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ON A MODEL “SUM OF SQUARES” OPERATOR

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Dedicated to Jorge Hounie, as a token of
our appreciation and friendship.

Abstract. We study the real analytic and Gevrey regularity of the solutions to a type of “sum of squares” model operator, see (1.1), in two variables and obtain a result in agreement with Treves conjecture.

1. Introduction

The purpose of this paper is the study of a model operator being a “sum of squares” of vector fields in two variables. More precisely we are interested in the Gevrey or real analytic regularity of the solutions when the coefficients are real analytic.

The reason why we restrict ourselves to a two dimensional case deserves some elucidation.

First of all we always assume that Hörmander’s bracket condition ([21], [24]) is satisfied, i.e. the iterated commutators of the vector fields generate a three dimensional vector space. As a consequence all the operators we consider are C^∞ hypoelliptic. Our main concern is thus their real analytic (C^ω) or Gevrey s (G^s) hypoellipticity.

In 1972 and then in 1973 Baouendi and Goulaouic, [2], and Oleĭnik and Radkevič, [32], showed that C^∞ hypoellipticity does not imply C^ω hypoellipticity by studying in detail two model operators.

On the other hand in 1973 Derridj, [19], showed that a sum of squares operator with analytic coefficients is always G^r hypoelliptic, where r denotes the maximum length (number of vector fields) of the iterated brackets needed to satisfy Hörmander condition.

In 1978 Treves, [35], and in 1980 Tartakoff, [34], showed with different proofs that for a sum of squares with real analytic coefficients, satisfying Hörmander condition, we have C^ω hypoellipticity if the characteristic variety is a symplectic manifold and if the principal symbol vanishes exactly to the second order on the characteristic manifold. We recall that a submanifold, M, of $T^*\mathbb{R}^n \setminus \{0\}$, the cotangent bundle minus the zero section, is symplectic if the

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restriction to M of the symplectic form $d\sigma = d\xi \wedge dx = \sum_{j=1}^{n} d\xi_j \wedge dx_j$ has rank $\dim M$. Moreover we say that the principal symbol, p, vanishes exactly to the second order on $\operatorname{Char}(P)$ if $\ker F_p(\rho) = T_\rho \operatorname{Char}(P)$, for $\rho \in \operatorname{Char}(P)$. Here $F_p(\rho)$ denotes the fundamental matrix of p, which we may define in coordinates as $F_p(\rho) = dH_p(\rho)$, and $H_p(\rho) = (\partial_\xi p(\rho), -\partial_x p(\rho))$ denotes the Hamilton vector field of p at ρ.

In 1996 Treves formulated a conjecture for the real analytic hypoellipticity of a sum of squares operator, based on a real analytic stratification of the characteristic variety of p in [37]. According to the conjecture we have C^ω hypoellipticity if and only if the strata of the stratification defined by Treves are real analytic symplectic submanifolds.

In 2017, 2018 it has been proved, [8], [1], that Treves conjecture is false, in dimension $n \geq 4$, in its sufficient part.

We believe that the same is true in dimension $n = 3$, even though the Hamilton leaves of the possible strata—no definition for such strata yet—project injectively onto the fibers of the cotangent bundle.

Finally, we believe that the conjecture is true in the two dimensional case.

In this paper we study a model operator in two variables and show that its behaviour is in agreement with the conjecture, even though we do not provide a proof of the optimality, which, to our knowledge, is a difficult open problem.

More precisely we consider, for $(x,y) \in \mathbb{R}^2$, the operator

$$P(x,y,D_x,D_y) = D_x^2 + \left(f(x,y)D_y\right)^2 = X_1^2 + X_2^2, \tag{1.1}$$

where $D_x = i^{-1}\partial_x$ and analogously for D_y. Moreover

$$f(x,y) = y^{2a}x^m + x^\ell, \quad a, m, \ell \in \mathbb{N} = \{1, 2, \ldots\}, \tag{1.2}$$

with the condition that

$$|\ell - m| \text{ is an even integer.} \tag{1.3}$$

We immediately see that the Hörmander condition is satisfied with brackets of length $\ell + 1$.

Note that if $a = 0$ the operator in (1.1) is a Grušin operator whose analytic hypoellipticity has been proved in [20], Theorem 5.1.

The characteristic variety of P is $\operatorname{Char}(P) = \{(x,y) \mid \xi = 0, f(x,y) = 0, \eta \neq 0\}$. In other words

$$\operatorname{Char}(P) = \{(0,y;0,\eta) \mid \eta \neq 0\}$$

is a symplectic real analytic submanifold of codimension 2.

Assume that $\ell \leq m$. Then we may write $f(x,y) = x^\ell(1 + y^{2a}x^{m-\ell})$, with $m - \ell \geq 0$ and even. In this case, according to Treves conjecture, there is only one type of stratum coinciding with (the connected components of) the characteristic manifold.

If, on the other hand, $\ell > m$, then $f(x,y) = x^m(y^{2a} + x^{\ell-m})$ and we see that the characteristic manifold is symplectic and is made up by a nontrivial Treves stratification. In fact taking the Poisson bracket $\{X_1, X_2\}$, where X_i

denotes the symbol of the vector field X_i, and $\{\cdot,\cdot\}$ the Poisson bracket, we see that $\{X_1, X_2\} = 0$ on the characteristic set. Hence taking the iterated Poisson bracket

$$\underbrace{\{X_1, \{X_1, \{\cdots \{X_1}_{m \text{ times}}, X_2\} \cdots\}\}\}_{|\mathrm{Char}(P)} = m! y^{2a} \eta.$$

This vanishes when $y = 0$ so that the stratification is given by

$$\Sigma_{1,\pm} = \{(0, y; 0, \eta) \mid y \neq 0, \pm\eta > 0\}; \tag{1.4}$$

and

$$\Sigma_{2,\pm} = \{(0, 0; 0, \eta) \mid \pm\eta > 0\}. \tag{1.5}$$

Since we have

$$\underbrace{\{X_1, \{X_1, \{\cdots \{X_1}_{\ell \text{ times}}, X_2\} \cdots\}\}\}_{|\Sigma_{1,\pm}} = \ell! \eta \neq 0,$$

which is the generating bracket satisfying Hörmander condition. Hence we see that Hörmander condition is satisfied at length $m + 1$ on $\Sigma_{1,\pm}$ and at length $\ell + 1$ on $\Sigma_{2,\pm}$.

