cdot)\lambda_{p,q}(\cdot)$. This nonlinear version of it, was recently proved by Brasco in [4], where he shows, with somehow simplified arguments, that

$$T_p^\theta(\Omega)\lambda_{p,q}(\Omega) \geq T_p^\theta(B)\lambda_{p,q}(B)$$

whenever B is a ball, $1 < p < +\infty$, and $1 < q < Np/(N-p)$ if $p < N$, $1 \leq q < +\infty$ if $p \geq N$. Again, $\theta = \theta(N, p, q)$ is chosen so to make the inequality homogeneous and equality can hold only if Ω is a ball, up to a set of p-capacity zero. We notice that the above class of parameters does not include the case $T_p^\theta(\cdot)\lambda_{r,q}(\cdot)$ with $p \neq r$ and in particular the interesting case $p = 2$, $r = q = 1$ which involves the Cheeger constant. This latter choice of parameters has a substantial difference from the other cases: for $r > 1$ it is not difficult to show the existence of a minimizer in the definition (1.1) of $\lambda_{r,q}(\Omega)$, while for $r = 1$ this is not the case, since a minimizing sequence may relax to a function with jumps, not belonging to $W^{1,1}(\Omega)$, so that a minimizer must be searched in the space of functions with bounded variation $BV(\Omega)$. Moreover, the positive level sets of the BV minimizers turn out to be optimal sets for $h_1(\Omega)$ (more details on the Cheeger constant are given in Section 3 below).

We stress here that the existence of a minimizer for the functional $T_p^\theta(\cdot)\lambda_{1,q}(\cdot)$ is false if stated among merely bounded open sets of $\mathbb{R}^N$, due to the fact that the functional $\lambda_{1,q}(\cdot)$ is sensible to nontrivial 1-capacitary modifications of a set (see Section 2 for a survey on capacities), while $T_p(\cdot)$ to p-capacitary ones. More precisely: it is always possible to find a (compact) set $K \subset B$ with $\mathrm{cap}_1(K) = 0$ and $\mathrm{cap}_p(K) > 0$ for $p > 1$ so that $\lambda_{1,q}(B \setminus K) = \lambda_{1,q}(B)$ but $T_p(B) > T_p(B \setminus K)$. Moreover we note that the q-Cheeger set of $B \setminus K$ is exactly B.
It is also possible to find a sequence of compacts K_n with the same property above so that $T_p(B \setminus K_n) \to 0$ as $n \to \infty$, while $\lambda_{1,q}(B \setminus K_n) = \lambda_{1,q}(B)$. Therefore we have

$$\inf\left\{T_p(\Omega)^\theta\lambda_{1,q}(\Omega)\ :\ \Omega \subset \mathbb{R}^N,\ \text{open, bounded}\right\} = 0,$$

and the infimum is not attained by any set.
This same observation holds for all functionals $T_p(\cdot)\lambda_{r,q}(\cdot)$ whenever $p \neq r$.

In this paper we aim to study the extension of (1.3) to the functionals $T_2^{\frac{1}{N+2}}(\cdot)\lambda_{1,1}(\cdot)$, among convex sets. Restricting to the class of convex sets yield to a well posed problem. Our first result is about existence of minimizers. For sake of simplicity, and to keep the number of parameters reasonable, the case $p = 2$ in this paper. Most of the examples are extendable to nonlinear cases.

⟨thm:existence⟩ **Theorem 1.1.** *There exists a minimizer for the problem*

$$\min\left\{T_2(\Omega)^{\frac{1}{N+2}}h_1(\Omega) : \Omega \subset \mathbb{R}^N,\ \textit{convex body}^1\right\}.$$

?eq:existence?

A definitely major task is to show that also in this case the ball is the unique minimizer. We are not able to accomplish this task and we propose it as an open problem.

⟨conjecture⟩ **Conjecture 1.2.** *[Cheeger-Kohler-Jobin inequality] Let $\Omega \subset \mathbb{R}^N$ be an open and convex set and $B \subset \mathbb{R}^N$ be a ball of unit volume. The following inequality holds true:*

$$T_2(\Omega)^{\frac{1}{N+2}} h_1(\Omega) \geq T_2(B)^{\frac{1}{N+2}} h_1(B)\,. \tag{1.4}$$ intro-KJq

Moreover, equality holds in (1.4) *if and only if Ω is equal to a ball, up to a set of 2-capacity zero.*

In order to justify such a conjecture, we provide a proof of it subordinated to a technical assumption. For the precise definition and motivation of the assumption we refer to section 5. Here we limit ourselves to explain the nature of it.

The proof of the Kohler-Jobin makes use of a quite deep rearrangement technique which, after the work of Brasco in [4], holds as long as the parameter p in the definition of $\lambda_{p,q}$ is strictly greater than 1. Such a rearrangement is done in a way such that, given a suitably chosen function u,:

(1) The numerator in the infimum defining $\lambda_{p,q}$ does not increases as u is rearranged;
(2) its denominator increases;
(3) the set $\{u > 0\}$ is transformed into a ball such that its torsion does not exceed $T_2(\Omega)$.

By suitably choosing the function u, this leads immediately to the Kohler-Jobin (and Brasco's) inequality.

In this paper via an approximation argument we are able to construct an abstract rearrangement satisfying (1) and (2), but we are not able to show that (3) is satisfied too (yet we believe so).

The very precise statement of the hypothesis is given in 5.6, in Section 5.

Remark 1.3. Notice that the nature of the Cheeger constant may suggest to use level-by-level rearrangements, for instance by exploiting coarea type formulas. Nevertheless such an approach appears doomed to fail, as the Kohler-Jobin rearrangement changes the height of each rearranged set.

The subordinated result we have is the following.

derassumption⟩ **Theorem 1.4.** *Let $\Omega \subset \mathbb{R}^N$ be a convex body and $B \subset \mathbb{R}^N$ be a ball of unit volume. If there exists a rearrangement $u \mapsto u^\sharp$ satisfying Hypothesis 5.6 then the minimality of the ball statement in Conjecture 1.2 holds.*

Remark 1.5. Note that we are not able to show, even subordinated to Hypothesis 5.6, nothing about the uniqueness of minimizers.

As a consequence, we are able to offer a new proof of the following quantitative version of (1.2).

⟨thm1⟩ **Corollary 1.6.** *There exists a dimensional constant $\sigma(N) > 0$ such that for any $\Omega \subset \mathbb{R}^N$ convex body for any ball $B \subset \mathbb{R}^N$, we have*

$$|\Omega|^{\frac{1}{N}} h_1(\Omega) - |B|^{\frac{1}{N}} h_1(B) \geq \sigma(N)\alpha(\Omega)^2\,, \tag{1.5}$$ qscheeger

where $\alpha(\Omega)$ denotes the Fraenkel asymmetry of Ω, see (2.2)*. Moreover, the power 2 in* (1.5) *is sharp, in the sense that it can not be replaced by any lower number.*

This improvement of the Cheeger inequality was first showed by Figalli, Maggi, and Pratelli, among open sets: in [12] they provide a short proof of this fact, based on the quantitative version of the isoperimetric inequality [14, 11, 9].

