

RESEARCH ARTICLE

On finite generation in magnitude (co)homology and its torsion

Luigi Caputi¹ | Carlo Collari² 

¹Dipartimento di Matematica, Università di Bologna, Bologna, Italy

²Dipartimento di Matematica, Università di Pisa, Pisa, Italy

Correspondence

Carlo Collari, Dipartimento di Matematica, Università di Pisa, 56127 Pisa, Italy.

Email: carlo.collari@dm.unipi.it

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Abstract

The aim of this paper is to apply the framework developed by Sam and Snowden to study structural properties of graph homologies, in the spirit of Ramos, Miyata and Proudfoot. Our main results concern the magnitude homology of graphs introduced by Hepworth and Willerton, and we prove that it is a finitely generated functor (on graphs of bounded genus). More precisely, for graphs of bounded genus, we prove that magnitude cohomology, in each homological degree, has rank which grows at most polynomially in the number of vertices, and that its torsion is bounded. As a consequence, we obtain analogous results for path homology of (undirected) graphs.

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INTRODUCTION

Magnitude is a numerical invariant of isometry classes of metric spaces, and it finds its motivations in the study of species similarity [33] — see [23] for a general overview on the topic. From the mathematical point of view, the definition of magnitude was motivated, and naturally arose, from considerations on Euler characteristic in enriched category theory [21] — see also [23, 26]. Aside from category theory and metric geometry, magnitude has also many interesting connections with

other areas of mathematics, such as differential geometry [34], minimal surfaces [10], geometric measure theory and potential theory [24].

Graphs, endowed with the shortest path distance, are prominent examples of metric spaces. In this context, magnitude was studied first in [22]. Leinster showed that magnitude of graphs has a wealth of interesting properties. Among those, we have that magnitude is a rational function which can be expressed as an integral power series, which is multiplicative with respect to the Cartesian product of graphs, and that it satisfies an inclusion–exclusion formula. Remarkably, magnitude shares similar features with — and yet is not determined by — the Tutte polynomial, see [22].

Rather than on magnitude itself, here we focus on its categorification: magnitude (co)homology, as defined by Hepworth and Willerton [14, 17], see also [25]. As with magnitude, also its categorification has attracted some attention — see, for instance, the recent papers [1, 2, 12, 32]. In this context, categorification means to associate to a numerical (or polynomial) invariant a whole homology theory, whose Euler characteristic recovers the original invariant. The simplest example of categorification is given by the classical Euler–Poincaré characteristic of simplicial complexes, which is categorified by simplicial homology. The categorification of the Jones polynomial introduced by Khovanov [20] has shown the advantage of the homological and categorical approaches to the study of (polynomial) invariants of graphs, knots, and so on. After Khovanov’s discovery, the interest in categorification of knots and graph invariants, and their properties, skyrocketed. Among the well-known graph invariants which have been categorified using Khovanov’s framework, we can find the chromatic polynomial [13], and the Tutte polynomial [18]. Magnitude homology follows similar ideas. A consequence of the categorification process is that it brings more refined invariants. This is also the case for magnitude homology: there exist graphs with the same magnitude, but non-isomorphic magnitude homology groups [12, Appendix A]. Furthermore, the categorical approach allows us to explain some intrinsic properties of the magnitude of graphs, in homological terms; for example, the multiplicative property with respect to Cartesian products descends from a Künneth theorem, and a Mayer–Vietoris theorem categorifies the inclusion–exclusion formula.

Statement of results

We now come to the main point of this paper. It was asked in [17] whether there are graphs with torsion in their magnitude homology. This question was positively answered, first by Kaneta and Yoshinaga [19], and then extended by Sazdanovic and Summers [32]. Computations show that any finitely generated Abelian group appears as a subgroup of the magnitude homology of a graph [32, Theorem 3.14]. However, to get such torsion, Sazdanovic and Summers increased the combinatorial complexity of the graphs. This means that, to get new torsion, they increase the combinatorial genus of the graphs (also known as circuit rank, see text before Remark 1.7 for the precise definition). It is not clear from their proof whether this behaviour is a structural property of magnitude homology, or if it depends on the methods developed in [19, 32]. Instead of working with magnitude homology, here we use its cohomological version [14], and we will confine ourselves within the category of finite, connected, undirected graphs. As magnitude homology and cohomology are related by a short exact sequence [14, Remark 2.5], passing to cohomology is not a restriction.

The main goal of this work is to prove that, to get torsion of higher order in magnitude cohomology, it is indeed necessary to increase the combinatorial complexity of the graphs. Therefore, this behaviour is somehow due to a structural property of magnitude (co)homology. To achieve

such result, we borrow methods from representation stability [7] of (combinatorial) categories, as recently developed by Sam and Snowden [31]. The categorical viewpoint enables us to gain a deeper understanding of the behaviour of magnitude (co)homology (and of its torsion), by looking at combinatorial properties of the category of graphs considered. Our ideas were inspired by works from Ramos, Miyata and Proudfoot [28–30], who proved similar statements for matching complexes and unordered configuration spaces of graphs. Recall that the magnitude cohomology of a graph G , with coefficients in a commutative ring R , is a bigraded R -module $\text{MH}_*^k(G; R) = \bigoplus_{k,l} \text{MH}_l^k(G; R)$, where k is the cohomological grading, and l is the length, see Definition 2.1. The main result of the paper is the following — cf. Theorem 3.5.