In particular we see that the $\Sigma_{1,\pm}$ are symplectic submanifolds of dimension two and the $\Sigma_{2,\pm}$ are one dimensional, and hence non symplectic. Moreover since the η half line is the integral curve of H_y the strata $\Sigma_{2,\pm}$ can be thought of as the Hamilton leaves of $\Sigma_{2,\pm}$.

In this paper we prove the following theorem

Theorem 1.1. *Let P be given as in* (1.1)–(1.3). *Then,*

i) If $\ell \leq m$, P is analytic hypoelliptic.
ii) If $\ell > m$, P is Gevrey s hypoelliptic for any $s \geq s_0$, where

$$s_0 = \left(1 - \frac{1}{2a} \frac{\ell - m}{\ell + 1}\right)^{-1} > 1. \tag{1.6}$$

A few remarks are in order. First we observe that the above statement is in agreement with the Treves conjecture: when the Poisson strata are symplectic we get analytic hypoellipticity.

When the Poisson strata are not symplectic we prove Gevrey s_0 hypoellipticity. We believe that this number is actually optimal, meaning that for any s, $1 \leq s < s_0$ there are solutions to the equation $Pu = f$, with $f \in C^\omega$, such that $WF_s(u) \neq \emptyset$. However, since a proof of this fact is long and highly technical, we do not attempt to provide one in this paper. We refer to [16] and [12] for results in similar, although particular, cases.

Assumption (1.3) is not necessary for our conclusion, meaning that we could drop it. It implies that in the case $\ell > m$ the vector field X_2 vanishes only on the line $x = 0$ outside of the origin. In general we could have cases where X_2 vanishes on a certain number of analytic real curves outside of the origin—see e.g. [10] for a description of the geometry of the characteristic variety in two variables. In any case the points on those curves are points of analytic regularity for the solution, so that their analytic wave front set projects onto the origin $x = 0, y = 0$.

Unfortunately including such a case would have forced us to produce a much longer and technical proof and we decided to sacrifice generality to attain a greater level of clarity. We refer to [4] for a result in this direction.

The proof of Theorem 1.1 is contained in the next two sections.

2. Proof of Theorem 1.1 in case *i)*

The basic tool for the proof is the so called *a priori* subelliptic estimate for P—see [21], [33] for a general proof.

Let Ω denote an open set in $\mathbb{R}^2$, $(0,0) \in \Omega$, then the subelliptic estimate for P in case *i)* is

$$\|u\|^2_{\frac{1}{\ell+1}} + \sum_{j=1}^{2} \|X_j u\|_0^2 \leq C\left(\langle Pu, u\rangle + \|u\|_0^2\right), \tag{2.1}$$

where $u \in C_0^\infty(\Omega)$, C is a positive constant and $\|\cdot\|_s$ denotes the Sobolev norm of order s in $\mathbb{R}^2$ (when $s = 0$ we get the L^2 norm.)

To prove the real analyticity of the solutions to $Pu = g$, where g is a real analytic function in Ω, it is enough to show that

$$\|\varphi D_y^p u\|_{\frac{1}{\ell+1}} + \sum_{j=1}^{2} \|X_j \varphi D_y^p u\|_0 \leq C_0^{p+1} p!, \tag{2.2}$$

for any $p \in \mathbb{N}$, where φ is an Ehrenpreis cutoff function supported in Ω (see the definition below.) We point out that φD_y^p is microlocally elliptic on the characteristic variety near the origin, so that (2.2) implies the real analyticity of u.

Definition 2.1 ([18], [22]). *For any natural number p, denote by φ_p a function in $C_0^\infty(\mathbb{R}^2)$, $\varphi_p(x,y)$, such that*

- **(S)** *φ_p is equal to 1 in a small neighborhood, W, of the origin and supported in W', with $W \subset W' \Subset \Omega$.*

We say that φ_p is an Ehrenpreis sequence of cutoff functions if there is a positive constant R such that for $|\alpha| \leq R(p+1)$ we have, for every p

$$|\partial_x^{\alpha_1} \partial_y^{\alpha_2} \varphi_p(x,y)| \leq C_\varphi^{|\alpha|+1} p^{|\alpha|}, \tag{2.3}$$

where $C_\varphi > 0$ and independent of p.

We now recall how to localize the function u in order to take advantage of the a priori estimate (2.1) and of the analytic regularity of Pu.

We observe that $\varphi_p(x,y)$ can be constructed as a product $\varphi_{1,p}(x)\varphi_{2,p}(y)$ of two Ehrenpreis type cutoff functions, each having support in a neighborhood of the origin of the line. Since, in case *i)*, $\mathrm{Char}(P) = \{(0,y;0,\eta) \mid \eta \neq 0\}$, we see that any x-derivative produces a function $(\partial_x^{\alpha_1}\varphi_{1,p}(x))\varphi_{2,p}(y)$

supported far from $x = 0$, where P is an elliptic operator and hence an analytic hypoelliptic operator. From the equation

$$Pu = g \in C^{\omega}(\Omega),$$

we have that $u \in C^{\infty}(\Omega)$ and, setting $\tilde{\Omega} = \Omega \cap \{(x,y) \in \Omega \mid x \neq 0\}$, $u \in C^{\omega}(\tilde{\Omega})$. As a consequence, defining $v = \varphi_{1,p}(x)u$, we have

$$Pv = \varphi_{1,p}(x)Pu + [P, \varphi_{1,p}(x)]u = \varphi_{1,p}(x)g + [D_x^2, \varphi_{1,p}(x)]u.$$

The r.h.s. of the above equation satisfies estimates of type (2.3), with a different constant, independent of p, for $\alpha_1 \leq R(p+1)$.

We may thus assume that

$$Pu = g,$$

where the derivatives of g have analytic growth rate for $|\alpha| \leq R(p+1)$, and u is compactly supported with respect to x.

Denoting by $\varphi_p(y)$ the function $\varphi_{2,p}(y)$, by (2.1) we have

(2.4)

$$\|\varphi_p D_y^p u\|_{\frac{1}{\ell+1}} + \sum_{j=1}^{2} \|X_j \varphi_p D_y^p u\|_0 \leq C\left(\langle P\varphi_p D_y^p u, \varphi_p D_y^p u\rangle + \|\varphi_p D_y^p u\|_0^2\right).$$

Let us treat first the “error term” $\|\varphi_p D_y^p u\|_0$ in the r.h.s. above.