The approach in this paper is quite different, and borrows an idea from [5], where Brasco, De Philippis, and Velichkov show that $\lambda_{2,2}(\cdot)$ satisfies a (asimptotically sharp) quantitative estimate of the form

$$|\Omega|^{\frac{2}{N}} \lambda_{2,2}(\Omega) - |B|^{\frac{2}{N}} \lambda_{2,2}(B) \geq C(N)\alpha(\Omega)^2\,,$$

by relating the stability of $\lambda_{2,2}(\cdot)$ to that of the torsional rigidity $T_2(\cdot)$. In the very same spirit, we obtain (1.5) by combining the Cheeger–Kohler-Jobin inequality (1.4), together with the quantitative stability of the 2-torsional rigidity provided in [5]. Though this result is weaker than the one in [12], we believe it is an interesting application of Conjecture 1.2, and it should be a stimulus toward proving (or disproving) it.

Plan of the paper. The article is organized as follows. After some preliminaries in Section 2 about the geometric measure theory notions needed to define the Cheeger constant and some recalls on capacities, in Section 3 we survey the (several) different definitions of the Cheeger constant appearing in literature. This is in some sense of independent interest, even if it can not have any claim of actual novelty. Section 4 is devoted to the proof of Theorem 1.1, while in Section 6 we discuss about Kohler-Jobin rearrangement and of Conjecture 1.2 and Theorem 1.4. Section 7 deal with the proof of the quantitative estimate of Corollary 1.6.

2. Preliminaries

⟨section2⟩ In this section we collect some well known facts on geometric measure theory, which will serve our scopes later in the paper. We refer to [2, Chapter 3] for more details. Then we discuss some features and links of the several possible definitions of the Cheeger constant.

2.1. **The perimeter and its properties.** The measure theoretic perimeter (shortly: perimeter) of a Borel set $E \subset \mathbb{R}^N$ is the quantity

$$P(E) := \sup\left\{ \int_E \nabla \cdot \phi \,\mathrm{d}x \;:\; \phi \in C^1_c(\mathbb{R}^N, \mathbb{R}^N),\ \|\phi\|_\infty \leq 1 \right\}.$$

If $P(E) < +\infty$ then we say that E has finite perimeter. Equivalently, it can be defined in the setting of functions of bounded variation as the distributional derivative of characteristic functions. We recall that if $\Omega \subset \mathbb{R}^N$ is an open set, $u \in L^1(\Omega)$ is a function of bounded variation, $u \in BV(\Omega)$ if the distributional derivative Du of u is an $\mathbb{R}^N$-valued finite Radon measure. If $E \subset \mathbb{R}^N$ is a set of finite perimeter, then $\chi_E \in BV(\mathbb{R}^N)$ and $P(E) = |D\chi_E|(\mathbb{R}^N) =: \|D\chi_E\|_{TV(\mathbb{R}^N)}$.

Whenever it exists, the quantity

$$[0,1] \ni \theta_E(x) := \lim_{r\to 0} \frac{|E \cap B_r(x)|}{|B_r(x)|},$$

is called the density of a Borel set E at x. We denote by E^t the subset of points of $\mathbb{R}^N$ such that $\theta_E(x) = t$, and we call essential boundary the set $\partial^e E = E \setminus (E^0 \cup E^1)$. Eventually, we define the reduced boundary of E as the set $\partial^* E \subset \partial^e E$ of points of the essential boundary such that the measure theoretic inner unit normal

$$\nu_E(x) := \lim_{r\to 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))}$$

exists.

The geometry of the boundary of sets of finite perimeter is described in the two cornerstones of the theory of sets of finite perimeter: the De Giorgi's and the Federer's structure theorems.

Theorem 2.1 (De Giorgi's Structure Theorem)**.** *Let E be a set of finite perimeter. Then $\partial^* E$ is $\mathcal{H}^{N-1}$−rectifiable and $P(E) = \mathcal{H}^{N-1}(\partial^* E)$. Moreover, if $x \in \partial^* E$, then $(E - x)/r$ converges in L^1_{loc} to the hyperspace orthogonal to $\nu_E(x)$, as $r \to 0$. Eventually, the following divergence formula holds true*

$$\int_E \nabla \cdot \phi \,\mathrm{d}x = -\int_{\partial^* E} \phi \cdot \nu_E \,\mathrm{d}\mathcal{H}^{N-1}(x),$$

for any vector field $\phi \in C^1_c(\mathbb{R}^N, \mathbb{R}^N)$.

Theorem 2.2 (Federer's Structure Theorem)**.** *Let E be a set of finite perimeter. Then $\partial^* E \subset E^{1/2}$ and $\mathcal{H}^{N-1}(\partial^e E \setminus \partial^* E) = 0$. In particular $\partial^* E$, $E^{1/2}$, and $\partial^e E$ are equivalent, up to a $\mathcal{H}^{N-1}$−negligible set.*

2.2. **The isoperimetric inequality and its quantitative improvement.** The Kohler-Jobin inequality is based on the simple principle of slicing the energy functionals defining $\lambda_{p,q}$ and T_p horizontally, and then rearrange the level sets of the involved functions in a suitable way. The energies before and after rearrangement are compared exploiting the isoperimetric inequality, which states the following: for any set E of finite measure,

$$P(E) - P(B) \geq 0$$

whenever B is a ball of measure $|E|$, with equality if and only if E coincides with B up to a negligible set, or, equivalently, by homogeneity for all $E \subset \mathbb{R}^N$ of finite measure there holds

$$P(E) - N\omega_N^{1/N} |E|^{(N-1)/N} \geq 0.$$

Here ω_N denotes the measure of the ball with unit radius in $\mathbb{R}^N$, and again equality holds if and only if E is a ball. While dealing with $\lambda_{1,q}$ (more precisely, with the equality cases in (1.4)), we need to exploit a stronger version of the isoperimetric inequality, proved about a decade ago in [14]: there exists a dimensional constant C_N such that, for any set $E \subset \mathbb{R}^N$ of finite measure, it holds

$$\frac{P(E) - N\omega_N^{1/N} |E|^{(N-1)/N}}{N\omega_N^{1/N} |E|^{(N-1)/N}} \geq C_N \alpha(E)^2, \tag{2.1}$$ isoquantitativ

where $\alpha(E)$ is the *Fraenkel asymmetry* of the set E,

$$\alpha(E) := \inf_{x\in\mathbb{R}^N} \left\{ \frac{|E \Delta (B + x)|}{|\Omega|} \;:\; B \subset \mathbb{R}^N \text{ is a ball, } |B| = |E| \right\}, \tag{2.2}$$ fraenky

where $U\Delta V$ stands for the symmetric difference between the sets U and V.
It is worth stressing that the exponent 2 in the quantitative estimate (2.1) is *sharp*, in the sense that it can not be replaced by any lower number.