Proposition 2.2 ([7]). *Using the above notation we have the inequality*

$$\|\varphi_p D_y^p u\| \leq C p^{-\frac{1}{m+1}} \|\varphi_p D_y^p u\|_{\frac{1}{m+1}} + C^{p+1} p^p, \tag{2.5}$$

where C is positive and independent of p.

Proof. Let χ be a smooth function such that $\chi(t) = 1$ if $|t| \geq 2$ and $\chi(t) = 0$ if $|t| \leq 1$. Consider the pseudodifferential operator $\chi(p^{-1}D_y)$. We have that $\chi(p^{-1}D_y) \in OPS^0_{0,0}$, the Hörmander (ρ, δ)–class of order 0 with $\rho = \delta = 0$. Then

$$\|\varphi_p D_y^p u\| \leq \|(1-\chi(p^{-1}D_y))\varphi_p D_y^p u\| + \|\chi(p^{-1}D_y)\varphi_p D_y^p u\|. \tag{2.6}$$

Consider the first summand on the r.h.s. above. Even though $1-\chi$ has a compact support, we have no composition formula for χ. Observe that

$$\varphi_p D_y^p u = \sum_{s=0}^{p} (-1)^s \binom{p}{s} D_y^{p-s}\left(\varphi_p^{(s)} u\right), \tag{2.7}$$

so that

(2.8)

$$\|(1-\chi(p^{-1}D_y))\varphi_p D_y^p u\| \leq \sum_{s=0}^{p} \binom{p}{s} \left\|(1-\chi(p^{-1}D_y)) D_y^{p-s}\left(\varphi_p^{(s)} u\right)\right\|.$$

We immediately verify that

$$\sigma\left((1-\chi(p^{-1}D_y))D_y^{p-s}\right) = (1-\chi(p^{-1}\eta))\eta^{p-s} \in S^0_{0,0},$$

because $p^{-1}|\eta| \leq 2$ in the support of $1-\chi$. Here $\sigma(A)$ denotes the symbol of the pseudodifferential operator A.

We also verify that

$$\max_{0\leq\alpha\leq\ell}\sup_{\eta}\left|\partial_\eta^\alpha\left(1-\chi(p^{-1}\eta)\eta^{p-s}\right)\right| \leq C^{p-s+1}p^{p-s}.$$

Here $C>0$ is a suitable constant independent of p and ℓ is given in Theorem A.2. This bounds the $S^0_{0,0}$–seminorms of $(1-\chi(p^{-1}D_y))D_y^{p-s}$ needed to apply the Calderón–Vaillancourt theorem (see Thm. A.2 in the appendix) so that we obtain that

$$\|(1-\chi(p^{-1}D_y))D_y^{p-s}\|_{\mathscr{L}(L^2,L^2)} \leq C^{p-s+1}p^{p-s},$$

for a new positive constant C. Hence, using the definition of the cutoff function φ_p, we deduce

$$\begin{aligned}\|(1-\chi(p^{-1}D_y))\varphi_p D_y^p u\| &\leq \sum_{s=0}^{p}\binom{p}{s}C^{p-s+1}p^{p-s}\|\varphi_p^{(s)}u\| \\ &\leq C^{p+1}C_\varphi p^p\|u\|\sum_{s=0}^{p}\binom{p}{s}\left(\frac{C_\varphi}{C}\right)^s \\ &\leq C_1^{p+1}p^p,\end{aligned}$$

where C_1 is independent of p if we choose $C\geq C_\varphi$, which is always possible.

Consider now the second summand in (2.6). We have

$$\begin{aligned}&\|\chi(p^{-1}D_y)\varphi_p D_y^p u\| \\ &\qquad = p^{-\frac{1}{m+1}}\|p^{\frac{1}{m+1}}\chi(p^{-1}D_y)D^{-\frac{1}{m+1}}\circ D^{\frac{1}{m+1}}\varphi_p D_y^p u\|,\end{aligned}$$

where $D^s = \mathrm{Op}\left((1+|\xi|^2+\eta^2)^{s/2}\right)$, for any $s\in\mathbb{R}$. Again using the support of χ we see that

$$\begin{aligned}&\sigma\left(p^{\frac{1}{m+1}}\chi(p^{-1}D_y)D^{-\frac{1}{m+1}}\right) \\ &\qquad = p^{\frac{1}{m+1}}\chi(p^{-1}\eta)(1+|\xi|^2+\eta^2)^{-\frac{1}{2(m+1)}} \in S^0_{0,0},\end{aligned}$$

with the $S^0_{0,0}$–seminorms uniformly bounded w.r.t. p. Thus the Calderón–Vaillancourt theorem yields

$$\|p^{\frac{1}{m+1}}\chi(p^{-1}D_y)D^{-\frac{1}{m+1}}\|_{\mathscr{L}(L^2,L^2)} \leq C,$$

where C is a positive constant independent of p. So we have

$$\begin{aligned}\|\chi(p^{-1}D_y)\varphi_p D_y^p u\| &\leq Cp^{-\frac{1}{m+1}}\|D^{\frac{1}{m+1}}\varphi_p D_y^p u\| \\ &\leq Cp^{-\frac{1}{m+1}}\|\varphi_p D_y^p u\|_{\frac{1}{m+1}}.\end{aligned}$$

This completes the proof of the proposition. □

Consider next the scalar product on the r.h.s. of (2.4). Since $[X^2, A] = 2X[X, A] - [X, [X, A]]$, we have

$$\begin{aligned}(2.9)\quad \langle P\varphi_p D_y^p u, \varphi_p D_y^p u\rangle &= \langle \varphi_p D_y^p P u, \varphi_p D_y^p u\rangle + \langle [P, \varphi_p D_y^p] u, \varphi_p D_y^p u\rangle \\ &= \langle \varphi_p D_y^p P u, \varphi_p D_y^p u\rangle + 2\sum_{j=1}^{2} \langle [X_j, \varphi_p D_y^p] u, X_j^* \varphi_p D_y^p u\rangle \\ &\quad - \sum_{j=1}^{2} \langle [X_j, [X_j, \varphi_p D_y^p]] u, \varphi_p D_y^p u\rangle \\ &= \langle \varphi_p D_y^p g, \varphi_p D_y^p u\rangle + 2\langle [X_2, \varphi_p D_y^p] u, X_2^* \varphi_p D_y^p u\rangle \\ &\quad - \langle [X_2, [X_2, \varphi_p D_y^p]] u, \varphi_p D_y^p u\rangle,\end{aligned}$$

since X_1 commutes with $\varphi_p D_y^p$, due to our choice of the cutoff function φ_p.

Since the derivatives of g up to the order $R(p+1)$ have analytic growth rate on the support of φ_p we see that

$$\left|\langle \varphi_p D_y^p g, \varphi_p D_y^p u\rangle\right| \leq \|\varphi_p D_y^p u\|_0^2 + \left(C_g^{1+p} p!\right)^2,$$

where $C_g > 0$ and the first term is treated using Proposition 2.2. Let us consider the second term in the expression of the scalar product in (2.9). Writing $X_2^* = X_2 + g_2$ we have

$$\begin{aligned}(2.10)\quad &\langle [X_2, \varphi_p D_y^p] u, X_2^* \varphi_p D_y^p u\rangle \\ &\qquad = \langle [X_2, \varphi_p D_y^p] u, X_2 \varphi_p D_y^p u\rangle + \langle [X_2, \varphi_p D_y^p] u, g_2 \varphi_p D_y^p u\rangle.\end{aligned}$$

Consider the first term on the r.h.s. of the above relation:

$$\left|\langle [X_2, \varphi_p D_y^p] u, X_2 \varphi_p D_y^p u\rangle\right| \leq \delta \|X_2 \varphi_p D_y^p u\|_0^2 + C_\delta \|[X_2, \varphi_p D_y^p] u\|_0^2,$$

where δ is small in such a way that its coefficient can be absorbed on the l.h.s. of (2.4). Consider then the second term above.

$$[X_2, \varphi_p D_y^p] = (x^m y^{2a} + x^\ell)\varphi_p' D_y^p + x^m \varphi_p [y^{2a}, D_y^p] D_y,$$

where $\varphi_p' = D_y \varphi_p$. Since

$$(2.11)\qquad \varphi_p' D_y^p = \sum_{r=0}^{p-1} (-1)^r D_y \varphi_p^{(r+1)} D_y^{p-1-r} + (-1)^p \varphi_p^{(p+1)},$$

and

$$(2.12)\qquad \left[y^{2a}, D_y^p\right] = -\sum_{k=1}^{2a} \binom{p}{k} \frac{(2a)!}{(2a-k)!} (-i)^k y^{2a-k} D_y^{p-k},$$

we have

(2.13)

$$[X_2, \varphi_p D_y^p] = \sum_{r=0}^{p-1} (-1)^r X_2 \varphi_p^{(r+1)} D_y^{p-1-r} + (-1)^p (x^m y^{2a} + x^\ell) \varphi_p^{(p+1)}$$

$$\begin{aligned}
&- \sum_{k=1}^{2a} \binom{p}{k} \frac{(2a)!}{(2a-k)!} (-i)^k y^{2a-k} x^m \varphi_p D_y^{p-k+1} \\
= &\sum_{r=0}^{p-1} (-1)^r X_2 \varphi_p^{(r+1)} D_y^{p-1-r} + (-1)^p \left(x^m y^{2a} + x^\ell\right) \varphi_p^{(p+1)} \\
&- \sum_{k=1}^{2a} \sum_{r=0}^{p-k} \binom{p}{k} \frac{(2a)!}{(2a-k)!} (-i)^k (-1)^r y^{2a-k} x^{m-\ell} \\
&\qquad \cdot \left(1 + y^{2a} x^{m-\ell}\right)^{-1} X_2 \varphi_p^{(r)} D_y^{p-k-r} \\
&- \sum_{k=1}^{2a} \binom{p}{k} \frac{(2a)!}{(2a-k)!} (-i)^k y^{2a-k} x^m (-1)^{p-k+1} \varphi_p^{(p-k+1)} .
\end{aligned}$$

Let us now get back to (2.10).

$$\begin{aligned}
\left| \langle [X_2, \varphi_p D_y^p] u, X_2^* \varphi_p D_y^p u \rangle \right| \leq \delta \| X_2 \varphi_p D_y^p u \|_0^2& \\
+ \gamma \| \varphi_p D_y^p u \|_0^2 + C'_\delta \| [X_2, \varphi_p D_y^p] u \|_0^2&,
\end{aligned}$$

where $\gamma > 0$ and C'_δ denotes a new positive constant depending on δ. The first term—as already said—can be absorbed on the l.h.s. of (2.4) provided δ is small. The second is dealt with using Proposition 2.2. Hence let us consider the third.

By (2.13) we have

$$\begin{aligned}
\| [X_2, \varphi_p D_y^p] u \|_0 \leq &\sum_{r=0}^{p-1} \| X_2 \varphi_p^{(r+1)} D_y^{p-1-r} u \|_0 + C_X \Bigg(\| \varphi_p^{(p+1)} u \|_0 \\
&+ \sum_{k=1}^{2a} \sum_{r=0}^{p-k} p^k \| X_2 \varphi_p^{(r)} D_y^{p-k-r} u \|_0 + \sum_{k=1}^{2a} p^k \| \varphi_p^{(p-k+1)} u \|_0 \Bigg),
\end{aligned}$$

where C_X denotes a positive constant depending on the problem data only, that is small if we suitably shrink the open set Ω.

As we can see the second and fourth term above are bounded by $C_0^{p+1} p!$ due to the properties of the Ehrenpreis sequence φ_p, while the first and third terms contain norms that can be bounded once more using the subelliptic estimate (2.4).