2.3. **Sobolev capacities.** We briefly recall the p-capacity of a measurable set $E \subset \mathbb{R}^N$, for $1 \le p < N$. We refer for more properties and details about it to [10]. We define

$$\mathrm{cap}_p(E) := \inf\left\{\int_{\mathbb{R}^N} |\nabla u|^p\,dx \;:\; u \ge 0,\ u \in L^{p^*}(\mathbb{R}^N),\ \nabla u \in L^p(\mathbb{R}^N;\mathbb{R}^N),\ \{u \ge 1\}^\circ \supset E\right\},$$

and clearly for all $K \subset \mathbb{R}^N$ compact it becomes

$$\mathrm{cap}_p(K) = \inf\left\{\int_{\mathbb{R}^N} |\nabla u|^p\,dx \;:\; u \in C_c^\infty(\mathbb{R}^N),\ u \ge \chi_K\right\}.$$

The only properties that we highlight here are the link of the p-capacity with Hausdorff and Lebesgue measures of a set. Namely that there exist constants $C_1(N,p), C_2(N,p) > 0$ such that, for all $E \subset \mathbb{R}^N$

$$\mathrm{cap}_p(E) \le C_1 \mathcal{H}^{N-p}(E), \qquad \mathcal{L}^N(E) \le C_2 \mathrm{cap}_p(E)^{\frac{N}{N-p}}.$$

3. On the different definitions of the Cheeger constant

⟨sec:cheeger⟩ A main role in this paper is played by the Cheeger constant of a set. Such a constant was introduced in [8] to obtain lower bounds for the first eigenvalue of the Laplace-Beltrami operator. While its original definition is given on Riemannian compact manifolds without boundary, it has lately found many applications in the euclidean setting, where it can be defined in several ways. Here we briefly survey such different definitions (for more details, see the recent paper [19]), and we offer a criterion under which all the corresponding constants coincide. This allows us to switch from one definition to the other in the rest of the paper.

zioni cheeger⟩ **Definition 3.1.** Let Ω be an open, bounded set in $\mathbb{R}^N$. Then the Cheeger constant is either

$$h_1(\Omega) := \inf\left\{\frac{P(E)}{|E|} : E \subset \overline{\Omega}\right\}, \text{ or}$$

$$h(\Omega) := \inf\left\{\frac{P(E)}{|E|} : E \Subset \Omega\right\}, \text{ or}$$

$$\lambda_1(\Omega) := \inf\left\{\frac{\|Du\|_{TV(\mathbb{R}^N)}}{\|u\|_{L^1(\Omega)}} \;:\; u \in BV(\overline{\Omega}) \setminus \{0\},\ u|_{\mathbb{R}^N\setminus\overline{\Omega}} = 0\right\}, \text{ or}$$

$$\lambda_{1,1}(\Omega) := \inf\left\{\frac{\|\nabla u\|_{L^1(\mathbb{R}^N)}}{\|u\|_{L^1(\Omega)}} : u \in C_c^\infty(\Omega) \setminus \{0\}\right\}.$$

We say that E_Ω is a *Cheeger set* for Ω if $h_1(\Omega) = \frac{P(E_\Omega)}{|E_\Omega|}$ and that Ω is *self–Cheeger* if $\Omega = E_\Omega$ up to sets of 1-capacity zero.

The first definition h_1 is the one in the spirit of the original work by Cheeger [8]. By coarea formula, it is not difficult to show that any minimizer in the third and fourth definition has the property that each of its level sets is a Cheeger set of Ω, that is, a minimizer of $h_1(\Omega)$, see [13]. Nonetheless, the above defined constants are not the same in general.

Example 3.2. Let us consider $B(0,1)$, the ball centered at 0 of radius 1 in $\mathbb{R}^2$. Let $U = [-1,1] \times \{0\}$, and let $\Omega = B \setminus U$. One can easily show by the minimality of the ball that $h(\Omega) > h(B)$. By a more delicate argument one might show also that if one removes $U = [-\tau, \tau]$ from B, being $\tau > 0$, one gets

$$h(\Omega) \ge \frac{P(E)}{|E|} + 2\mathcal{H}^1(U) \ge h_1(B) + 2\tau$$

as long as τ is small enough. The very same example holds for $\lambda_{1,1}$ in place of h.

Notice that in the previous examples we removed a *(N-1)-dimensional manifold* from a regular set (the ball). We wish to investigate if this condition is somehow close to be sharp. The answer happens to be positive, as shown in the next proposition.

B: il seguente lemma di esistenza secondo me e' falso per almeno due costanti su 4... ma non credo ci serva a nulla in realta'.. Mettere prima un pre lemma con esistenza insieme ottimo per la cheeger?

e:existcheeger⟩ **Lemma 3.3.** *Let Ω be an open, bounded set of $\mathbb{R}^N$, then all the infima in Definition 3.1 are actually minima.*

Proof. We show the existence of a minimum for h_1, the other cases are similar. It is clear that $h_1(\Omega) < +\infty$. Let $(E_n) \subset \overline{\Omega}$ be a minimizing sequence for $h_1(\Omega)$ such that

$$h_1(\Omega) - \frac{1}{n} \leq \frac{P(E_n)}{|E_n|} \leq h_1(\Omega) + \frac{1}{n},$$

then using the isoperimetric inequality, we obtain

$$N\omega_N^{1/N}|E_n|^{1-\frac{1}{N}} \leq P(E_n) \leq (h_1(\Omega) + \frac{1}{n})|E_n|, \qquad \text{hence,} \qquad |E_n|^{1/N} \geq C,$$

$$P(E_n)^{1+\frac{1}{N-1}} \leq N^{1+\frac{1}{N-1}}\omega_N^{\frac{1}{N-1}}|E_n|(h_1(\Omega) + \frac{1}{n}) \leq N^{1+\frac{1}{N-1}}\omega_N^{\frac{1}{N-1}}P(E_n), \qquad \text{hence,} \qquad P(E_n) \leq C.$$

By the equiboundedness of the perimeter, we can find a nonrelabeled subsequence and a nonempty set of finite perimeter $E \subset \overline{\Omega}$ such that $E_n \to E$ in L^1. By lower semicontinuity of the perimeter, we conclude

$$h_1(\Omega) = \liminf_{n\to+\infty} \frac{P(E_n)}{|E_n|} \leq \frac{P(E)}{|E|}.$$

□

?⟨equivdefh⟩? **Proposition 3.4.** *Let Ω be an open, bounded set of $\mathbb{R}^N$ such that*

$$P(\Omega) = \mathcal{H}^{N-1}(\partial\Omega).$$

Then $h_1(\Omega) = h(\Omega) = \lambda_1(\Omega) = \lambda_{1,1}(\Omega)$.