Let us now examine the double commutator term in (2.9). To keep the notation simple we write $f(x, y) = y^{2a} x^m + x^\ell$, so that $X_2 = f D_y$. First we need an expression for the double commutator. To keep it short, we write $f^{(k)}$ for $D_y^k f$, since only y-derivatives are involved at this stage.

$$[X_2, [X_2, \varphi_p D_y^p]] = [f D_y, f \varphi'_p D_y^p - \sum_{k=1}^{p} \binom{p}{k} \varphi_p f^{(k)} D_y^{p-k+1}] \tag{2.14}$$

$$
\begin{aligned}
&= f[D_y, f\varphi_p']D_y^p + f\varphi_p'[f, D_y^p]D_y - \sum_{k=1}^{p} \binom{p}{k} [fD_y, \varphi_p f^{(k)} D_y^{p-k+1}] \\
&= f[D_y, f\varphi_p']D_y^p - f\varphi_p'[D_y^p, f]D_y - \sum_{k=1}^{p} \binom{p}{k} f[D_y, \varphi_p f^{(k)}] D_y^{p-k+1} \\
&\qquad - \sum_{k=1}^{p} \binom{p}{k} \varphi_p f^{(k)} [f, D_y^{p-k+1}] D_y \\
&= f(f\varphi_p')' D_y^p - \sum_{k=1}^{p} \binom{p}{k} f\varphi_p' f^{(k)} D_y^{p-k+1} - \sum_{k=1}^{p} \binom{p}{k} f(\varphi_p f^{(k)})' D_y^{p-k+1} \\
&\qquad + \sum_{k=1}^{p} \sum_{k_1=1}^{p-k+1} \binom{p}{k} \binom{p-k+1}{k_1} \varphi_p f^{(k)} f^{(k_1)} D_y^{p-k-k_1+2} \\
&\qquad\qquad = \sum_{j=1}^{4} T_j.
\end{aligned}
$$

Thus we have to examine the terms

$$
\langle [X_2, [X_2, \varphi_p D_y^p]] u, \varphi_p D_y^p u \rangle = \sum_{j=1}^{4} \langle T_j u, \varphi_p D_y^p u \rangle .
$$

Let us start with the summand involving T_1. Applying (2.11), we have

$$
\begin{aligned}
\langle T_1 u, \varphi_p D_y^p u \rangle &= \langle (f\varphi_p')' D_y^p u, f\varphi_p D_y^p u \rangle \\
&= \sum_{r=0}^{p-1} (-1)^r \langle (f\varphi_p')^{(r+1)} D_y^{p-r-1} u, D_y f \varphi_p D_y^p u \rangle \\
&\qquad + (-1)^{p+1} \langle (f\varphi_p')^{(p+1)} u, f\varphi_p D_y^p u \rangle \\
&= \sum_{r=0}^{p-1} \sum_{\alpha=0}^{r+1} (-1)^r \binom{r+1}{\alpha} \Big(\langle f^{(\alpha)} \varphi_p^{(r+2-\alpha)} D_y^{p-r-1} u, X_2 \varphi_p D_y^p u \rangle \\
&\quad + \langle f^{(\alpha)} \varphi_p^{(r+2-\alpha)} D_y^{p-r-1} u, f' \varphi_p D_y^p u \rangle \Big) + (-1)^{p+1} \langle (f\varphi_p')^{(p+1)} u, f\varphi_p D_y^p u \rangle .
\end{aligned}
$$

We are going to treat each term above by essentially pulling back a y-derivative by means of (2.11). We do this in detail here and will then consider such procedure as routine for the remaining terms.

Consider the scalar product $\langle f^{(\alpha)} \varphi_p^{(r+2-\alpha)} D_y^{p-r-1} u, X_2 \varphi_p D_y^p u \rangle$. The second factor can be absorbed on the l.h.s. of (2.4), so that we need to estimate the norm of the (square of the) first factor.

As we saw above, only a fixed number of derivatives of f are non zero and moreover we have that $|f^{(\alpha)}| \leq C_f |f|$, since $\ell \leq m$. We note explicitly that this fact has already been used for the estimate of the simple commutator term.

Hence for the first set of summands we have

$$
\begin{aligned}
\text{(2.15)}\quad & \binom{r+1}{\alpha} \| f^{(\alpha)} \varphi_p^{(r+2-\alpha)} D_y^{p-r-1} u \|_0 \\
&\le (r+1)^{\alpha} \Bigg(\sum_{r_1=0}^{p-2-r} \| f^{(\alpha)} D_y \varphi_p^{(r+2-\alpha+r_1)} D_y^{p-r-r_1-2} u \|_0 \\
&\qquad + \| f^{(\alpha)} \varphi_p^{(r+2-\alpha+p-1-r)} u \|_0 \Bigg) \\
&\le C_f (r+1)^{\alpha} \Bigg(\sum_{r_1=0}^{p-2-r} \| X_2 \varphi_p^{(r+2-\alpha+r_1)} D_y^{p-r-r_1-2} u \|_0 \\
&\qquad + \| \varphi_p^{(r+2-\alpha+p-1-r)} u \|_0 \Bigg).
\end{aligned}
$$

The last summand is estimated by

$$
C_f C_\varphi^{p-\alpha+1} \|u\|_0 p^{p+1} \le C_f (2C_\varphi)^{p+1} \|u\|_0 p^p .
$$

As for the other summands we see that the order of D_y drops by $r + r_1 + 2$, but there is a gain of p^{r+r_1+2} in the cutoff functions as well as in the term $(r+1)^{\alpha}$. Thus the above quantity exhibits an analytic growth rate.

Next we deal with the term involving T_4. The other terms can then be treated along the same lines. Thus

$$
\begin{aligned}
& \langle T_4 u, \varphi_p D_y^p u \rangle \\
&\quad = \sum_{k=1}^{p} \sum_{k_1=1}^{p-k+1} \binom{p}{k} \binom{p-k+1}{k_1} \langle f^{(k_1)} \varphi_p D_y^{p-k-k_1+2} u, f^{(k)} \varphi_p D_y^p u \rangle .
\end{aligned}
$$

We point out explicitly that since $f^{(k)} \not\equiv 0$ for $k \le 2a$, it is enough to estimate a single summand.

For the sake of brevity we may bound the two binomials by p^{k+k_1} and work on a single summand, k and k_1 being understood to run on their intervals.

Therefore, using (2.11), we have

$$
\begin{aligned}
& p^{k+k_1} \left| \langle f^{(k_1)} \varphi_p D_y^{p-k-k_1+2} u, f^{(k)} \varphi_p D_y^p u \rangle \right| \\
&\quad \le p^{k+k_1} \Bigg(\sum_{r=0}^{p-k-k_1+1} \left| \langle f^{(k_1)} D_y \varphi_p^{(r)} D_y^{p-k-k_1+1-r} u, f^{(k)} \varphi_p D_y^p u \rangle \right| \\
&\qquad + \left| \langle f^{(k_1)} \varphi_p^{(p-k-k_1+2)} u, f^{(k)} \varphi_p D_y^p u \rangle \right| \Bigg) \\
&\le p^{k+k_1} \Bigg(\sum_{r=0}^{p-k-k_1+1} \sum_{r_1=0}^{p-1} \left| \langle f^{(k_1)} D_y \varphi_p^{(r)} D_y^{p-k-k_1+1-r} u, f^{(k)} D_y \varphi_p^{(r_1)} D_y^{p-1-r_1} u \rangle \right|
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{r=0}^{p-k-k_1+1} \left|\langle f^{(k_1)} D_y \varphi_p^{(r)} D_y^{p-k-k_1+1-r} u, f^{(k)} \varphi_p^{(p)} u\rangle\right| \\
&+ \sum_{r_1=0}^{p-1} \left|\langle f^{(k_1)} \varphi_p^{(p-k-k_1+2)} u, f^{(k)} D_y \varphi_p^{(r_1)} D_y^{p-1-r_1} u\rangle\right| \\
&\qquad\qquad + \left|\langle f^{(k_1)} \varphi_p^{(p-k-k_1+2)} u, f^{(k)} \varphi_p^{(p)} u\rangle\right| \Bigg)
\end{aligned}
$$

We may now apply the Cauchy–Schwartz inequality to each scalar product and keep in mind that, if it's not zero, we have $|f^{(k)}| \leq C_f |f|$, where C_f denotes a suitable positive constant.

Eventually we obtain

$$
\begin{aligned}
p^{k+k_1} & \left|\langle f^{(k_1)} \varphi_p D_y^{p-k-k_1+2} u, f^{(k)} \varphi_p D_y^{p} u\rangle\right| \\
&\leq C_f^2 \Bigg(\sum_{r=0}^{p-k-k_1+1} \sum_{r_1=0}^{p-1} p^{k+k_1-1} \|X_2 \varphi_p^{(r)} D_y^{p-k-k_1+1-r} u\|_0 \\
&\qquad\qquad \cdot p \|X_2 \varphi_p^{(r_1)} D_y^{p-1-r_1} u\|_0 \\
&+ \sum_{r=0}^{p-k-k_1+1} p^{k+k_1-1} \|X_2 \varphi_p^{(r)} D_y^{p-k-k_1+1-r} u\|_0 \cdot p \|\varphi_p^{(p)} u\|_0 \\
&+ \sum_{r_1=0}^{p-1} p^{k+k_1-1} \|\varphi_p^{(p-k-k_1+2)} u\|_0 \cdot p \|X_2 \varphi_p^{(r_1)} D_y^{p-1-r_1} u\|_0 \\
&\qquad\qquad + p^{k+k_1-1} \|\varphi_p^{(p-k-k_1+2)} u\|_0 \cdot p \|\varphi_p^{(p)} u\|_0 \Bigg).
\end{aligned}
$$

The above norms exhibit an analytic growth rate and choosing C_0 in (2.2) large enough compared to C_φ in (2.3), we can safely sum the various contributions. Hence, as in the case of the simple commutator, we can conclude.

We leave the details to the reader.

3. Proof of Theorem 1.1 in case *ii)*

Assume now $\ell > m$. Again we are going to use the subelliptic estimate to prove the second statement of Theorem 1.1.

As a preliminary observation we point out that if $(x_0, y_0) \neq (0,0)$ and Ω is a neighborhood of (x_0, y_0), with $(0,0) \notin \Omega$, then, by well known facts, see [17] and [30], we have that the solutions of $Pu = g \in C^\omega(\Omega)$ are also in $C^\omega(\Omega)$.

In fact, if (x_0, y_0) does not belong to the space projection of the characteristic manifold, so does Ω and P is microlocally elliptic, which implies C^ω hypoelliptic.

On the other hand if (x_0, y_0) belongs to the space projection of the first strata, $\Sigma_{1,\pm}$, see (1.4), which means that $y_0 \neq 0$, then it is well known that P is analytic, and hence G^s, for every $s \geq 1$, micro hypoelliptic on Ω.

This allows us to simplify a bit the proof by eliminating the cutoff function φ_p. Let us consider the equation

$$Pu = g \in G^{s_0}(\Omega).$$

From the above remarks we know that $u \in G^{s_0}(\Omega \setminus \{(0,0)\})$. Let $\varphi \in G_0^{s_0}(\Omega)$, $\varphi \equiv 1$ in a neighborhood, U, of the origin, $U \Subset \Omega$. Let $v = \varphi u$. Then

$$Pv = \varphi P u + [P, \varphi] u.$$

The r.h.s. of the above equation is in $G_0^{s_0}(\Omega)$ since the first order differential operator $[P, \varphi] = 0$ in U and u is Gevrey s_0 outside $(0,0)$, as mentioned above. Hence writing u instead of v, we may assume that u solves

$$Pu = g \in G_0^{s_0}(\Omega), \tag{3.1}$$

and $u \in C_0^\infty(\Omega)$, supported in a neighborhood of the origin.

As above we are going to show that

$$\|D_y^p u\|_{\frac{1}{\ell+1}} + \sum_{j=1}^{2} \|X_j D_y^p u\|_0 \leq C_0^{p+1} p!^{s_0}, \tag{3.2}$$

where s_0 is defined in (1.6).

By (2.1) we have

$$\|D_y^p u\|^2_{\frac{1}{\ell+1}} + \sum_{j=1}^{2} \|X_j D_y^p u\|_0^2 \leq C \left(\langle P D_y^p u, D_y^p u \rangle + \|D_y^p u\|_0^2 \right). \tag{3.3}$$

Consider the error term first. Since

$$\begin{aligned} \|D_y^p u\|_0 \leq \|D_y^{p-\frac{1}{\ell+1}} u\|_{\frac{1}{\ell+1}} &\leq \delta_1^{\frac{p(\ell+1)}{p(\ell+1)-1}} \|D_y^p u\|_{\frac{1}{\ell+1}} + \delta_1^{-p(\ell+1)} \|u\|_{\frac{1}{\ell+1}} \\ &= \delta \|D_y^p u\|_{\frac{1}{\ell+1}} + C_\delta^p \|u\|_{\frac{1}{\ell+1}}. \end{aligned} \tag{3.4}$$

The first term can be absorbed on the left of (3.3) if δ is chosen small. The other term is harmless in any Gevrey type estimate.