Proof. We claim that, since $P(\Omega) = \mathcal{H}^{N-1}(\partial\Omega)$, the same happens for a minimizer E of $h_1(\Omega)$, which exists by Lemma 3.3. To show this, we first split the reduced boundary of E in the following way

$$\partial^*E = (\partial^*E \cap \partial\Omega) \cup (\partial^*E \setminus \partial\Omega).$$

From De Giorgi's Theorem, it is well known that outside the contact points, i.e. in $\partial^*E \setminus \partial\Omega$, the set E is regular, so that $\mathcal{H}^{N-1}(\partial^*E \setminus \partial\Omega) = \mathcal{H}^{N-1}(\partial E \setminus \partial\Omega)$, see for example [19, Proposition 3.5 (iv)]. As for the contact points, we clearly have $\mathcal{H}^{N-1}(\partial^*E \cap \partial\Omega) \leq \mathcal{H}^{N-1}(\partial E \cap \partial\Omega)$. On the other hand, $\mathcal{H}^{N-1}$–a.e. $x \in \partial E \cap \partial\Omega$ belongs to $\partial E \cap \partial^*\Omega$, from our hypothesis on Ω and Federer's Theorem. Finally, thanks to [19, Prop. 3.5, point (vii)], any $x \in \partial^*\Omega \cap \partial E$ belongs to ∂^*E (namely: ∂E meets $\partial\Omega$ tangentially), so that $\mathcal{H}^{N-1}(\partial^*E \cap \partial\Omega) = \mathcal{H}^{N-1}(\partial E \cap \partial\Omega)$. Summing up all the informations, we have

$$\begin{aligned} P(E) &= \mathcal{H}^{N-1}(\partial^*E) \\ &= \mathcal{H}^{N-1}(\partial^*E \cap \partial\Omega) + \mathcal{H}^{N-1}(\partial^*E \setminus \partial\Omega) = \mathcal{H}^{N-1}(\partial^*E \cap \partial\Omega) + \mathcal{H}^{N-1}(\partial E \setminus \partial\Omega) \\ &= \mathcal{H}^{N-1}(\partial E \cap \partial\Omega) + \mathcal{H}^{N-1}(\partial E \setminus \partial\Omega) = \mathcal{H}^{N-1}(\partial E). \end{aligned}$$

This shows the claim. From this fact, we can exploit [25, Theorem 1.1], which assures the existence of a sequence of smooth sets E_n compactly contained inside E, which approximate E both in L^1 and in perimeter. In particular,

$$h_1(\Omega) \leq h(\Omega) \leq \lim_{n\to+\infty} \frac{P(E_n)}{|E_n|} \leq \frac{P(E)}{|E|} = h_1(\Omega).$$

Moreover, since $E_n \Subset E$ we can construct functions $u_n \in W_0^{1,1}(\Omega)$ such that

$$\frac{\int_\Omega |\nabla u_n|\,\mathrm{d}x}{\int_\Omega |u_n|\,\mathrm{d}x} = \frac{P(E)}{|E|} + o_n(1),$$

which easily entails that $\lambda_{1,1}(\Omega) = h(\Omega) = h_1(\Omega)$. A possible construction of the sequence (u_n) above is the following: given an optimal function u for λ_1 (its existence can be easily proven by means of the direct method in the Calculus of Variations), then we define $u_n = \rho_{\varepsilon/2} * (u\chi_{E_n^\varepsilon})$, where A^ε is the set of points of A whose distance from ∂A is larger than ε, and ρ_t is a positive mollifying kernel of total mass 1. Notice that such a construction is admissible since $\mathrm{dist}(\partial E, \partial E_n) > 0$.

The proof of the fact that $\lambda_1(\Omega) = \lambda_{1,1}(\Omega)$ is analogous. The main difference in the approximation argument is that one must use [25, Theorem 1.2] instead of [25, Theorem 1.1]. We thus skip the details. □

Remark 3.5. Notice that, for sets of finite perimeter, the inequality $P(E) \leq \mathcal{H}^{N-1}(\partial E)$ is always true, but the equality does not hold in general as long as the $\mathcal{H}^{N-1}-$ measure of $E^0 \cap \partial E$ and $E^1 \cap \partial E$ is non zero, as a consequence of Federer's Theorem. The condition $\mathcal{H}^{N-1}(E^0 \cap \partial E) > 0$ is quite pathological. Indeed, due to the fact that sets of finite relative perimeter satisfy density estimates on their boundary, whenever this happens to be true, then E can not even support a relative isoperimetric inequality. For a proof see [24, Lemma 3.5]. On the other hand, the condition $\mathcal{H}^{N-1}(E^1 \cap \partial E) > 0$ can hold even for self–Cheeger sets, that is those sets who are minimizer of their Cheeger constant h_1, as shown in [20, Section 2].

Some important and difficult topics related to the Cheeger problem are, given an open, bounded set $\Omega \subset \mathbb{R}^N$, to study if its Cheeger set is unique, its regularity and whether Ω is self–Cheeger or not. In particular, for our purposes, it is important to know that if Ω is convex, then E_Ω has the same regularity. This result has been proved by Alter and Caselles [1, Theorem 1], we recall it for completeness.

Theorem 3.6 (Alter-Caselles)**.** *There is a unique Cheeger set inside any non-trivial convex body in $\mathbb{R}^N$. The Cheeger set is convex and of class $C^{1,1}$.*

4. Proof of Theorem 1.1

ofthmexistence⟩ This section is devoted to the proof of the existence result of a minimizer for the shape functional $T_2^{\frac{1}{N+2}} h_1$ among open convex sets.

A key tool will be the adaptation of the well-known John's Lemma [16]. Andrebbe enunciato per i convex bodies.

⟨John⟩ **Lemma 4.1.** *Let $\Omega \subset \mathbb{R}^N$ be a convex body. Then there exists a parallelepiped K such that*

$$K \subset \Omega \subset C(N)K,$$

where $C(N) > 0$ is a constant which only depends on the dimension N. In particular, $|\Omega| \sim |K|$ and $\operatorname{diam}(\Omega) \sim \operatorname{diam}(K)$[2].

Proof of Theorem 1.1. Let $\{\Omega_n\}_n$ be a minimizing sequence for $T_2^{\frac{1}{N+2}} h_1$ among convex sets of $\mathbb{R}^N$. Since the functional is scale invariant, we may assume that $|\Omega_n| = 1$ for every $n \in \mathbb{N}$. Let us prove that the diameters $\{\operatorname{diam}(\Omega_n)\}_n$ are uniformly bounded. Assume by contradiction that this is not true, hence there exists a subsequence (not relabeled) with diameters diverging to $+\infty$.

Let us consider the associated sequence K_n of parallelepipeds provided by Lemma 4.1. By construction, we have that

$$|K_n| \sim 1, \quad \operatorname{diam}(K_n) \to +\infty.$$

It is now convenient to rewrite K_n as a scaling of a parallelepid of volume 1, as follows:

$$K_n = |K_n|^{1/N} Q_n \quad \text{with} \quad |Q_n| = 1.$$

Without loss of generality (up to a rotation and a translation), we may take Q_n of the form

$$Q_n := (0, \ell_1^{(n)}) \times \ldots \times (0, \ell_N^{(n)}),$$

with

$$\Pi_{i=1}^N \ell_i^{(n)} = 1$$

and with ordered sides

$$0 < \ell_1^{(n)} \le \ell_j^{(n)} \quad \forall j = 2, \ldots, N.$$

The assumption $\operatorname{diam}(K_n) \to +\infty$ implies that also $\operatorname{diam}(Q_n) \to +\infty$. This fact, together with $|Q_n| = 1$, implies that for $n \to \infty$

$$\ell_N^{(n)} \to +\infty \quad \text{and} \quad \ell_1^{(n)} \to 0.$$

Using the monotonicity of T_2 and h_1 with respect to set inclusion, we infer that

$$T_2^{\frac{1}{N+2}}(\Omega_n) h_1(\Omega_n) \ge \frac{1}{C(N)} T_2^{\frac{1}{N+2}}(K_n) h_1(K_n) = \frac{1}{C(N)} T_2^{\frac{1}{N+2}}(Q_n) h_1(Q_n), \tag{4.1}$$ assurdo

where $C(N)$ is the constant appearing in Lemma 4.1. Let us bound from below the two functionals h_1 and T, computed at Q_n. On the one hand, we exploit the following estimate, proved in [3, Corollary 5.2]: for every $\Omega \subset \mathbb{R}^N$ open bounded convex set, there holds

$$h_1(\Omega) \ge \frac{1}{N} \frac{P(\Omega)}{|\Omega|}.$$

Taking $\Omega = Q_n$, recalling the assumption $|Q_n| = \Pi_i \ell_n^{(i)} = 1$, and computing

$$P(Q_n) = \sum_{i=1}^N \Pi_{i \ne j} \ell_j^{(n)} = \sum_{i=1}^N \frac{1}{\ell_i^{(n)}} \ge \frac{1}{\ell_1^{(n)}},$$

[2]Here and in the rest of the paper we say that $A \lesssim B$ if there exists a constant C depending only on N such that $A \le CB$. We say that $A \sim B$ if $A \lesssim B \lesssim A$

we infer that

$$h_1(Q_n) \geq \frac{1}{N} \frac{1}{\ell_1^{(n)}}. \tag{4.2}$$ h1n

Let us now pass to the torsional rigidity. By definition, for every $u \in H_0^1(Q_n) \setminus \{0\}$, we have

$$T_2(Q_n) \geq \frac{\left(\int_{Q_n} u(x)\,\mathrm{d}x\right)^2}{\int_{Q_n} |\nabla u(x)|^2\,\mathrm{d}x}.$$

Writing $\mathbb{R}^N \ni x = (x_1, \ldots, x_n)$, we take $u(x) := \Pi_{i=1}^N \sin(\pi x_i/\ell_i^{(n)})$. The function u clearly belongs to $H_0^1(Q_n) \setminus \{0\}$. The numerator of the Rayleigh quotient above is a constant, which only depends on N:

$$\int_{Q_n} u(x)\,\mathrm{d}x = \Pi_{i=1}^N \int_0^{\ell_i^{(n)}} \sin(\pi x_i/\ell_i^{(n)})\,\mathrm{d}x_i = \Pi_{i=1}^N \frac{\ell_i^{(n)}}{\pi} \int_0^{\pi} \sin(t)\,\mathrm{d}t = \left(\frac{2}{\pi}\right)^N \Pi_{i=1}^N \ell_i^{(n)} = \left(\frac{2}{\pi}\right)^N.$$

As for the denominator, since

$$(\nabla u(x))_i = \frac{\pi}{\ell_i^{(n)}} \cos(\pi x_i/\ell_i^{(n)}) \Pi_{j\neq i} \sin(\pi x_j/\ell_j^{(n)}),$$

we obtain

$$\int_{Q_n} |\nabla u(x)|^2\,\mathrm{d}x = \pi^2 \sum_{i=1}^N \frac{1}{\left(\ell_i^{(n)}\right)^2} \int_0^{\ell_i^{(n)}} \cos^2(\pi x_i/\ell_i^{(n)})\,\mathrm{d}x_i \Pi_{j\neq i} \int_0^{\ell_j^{(n)}} \sin^2(\pi x_j/\ell_j^{(n)})\,\mathrm{d}x_j = \frac{\pi^2}{2^N} \sum_{i=1}^N \frac{1}{\left(\ell_i^{(n)}\right)^2}.$$

All in all we obtain

$$T_2(Q_n) \geq \frac{4^N}{\pi^{N+2}} \left[\sum_{i=1}^N \frac{1}{\left(\ell_i^{(n)}\right)^2}\right]^{-1} \geq \frac{4^N}{\pi^{N+2}} \left[N \frac{1}{\left(\ell_1^{(n)}\right)^2}\right]^{-1} = \frac{4^N}{N\pi^{N+2}} \left(\ell_1^{(n)}\right)^2, \tag{4.3}$$ Tn

where, in the last inequality, we have used the assumption $\ell_i^{(n)} \geq \ell_1^{(n)}$.

By combining (4.2) with (4.3) we get

$$T_2^{\frac{1}{N+2}}(Q_n) h_1(Q_n) \geq \tilde{C}(N) \frac{1}{\left(\ell_1^{(n)}\right)^{\frac{N}{N+2}}}$$

where for brevity we have set $\tilde{C}(N) := 4^{\frac{N}{N+2}}/(\pi N^{1+\frac{1}{N+2}})$.

Letting $n \to \infty$, we conclude that $T_2^{\frac{1}{N+2}}(Q_n) h_1(Q_n)$ goes to $+\infty$. This is in contradiction with (4.1).

This implies that the minimizing sequence Ω_n with $|\Omega_n| = 1$ has uniformly bounded diameter. In order to conclude, we apply the Blaschke selection theorem: this guarantees the existence of a subsequence converging to a convex body Ω^*, with respect to the complementary Hausdorff distance; with respect to such a convergence, both T and h_1 are continuous (see, e.g., [15] for the continuity of T_2 and [21] for the continuity of h_1), implying that the shape Ω^* is a minimizer. This concludes the proof. □

5. Kohler-Jobin rearrangement technique

⟨sec-KJ⟩ Our strategy is based on the Kohler-Jobin radial rearrangement technique, later extended by Brasco to the nonlinear case $p \neq 2$. In this section we write a short explanation of this tool and its application in our setting (see Lemma 5.7). At the same time, we introduce some notations used in the sequel.

The cornerstone of the Kohler-Jobin inequality is a rearrangement inequality which acts as follows: given a non-negative function u in the usual Sobolev space $W_0^{1,p}(\Omega)$, one constructs a rearrangement u^* of u, belonging to $W_0^{1,p}(B)$ for some ball B, such that

$$\int_\Omega |\nabla u|^p \mathrm{d}x = \int_B |\nabla u^*|^p \mathrm{d}x \qquad \text{and} \qquad \int_\Omega |u|^q \mathrm{d}x \leq \int_B |u^*|^q \mathrm{d}x\,. \tag{5.1}$$ condizioni

A somehow natural idea, in order to obtain the Kohler-Jobin inequality, is to consider the function u^* such that any level set of u^* is a ball $B_{r(t)}$ centered at 0 with

$$T_p(B_{r(t)}) = T_p(\{u > t\})\,.$$

Unfortunately this idea fails. In particular, the second requirement in (5.1) can not hold in general: if $\{u > t\}$ is not a ball for too many values of t, then

$$\int_\Omega |u|^q \mathrm{d}x > \int_B |u^*|^q \mathrm{d}x\,,$$

from the Saint-Venant inequality (while nothing can be said on the L^p-norms of the gradients). This suggests that the rearrangement must somehow take into account the other level sets of u. The successful idea of Kohler-Jobin was to introduce the following modification of the torsional rigidity.