Consider now the term involving the scalar product in (3.3). We have

$$\begin{aligned} \langle P D_y^p u, D_y^p u \rangle &= \langle D_y^p P u, D_y^p u \rangle + \langle [P, D_y^p] u, D_y^p u \rangle \\ &= \langle D_y^p P u, D_y^p u \rangle + 2 \langle [X_2, D_y^p] u, X_2^* D_y^p u \rangle - \langle [X_2, [X_2, D_y^p]] u, D_y^p u \rangle. \end{aligned}$$

For the first term above we use the fact that $Pu = g \in G_0^{s_0}$:

$$\left| \langle D_y^p g, D_y^p u \rangle \right| \leq \|D_y^p u\|_0^2 + \left(C_g^{1+p} p!^{s_0} \right)^2,$$

Since $X_2^* = X_2 + f'$, using the same notations as in Section 2, and

$$[X_2, D_y^p] = -\sum_{k=1}^{p} \binom{p}{k} f^{(k)} D_y^{p-k+1},$$

we have

$$\langle [X_2, D_y^p]u, X_2^* D_y^p u\rangle = -\sum_{k=1}^{p} \binom{p}{k} \Big(\langle f^{(k)} D_y^{p-k+1} u, X_2 D_y^p u\rangle + \langle f^{(k)} D_y^{p-k+1} u, f' D_y^p u\rangle \Big).$$

The norm of the second factor in $\langle f^{(k)} D_y^{p-k+1} u, X_2 D_y^p u\rangle$ can be absorbed on the l.h.s. of (3.3), while the second factor of $\langle f^{(k)} D_y^{p-k+1} u, f' D_y^p u\rangle$ can be treated as in (3.4), so that we only need to estimate the norm $\|f^{(k)} D_y^{p-k+1} u\|_0$.

Modulo an inessential constant, keeping the binomial into account, we want to bound the quantity

$$p^k \|x^m y^{2a-k} D_y^{p-k+1} u\|_0. \tag{3.5}$$

We follow the same method we used in [9], but in this case it is simpler due to the lack of a cutoff function.

First observe that

$$p^k |y|^{2a-k} = p^{\theta k} |y|^{2a-k} p^{(1-\theta)k} \le C_0 \left((p^{\theta k} |y|^{2a-k})^{\lambda} + p^{(1-\theta)k\mu} \right),$$

where

$$\frac{1}{\lambda} + \frac{1}{\mu} = 1.$$

We point out explicitly that we are arguing in the case $k < 2a$. If $k = 2a$ we simply skip this step and go to the next. We omit the details. Choosing

$$\lambda = \frac{2a}{2a-k},$$

we have $\mu = \frac{2a}{k}$ and

$$p^k |y|^{2a-k} \le C_0 \left(p^{\theta \frac{2ak}{2a-k}} y^{2a} + p^{(1-\theta)2a} \right).$$

As a consequence

$$\begin{aligned} p^k \|x^m y^{2a-k} D_y^{p-k+1} u\|_0 \\ \le C_0 \big(p^{\theta \frac{2ak}{2a-k}} \|x^m y^{2a} D_y^{p-k+1} u\|_0 + p^{(1-\theta)2a} \|x^m D_y^{p-k+1} u\|_0 \big). \end{aligned} \tag{3.6}$$

Observe now that $|x|^m y^{2a} \le |x|^m (y^{2a} + x^{\ell-m})$ because of assumption (1.3). Hence we get

$$\begin{aligned} p^k \|x^m y^{2a-k} D_y^{p-k+1} u\|_0 \\ \le C_0 \big(p^{\theta \frac{2ak}{2a-k}} \|X_2 D_y^{p-k} u\|_0 + p^{(1-\theta)2a} \|x^m D_y^{p-k+1} u\|_0 \big). \end{aligned} \tag{3.7}$$

The first term can be resubjected to the procedure. It gives a Gevrey

$$G^{\theta\frac{2a}{2a-k}}.$$

The above result may be established using an induction argument on the order of y powers, y derivatives and powers of p. This is a classical technical step that, for the sake of brevity, we skip here and in what follows (see [9] for details.) We also point out that in the present case the constants are easy to control since they depend on the problem data and not on p due to the absence of cutoff functions.

Consider the second term. We write

$$p^{(1-\theta)2a}|x|^m = p^{\rho}\cdot p^{(1-\theta)2a-\rho}|x|^m,$$

and we use the same inequality used above:

$$p^{\rho}\cdot p^{(1-\theta)2a-\rho}|x|^m \le C_0\left(p^{\rho\sigma_1}+p^{((1-\theta)2a-\rho)\sigma_2}|x|^{m\sigma_2}\right),$$

with

$$\frac{1}{\sigma_1}+\frac{1}{\sigma_2}=1.$$

Choosing $\sigma_2=\frac{\ell}{m}$, and hence $\sigma_1=\frac{\ell}{\ell-m}$, we obtain

$$p^{\rho}\cdot p^{(1-\theta)2a-\rho}|x|^m \le C_0\left(p^{\rho\frac{\ell}{\ell-m}}+p^{((1-\theta)2a-\rho)\frac{\ell}{m}}|x|^{\ell}\right).$$

Plugging this into the second term on the r.h.s. of (3.7) we get

$$\begin{aligned} p^{(1-\theta)2a}\|x^m D_y^{p-k+1}u\|_0 &\le C_0\Big(p^{\rho\frac{\ell}{\ell-m}}\|D_y^{p-k+1}u\|_0 \\ &\quad + p^{((1-\theta)2a-\rho)\frac{\ell}{m}}\|x^{\ell}D_y^{p-k+1}u\|_0\Big) \\ &\le C_0\Big(p^{\rho\frac{\ell}{\ell-m}}\|D_y^{p-k+1-\frac{1}{\ell+1}}u\|_{\frac{1}{\ell+1}} + p^{((1-\theta)2a-\rho)\frac{\ell}{m}}\|X_2D_y^{p-k}u\|_0\Big). \end{aligned} \tag{3.8}$$

The last term yields

$$G^{((1-\theta)2a-\rho)\frac{\ell}{m}\frac{1}{k}},$$

as Gevrey regularity, while the first gives

$$G^{\rho\frac{\ell}{\ell-m}\left(k-\frac{\ell}{\ell+1}\right)^{-1}}.$$

Next we compute the parameters θ, ρ in such a way that the three Gevrey regularities coincide, i.e.

$$\theta\frac{2a}{2a-k}=\rho\frac{\ell}{\ell-m}\left(k-\frac{\ell}{\ell+1}\right)^{-1}=((1-\theta)2a-\rho)\frac{\ell}{m}\frac{1}{k}. \tag{3.9}$$

This yields

$$\theta=\frac{2a-k}{2a}\rho\frac{\ell}{\ell-m}\left(k-\frac{\ell}{\ell+1}\right)^{-1},$$

from which we can compute ρ in such a way that

$$\rho \frac{\ell}{\ell - m}\left(k - \frac{\ell}{\ell+1}\right)^{-1} = \left(1 - \frac{1}{2a}\frac{\ell - m}{\ell + 1}\right)^{-1},$$

which is the desired regularity.