Definition 5.1 ([4, 17])**.** Let Ω be an open, bounded set and $1 < p < +\infty$. We say that $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a *reference function for* Ω if $u \geq 0$ in Ω and

$$t \mapsto \frac{|\{x \in \Omega\ :\ u(x) > t\}|}{\int_{\{u=t\}} |\nabla u|^{p-1}\, \mathrm{d}\mathcal{H}^{N-1}} \in L^\infty([0, \|u\|_{L^\infty(\Omega)}])\,. \tag{5.2}$$ cattiva

We call $\mathcal{A}_p(\Omega)$ the set of all reference functions for Ω.
Then, for any $u \in \mathcal{A}_p(\Omega)$, the *modified torsional rigidity* is the functional, depending on Ω and u, defined by

$$T_{p,mod}(\Omega, u) = \left(\frac{p}{p-1} \sup \left\{ \int_\Omega g \circ u \,\mathrm{d}x - \frac{1}{p} \int_\Omega |\nabla g \circ u|^p \mathrm{d}x : g \in \mathrm{Lip}[0, \|u\|_{L^\infty(\Omega)}],\ g(0) = 0 \right\} \right)^{p-1}.$$

Remark 5.2. We note that we have the following inclusion of spaces for the reference functions:

$$\mathcal{A}_p(\Omega) \supset \mathcal{A}_q(\Omega), \qquad \text{for all } 1 < q \leq p < +\infty,$$

as, if $p > q$, by Hölder inequality, letting $\mu(t) = |\{u > t\}|$,

$$\frac{\mu(t)}{\int_{\{u=t\}} |\nabla u|^{q-1}\, \mathrm{d}\mathcal{H}^{N-1}} \leq \frac{\mu(t)}{\left(\int_{\{u=t\}} |\nabla u|^{p-1}\, \mathrm{d}\mathcal{H}^{N-1}\right)^{\frac{q-1}{p-1}} \mu(t)^{\frac{p-q}{p-1}}} = \left(\frac{\mu(t)}{\int_{\{u=t\}} |\nabla u|^{p-1}\, \mathrm{d}\mathcal{H}^{N-1}} \right)^{\frac{q-1}{p-1}}.$$

The features of $T_{p,mod}$ that will be used later are collected in the next Lemma (for the proof, we refer to [4, Proposition 3.8] and [17]).

⟨boiade⟩ **Lemma 5.3.** *Let $1 < p < +\infty$, $\Omega \subset \mathbb{R}^N$ be an open, bounded set, and $u \in \mathcal{A}_p(\Omega)$. Then*
 (i) *$T_{p,mod}(\Omega, u) \leq T_p(\Omega)$;*
 (ii) *if B is a ball such that $T_{p,mod}(\Omega, u) = T_p(B)$, then $|B| \leq |\Omega|$. Equality holds in the latter if and only if $\Omega = B$ and u is a radial function.*

The idea in [17, 18, 4] is then the following: given $u \in \mathcal{A}_p(\Omega)$, define u^* as the function such that for any $t \in [0, \|u\|_{L^\infty(\Omega)}]$, the set $\{u_p^* > \psi(t)\}$ is a ball with $p-$torsional rigidity equal to $T_{p,mod}(\{u > t\}, (u-t)_+)$, where for a function f we call $f_+ = \max\{f, 0\}$ the positive part of f. Here $\psi(t)$ is a suitably chosen decreasing function. Its definition is quite implicit and it is the core of the rearrangement. Here we only stress that in general if Ω is not a ball, ψ can not be the identity. With this construction, it is possible to show a rearrangement result as follows (for a proof we refer to [4, Proposition 4.1 and Remark 4.3] and [17]).

The following theorem contains the key features of the aforementioned rearrangement.

?⟨kjrt⟩? **Theorem 5.4** (Kohler-Jobin Rearrangement Theorem)**.** *Let $1 < p < +\infty$, $\Omega \subset \mathbb{R}^N$ be an open, bounded set, $u \in \mathcal{A}_p(\Omega)$, and B the origin centered ball such that $T_p(B) = T_{p,mod}(\Omega, u)$. Then, for every $q \geq 1$, there exists a radially symmetric decreasing function $u_p^* \in W_0^{1,p}(B)$, such that*

$$\int_\Omega |\nabla u|^p \mathrm{d}x = \int_B |\nabla u_p^*|^p \mathrm{d}x \quad \text{and} \quad \int_\Omega |u|^q \mathrm{d}x \leq \int_B |u_p^*|^q \mathrm{d}x\,.$$

Moreover, if $q > 1$, equality holds in the latter if and only if u is already a radially decreasing function.

In the sequel we will call such an u_p^* *Kohler-Jobin rearrangement of u.*

Remark 5.5. We note that while in [4, Proposition 4.1] the case $q = 1$ is not included, it is observed in [4, Remark 4.3] that it still holds (though the lack of strong convexity of the power does not allow to treat the equality cases).

We can not apply directly the Kohler-Jobin rearrangement for our aims, but a sort of limiting version will serve our purposes. To this scope let us fix $u \in \mathcal{A}_p(\Omega)$. By the last remark for all $r \in (1, p)$, $u \in \mathcal{A}_r(\Omega)$. We consider $u_r^* \in W_0^{1,r}(B^r) \subset BV_0(B^r) = \{u \in BV(B_r)\ :\ u = 0 \text{ outside of } B_r\}$, the Kohler-Jobin rearrangement of u, where B^r is a ball centered at the origin such that $T_r(B^r) = T_{r,mod}(\Omega, u)$. We extend to zero u_r^* outside B^r, if needed. By construction

$$\int_{B^r} |\nabla u_r^*| \,\mathrm{d}x = \int_\Omega |\nabla u| \,\mathrm{d}x \leq C,$$

thus (u_r^*) is uniformly bounded in $W_0^{1,1}(B_R) \subset BV_0(B_R)$, for some big ball B_R containing all the B^r, hence $u_r^* \rightharpoonup u^\#$ weakly in $BV_0(B_R)$ as $r \to 1$ for some $u^\sharp \in BV_0(\Omega)$. As a consequence there holds

$$\|Du^\#\|_{TV(B)} = \leq \liminf_{r\to 1} \int_{B^r} |\nabla u_r^*| \,\mathrm{d}x = \int_\Omega |\nabla u| \,\mathrm{d}x,$$

$$\int_B (u^\#)^q \,\mathrm{d}x = \lim_{r\to 1} \int_{B^r} (u_r^*)^q \,\mathrm{d}x \geq \int_\Omega u^q \,\mathrm{d}x,$$

for all $q \in [1, 1^*)$, where $B = \{u^\# > 0\}$. We are now in measure to formulate our hypothesis.