The next and last term to examine is

$$\langle [X_2, [X_2, D_y^p]]u, D_y^p u\rangle.$$

Since

$$[X_2, D_y^p] = -\sum_{k=1}^{p} \binom{p}{k} f^{(k)} D_y^{p-k+1},$$

and

$$\begin{aligned}
[X_2, f^{(k)} D_y^{p-k+1}] &= [f D_y, f^{(k)} D_y^{p-k+1}] \\
&= f f^{(k+1)} D_y^{p-k+1} - f^{(k)} \sum_{k_1=1}^{p-k+1} \binom{p-k+1}{k_1} f^{(k_1)} D_y^{p-(k+k_1)+2},
\end{aligned}$$

we may write

$$\begin{aligned}
(3.10)\quad [X_2, [X_2, D_y^p]] &= -\sum_{k=1}^{p} \binom{p}{k} f f^{(k+1)} D_y^{p-k+1} \\
&\quad + \sum_{k=1}^{p} \sum_{k_1=1}^{p-k+1} \binom{p}{k}\binom{p-k+1}{k_1} f^{(k)} f^{(k_1)} D_y^{p-(k+k_1)+2} \\
&= S_1 + S_2.
\end{aligned}$$

We recall that $f^{(k)} \not\equiv 0$ only if $0 \le k \le 2a$, so that it is enough to estimate each summand.

Hence

$$\begin{aligned}
\left|\langle S_1 u, D_y^p u\rangle\right| &\le \sum_{k=1}^{p} p^k \|f D_y^{p-k+1} u\|_0 \|f^{(k+1)} D_y^p u\|_0 \\
&\le C_1 \sum_{k=1}^{p} p^k \|X_2 D_y^{p-k} u\|_0 \|D_y^p u\|_0.
\end{aligned}$$

The second factor is treated as in (3.4) and the first has an analytic growth rate.

We recall here that only a finite number of y–derivatives of f appear, generating constants depending on the problem's data, but independent of p. We shall often use this fact without an explicit mention.

This sets S_1. Consider S_2. Again

$$(3.11)\quad \left|\langle S_2 u, D_y^p u\rangle\right|$$

$$\begin{aligned}
&\le \sum_{k=1}^{p}\sum_{k_1=1}^{p-k+1} \binom{p}{k}\binom{p-k+1}{k_1} \left|\langle f^{(k)} f^{(k_1)} D_y^{p-(k+k_1)+2} u, D_y^p u\rangle\right| \\
&\le C_a \sum_{k=1}^{p}\sum_{k_1=1}^{p-k+1} p^{k+k_1} \left|\langle x^m y^{2a-(k+k_1-1)} D_y^{p-(k+k_1-1)+1} u, x^m y^{2a-1} D_y^p u\rangle\right| \\
&\le C_a \sum_{k=1}^{p}\sum_{k_1=1}^{p-k+1} p^{k+k_1-1} \|x^m y^{2a-(k+k_1-1)} D_y^{p-(k+k_1-1)+1} u\|_0 \\
&\qquad\qquad \cdot p\|x^m y^{2a-1} D_y^p u\|_0.
\end{aligned}$$

Here C_a denotes a positive constant independent of p.

Next we estimate both terms

$$p^{k+k_1-1}\|x^m y^{2a-(k+k_1-1)} D_y^{p-(k+k_1-1)+1} u\|_0$$

and

$$p\|x^m y^{2a-1} D_y^p u\|_0,$$

with the same argument as that for (3.5).

This completes the proof of the theorem.

A. Appendix

For the sake of completeness we recall here some well-known facts used throughout the paper.

Definition A.1. *For any $m \in \mathbb{R}$, $\rho, \delta \in \mathbb{R}$ with $0 \le \delta \le \rho \le 1$, $\delta < 1$, we denote by $S^m_{\rho,\delta}$ the set of all the functions $p(x,\xi) \in C^\infty(\mathbb{R}^{2n})$ such that for every multi-index α, β there exits a positive constant $C_{\alpha,\beta}$ for which*

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \le C_{\alpha,\beta}\langle \xi\rangle^{m-\rho|\alpha|+\delta|\beta|},$$

where $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$.

We denote by $OPS^m_{\rho,\delta}$ the class of the corresponding pseudodifferential operators $P = p(x,D)$.

It is trivial to see that the symbol class $S^m_{\rho,\delta}$ equipped with the semi-norms

$$|p|_\ell^{(m)} = \max_{|\alpha+\beta|\le\ell} \sup_{(x,\xi)} \{|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)|\langle\xi\rangle^{-(m-\rho|\alpha|+\delta|\beta|)}\}, \quad \ell \in \mathbb{N}$$

is a Fréchet space.

The Calderón-Vaillancourt theorem shows the L^2-continuity properties of the pseudodifferential operators in the above classes (see [14] or, for a more general setting, [25] Chap. 7, Th.1.6). We state below a formulation of such a theorem for pseudodifferential operators of order zero.

Theorem A.2 (Calderón-Vaillancourt). *Let* $P = p(x,D) \in OPS^0_{\rho,\delta}$ *with* $\rho \leq \delta$, $\delta < 1$. *Then there exist a positive integer* ℓ *and a positive constant* M *(depending only on* n*) such that*

$$\|Pu\| \leq M|p|^{(0)}_{\ell}\|u\|, \quad \textit{for every } u \in L^2(\mathbb{R}^n).$$

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