⟨mainhp⟩ **Main hypothesis 5.6.** *If the ball $B_\sharp = \{u_\sharp > 0\}$ constructed above satisfies $B_\sharp \subset B_{\Omega,u}$ where $B_{p,\Omega,u}$ is the ball such that $T_p(B_{p,\Omega,u}) = T_{p,mod}(\Omega, u)$, then there holds*

$$B_\sharp \subseteq B_{p,\Omega,u}.$$

?eq:hp?

We shall apply Hypothesis 5.6 to a particular family $(\varphi_n)_n$ of functions in $\mathcal{A}_{\overline{p}}(\Omega)$, which are a minimizing sequence for the q-Cheeger constant of Ω, that is,

$$\frac{\int_\Omega |\nabla \varphi_n| \,\mathrm{d}x}{\left(\int_\Omega |\varphi_n|^q \mathrm{d}x\right)^{\frac{1}{q}}} \longrightarrow \lambda_{1,q}(\Omega), \qquad \text{as } n \to \infty\,.$$

The existence of such an approximating sequence φ_n is not guaranteed in general, but it can be found for Ω regular enough and q-self–Cheeger, as we prove in the next Lemma.

⟨apprAp⟩ **Lemma 5.7.** *Let $q \in [1, +\infty)$ and Ω be an open, bounded set with Lipschitz boundary and q-self–Cheeger. For $n \in \mathbb{N} \setminus \{0\}$ we call $\Omega_n := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > 1/n\}$, and we define*

$$\varphi_n(x) := \begin{cases} 1 & \text{if } x \in \Omega_n \\ n \operatorname{dist}(x, \partial\Omega) & \text{if } x \in \Omega \setminus \Omega_n. \end{cases}$$

For all $n \in \mathbb{N}$ such that $\Omega_n \neq \emptyset$, we have $\varphi_n \in \mathcal{A}_p(\Omega)$, moreover,

$$\mathcal{R}_q(\varphi_n) = \frac{\int_\Omega |\nabla \varphi_n| \,\mathrm{d}x}{\left(\int_\Omega |\varphi_n|^q \mathrm{d}x\right)^{\frac{1}{q}}} \to \lambda_{1,q}(\Omega)$$

as $n \to \infty$.

Proof. The fact that $\varphi_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is immediate because the functions are by construction Lipschitz continuous on a bounded set. Moreover, since

$$|\nabla \varphi_n(x)| = \begin{cases} 0 & \text{if } x \in \Omega_n \\ n & \text{if } x \in \Omega \setminus \Omega_n, \end{cases}$$

one easily deduces that $\varphi_n \in \mathcal{A}_p(\Omega)$, because for all $t \in (0, 1)$, the isoperimetric inequality entails that

$$\frac{|\{x \in \Omega \,:\, \varphi_n(x) > t\}|}{\int_{\{\varphi_n = t\}} |\nabla \varphi_n|^{p-1} \,\mathrm{d}\mathcal{H}^{N-1}} = \frac{|\{\varphi_n > t\}|}{n^{(p-1)}\, \mathcal{H}^{N-1}(\{\varphi_n = t\})} \leq \frac{C(N)}{n^{(p-1)}} |\{\varphi_n > t\}|^{1/N},$$

so condition (5.2) is satisfied. We compute now

$$\int_\Omega |\varphi_n|^q \,\mathrm{d}x = \int_{\Omega_n} 1 \,\mathrm{d}x + \int_{\Omega\setminus\Omega_n} |\varphi_n|^q \mathrm{d}x = |\Omega_n| + \int_{\Omega\setminus\Omega_n} |\varphi_n|^q \mathrm{d}x \leq |\Omega_n| + |\Omega \setminus \Omega_n| \longrightarrow |\Omega| \qquad \text{as } n \to \infty\,,$$

using also the fact that $\varphi_n \leq 1$. We observe moreover that

$$\int_\Omega |\nabla \varphi_n| \,\mathrm{d}x = n\, |\Omega \setminus \Omega_n| \to \mathcal{H}^{N-1}(\partial\Omega)$$

as $n \to \infty$. The conclusion of the Lemma easily follows from the assumption that Ω is q-self Cheeger. □

Remark 5.8. It is important to highlight that the hypothesis of Ω being q-self–Cheeger in Lemma 5.7 is crucial. In fact, otherwise, one would need to make all the approximation arguments on E_Ω, a q-Cheeger set of Ω, whose regularity is a priori not known.

6. Proof of Theorem 1.4

⟨section3⟩ This section is devoted to the proof of Theorem 1.4, which shows the validity of Conjecture 1.2, under an assumption concerning the existence of a suitable rearrangement.

6.1. **The Cheeger-Kohler-Jobin inequality for convex sets.** We begin with a simple but useful result, which states that the minimization of $\lambda_{1,1}(\cdot)T_p(\cdot)^\theta$ among convex sets can be equivalently performed in the subclass of self–Cheeger sets, that is, if $\lambda_{1,1}(\Omega) = \frac{P(\Omega)}{|\Omega|}$.

Lemma 6.1. *Let $p \in (1,\infty)$ and $\theta = \frac{1}{p+N(p-1)}$. Then*

$$\begin{aligned}&\inf\left\{\lambda_{1,1}(\Omega)T_p(\Omega)^\theta \ :\ \Omega \subset \mathbb{R}^N,\ \textit{open, bounded and convex}\right\}\\ &= \inf\left\{\lambda_{1,1}(\Omega)T_p(\Omega)^\theta \ :\ \Omega \subset \mathbb{R}^N,\ \textit{open, bounded, convex and self–Cheeger}\right\}.\end{aligned} \tag{6.1}$$ optselfcheeger

Proof. The left hand side is clearly smaller than or equal to the right hand side. In order to prove equality, we assume for sake of contradiction that the strict inequality holds. Then, given a convex set Ω, its Cheeger set E_Ω is convex too (see [1]) and by definition it satisfies $\lambda_{1,1}(E_\Omega) = \lambda_{1,1}(\Omega)$. Moreover we can consider E_Ω to be open without loss of generality, because the convexity entails $\lambda_{1,1}(E_\Omega) = \lambda_{1,1}(E_\Omega^\circ)$; hence $E_\Omega \subset \Omega$. If Ω is not self–Cheeger, then $\mathrm{cap}_1(\Omega \setminus E_\Omega) > 0$; in particular $\mathrm{cap}_p(\Omega \setminus E_\Omega) > 0$ for $p > 1$, so that $T_p(\Omega) > T_p(E_\Omega)$. All in all, E_Ω is a better competitor for $\lambda_{1,1}(\cdot)T_p(\cdot)^\theta$ in the left hand side in (6.1). Clearly E_Ω is self–Cheeger, thus the strict inequality does not hold and we have reached a contradiction. □

We are now in position to prove our main result. We actually show it for a generic T_p in place of T_2 as the proof is not burdened in any sense by this generalization.

Proof of Theorem 1.4. We consider a convex body $\Omega \subset \mathbb{R}^N$. Without loss of generality we suppose it is also self–Cheeger, thanks to Lemma 6.1. Furthermore, it allows us to apply Lemma 5.7, with $q = 1$, in order to build a sequence of non-negative functions $\varphi_n \in \mathcal{A}_p(\Omega)$, for $n \in \mathbb{N}$, such that

$$\mathcal{R}(\varphi_n) := \frac{\int_\Omega |\nabla\varphi_n|\,\mathrm{d}x}{\int_\Omega |\varphi_n|\mathrm{d}x} \longrightarrow \lambda_{1,1}(\Omega), \qquad \text{as } n \to \infty.$$?limite?

By applying the rearrangement described in the previous section we get for $\varphi_n \in \mathcal{A}_p(\Omega)$ the existence of a function $\varphi_b^\sharp$ such thah

$$\mathcal{R}(\varphi_n) \geq \mathcal{R}(\varphi_n^\sharp) = \frac{\int_{B_\sharp} |\nabla\varphi_n^\sharp|\,\mathrm{d}x}{\int_{B_\sharp} |\varphi_n^\sharp|\mathrm{d}x},$$

where, with a slight abuse of notation, we have denoted by $B_\sharp = \{\varphi_n^\sharp > 0\}$ is a ball. This yields, for $\theta = \theta(N,p) = \frac{1}{p+N(p-1)}$,

$$T_p(\Omega)^\theta\mathcal{R}(\varphi_n) \geq T_{p,mod}(\Omega,\varphi_n)^\theta\mathcal{R}(\varphi_n) = T_p(B_n)^\theta\mathcal{R}(\varphi_n) \geq T_p(B_n)^\theta\mathcal{R}(\varphi_n^\sharp) \geq T_p(B_n)^\theta\lambda_{1,1}(B_n), \tag{6.2}$$ catena

where the first inequality follows from Lemma 5.3, while the last one is true since $\varphi_n^\sharp$ is an admissible function for the infimum defining $\lambda_{1,1}(B_n)$. This is the only point where Hypothesis 5.6 is exploited.
We observe that the quantity on the right of the chain of inequalities in (6.2) is constant, since B_n is a ball and θ is taken so that the functional $T_p^\theta(\cdot)\lambda_{1,1}(\cdot)$ is scale invariant. Hence, passing to the limit as $n \to \infty$ on the left-hand side, we obtain

$$T_p(\Omega)^\theta\lambda_{1,1}(\Omega) = \limsup_{n\to\infty} T_p(\Omega)^\theta\mathcal{R}(\varphi_n) \geq T_p(B_n)^\theta\lambda_{1,1}(B_n) = T_p(B)^\theta\lambda_{1,1}(B),$$

where B is any ball.

□

7. Proof of the quantitative estimate for h_1

⟨sec-qe⟩ We offer here the proof of Corollary 1.6, assuming that Conjecture 1.2 holds. We remark that it is a modification of the combination of a Kohler-Jobin type inequality with a quantitative Saint-Venant inequality proposed in [5]. We recall that the quantitative Saint-Venant inequality proved in [5, Section 5, Proof of Main Theorem] reads as

$$T_2(B)|B|^{-\frac{N+2}{N}} - T_2(\Omega)|\Omega|^{-\frac{N+2}{N}} \geq \tau(N)\alpha(\Omega)^2,$$

where $\tau = \tau(N) > 0$ is a dimensional constant and α the Fraenkel asymmetry (2.2).

Proof of Corollary 1.6. Since inequality (1.5) is scale invariant, we may assume without loss of generality that Ω and B have the same measure, equal to 1. Moreover, for brevity of notation, we will denote by T the 2-torsional rigidity T_2. Thanks to the Cheeger–Kohler-Jobin estimate (1.4), we have

$$\frac{h_1(\Omega)}{h_1(B)} - 1 \geq \left(\frac{T(B)}{T(\Omega)}\right)^\theta - 1\,.$$

We now distinguish two cases: $T(B)/T(\Omega) > 2$ and $T(B)/T(\Omega) \in [1,2]$. In the former, exploiting the easy bound $\tau\alpha^2(\Omega) \leq T(B)$, we obtain

$$\frac{h_1(\Omega)}{h_1(B)} - 1 \geq 2^\theta - 1 \geq (2^\theta - 1)\frac{\tau\alpha^2(\Omega)}{T(B)}\,. \tag{7.1}$$ case1

In the latter, we use the concavity of the function $x \mapsto x^\theta$, being $0 < \theta = 1/(N+2) < 1$. For every $x \in [1,2]$, we have

$$x^\theta = ((2-x) + 2(x-1))^\theta \geq (2-x) + 2^\theta(x-1)\,, \tag{7.2}$$ concave

since $2-x$ and $x-1$ are both in $[0,1]$ and their sum is 1. By applying (7.2) to $x = T(B)/T(\Omega)$, we obtain

$$\begin{aligned}\frac{h_1(\Omega)}{h_1(B)} - 1 &\geq \left(2 - \frac{T(B)}{T(\Omega)}\right) + 2^\theta\left(\frac{T(B)}{T(\Omega)} - 1\right) - 1 = (2^\theta - 1)\left(\frac{T(B)}{T(\Omega)} - 1\right) \\ &\geq (2^\theta - 1)\frac{\tau\alpha^2(\Omega)}{T(\Omega)} \geq (2^\theta - 1)\frac{\tau\alpha^2(\Omega)}{T(B)}\,.\end{aligned} \tag{7.3}$$ case2

Finally, combining (7.1) and (7.3), we conclude the proof of (1.5) with

$$\sigma := \frac{\tau(2^\theta - 1)h_1(B)}{T(B)}\,,$$

where B denotes an N-dimensional ball of unit measure.
Eventually, we notice that the exponent 2 is sharp. Indeed it is enough to consider the family of ellipsoids

$$\Omega_\varepsilon = \left\{(x_1, \dots, x_N) : \sum_{i=1}^{N-1} x_i^2 + (1+\varepsilon)x_N^2 \leq 1\right\}.$$

A simple computation shows that $|\Omega_\varepsilon|^{\frac{N-1}{N}}P(\Omega_\varepsilon) - |B|^{\frac{N-1}{N}}P(B) \simeq \varepsilon^2$ while $\alpha(\Omega_\varepsilon) \simeq \varepsilon$ as $\varepsilon \to 0$. On the other hand, since $h_1(B) = N$,

$$h_1(\Omega_\varepsilon) - h_1(B) \leq \frac{P(\Omega_\varepsilon) - P(B)}{|\Omega_\varepsilon|} \simeq P(\Omega_\varepsilon) - P(B)\,.$$

This concludes the proof. □

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