Theorem 1. *For every pair of integers $k, g \geq 0$, there exists $m = m(g, k) \in \mathbb{Z}$ which annihilates the torsion subgroup of $\text{MH}_*^k(G; \mathbb{Z})$, for each graph G of genus at most g .*

Roughly speaking, the theorem says that the order of torsion classes in integral magnitude cohomology of connected graphs of genus at most g , in a fixed cohomological degree k , is bounded. In fact, to apply the ideas of Ramos, Miyata and Proudfoot to our context, we consider the category $\mathbf{CGraph}_{\leq g}$ of graphs with genus at most g (see Remark 1.7) and contractions. The opposite category of this category is quasi-Gröbner in the sense of [31], as proved in [29, Theorem 1.1]. Let \mathcal{V} be the $\mathbf{CGraph}_{\leq g}$ -module which associates to a graph G the free R -module generated by the vertices of G ; we call it the *vertex module* (see Definition 2.7). Then, in the spirit of [28–30], the main technical result of the paper is the following — cf. Theorem 2.11.

Theorem 2. *For any $k \geq 1$ natural number, the $\mathbf{CGraph}_{\leq g}^{\text{op}}$ -module $\text{Hom}_R(\mathcal{V}^{\otimes k}, R)$ is finitely generated.*

Using Theorem 2, we prove that magnitude cohomology is a finitely generated functor on the category $\mathbf{CGraph}_{\leq g}^{\text{op}}$, and the technology developed in [31] allows us to infer Theorem 1. As a by-product, we also obtain an estimate of the growth of the ranks of magnitude cohomology groups — cf. Corollary 3.1;

Theorem 3. *Let \mathbb{K} be a field, fix a cohomological grading k and genus $g \geq 0$. Then, there exists a polynomial $f \in \mathbb{Z}[t]$ of degree at most $g + k + 1$, such that, for all graphs G of genus at most g , we have*

$$\dim_{\mathbb{K}} \text{MH}_*^k(G; \mathbb{K}) \leq f(\#E(G)),$$

where $\#E(G)$ is the number of edges of G .

This last result says that the growth of the ranks of magnitude cohomology, in a fixed cohomological degree k , is at most polynomial, provided that we restrict to graphs of bounded genus — cf. Corollary 3.2. We remark here that the results are consistent with previous computations — see also Example 3.3, and the text thereafter — for example, for cycle graphs [12]. An immediate question is whether the polynomial nature of these estimates also holds for the category of graphs. Unfortunately, at the moment, it is not known if the category of graphs $\mathbf{CGraph}^{\text{op}}$ is a quasi-Gröbner category. Therefore, our result does not extend to the whole category of graphs, and an analogue of Theorem 3 may or may not hold. We also remark here that there are examples of cohomology theories of (directed) graphs whose rank grows exponentially in the number of

vertices — see, for example, the growth rate of multipath cohomology [4] with coefficients in an algebra, see [5, Table 2].

It was recently shown by Asao [1] that magnitude homology is related to another homology theory of directed graphs, the so-called *path homology* [11]. As a corollary of Asao's work, we can directly infer similar results to those above for path (co)homology of undirected graphs. For example, we get the following — see Corollary 3.6.

Theorem 4. *For each g, k positive integers, there exists an integer $d = d(g, k) \in \mathbb{Z}$ such that, for each graph G of genus g , the torsion part of the path cohomology $\text{PH}^k(G, \mathbb{Z})$ has exponent at most d .*

The main ingredient in the proof of the theorems is that magnitude cohomology is a finitely generated $\mathbf{C}\mathbf{Graph}_{\leq g}^{\text{op}}$ -module, in the sense of representation theory of categories — cf. Corollary 2.12. It would be interesting to see whether, also in the context of directed graphs, the usual homology theories considered in the literature are finitely generated.

Conventions

All graphs are assumed to be finite, connected and undirected, and are denoted in typewriter font, for example, G . Unless otherwise specified, R will denote a Noetherian commutative ring with identity. By \mathbb{N} , we will denote the set of non-negative integers.

1 | BASIC NOTIONS

In this section, we recall some basic notions needed in the follow-up.

1.1 | Finitely generated \mathbf{C} -modules

We start with some categorical notions as introduced in [7] in the context of FI-modules. We recall the notion of Noetherian modules and categories, following the general framework developed in [31].

Let \mathbf{C} be a (essentially) small category, and $R \neq 0$ be a commutative unital ring. A *representation* of the category \mathbf{C} , or a *\mathbf{C} -module over R* , is a functor $\mathcal{M} : \mathbf{C} \rightarrow R\text{-Mod}$ with values in the category of (left) R -modules. A map of \mathbf{C} -modules is a natural transformation of functors. Denote by $\mathbf{Rep}_R(\mathbf{C})$ the resulting category of \mathbf{C} -modules over R . The category $\mathbf{Rep}_R(\mathbf{C})$ is an Abelian category, and all the classical notions as submodules, kernels, cokernels, injections, surjections, can be defined pointwise. For example, a *submodule* of a \mathbf{C} -module \mathcal{M} is a \mathbf{C} -module $\mathcal{N} : \mathbf{C} \rightarrow R\text{-Mod}$ such that $\mathcal{N}(c)$ is a submodule of $\mathcal{M}(c)$, for each object c of \mathbf{C} , and such that the inclusion maps define a natural transformation. A submodule \mathcal{N} is called *proper*, if $\mathcal{N}(c)$ is a proper submodule of $\mathcal{M}(c)$ for at least one c , analogously for kernels and cokernels.

For a given \mathbf{C} -module \mathcal{M} , by an *element* of \mathcal{M} , we mean an element of $\mathcal{M}(c)$ for some object c of \mathbf{C} . If S is a subset of the disjoint union $\bigoplus_{c \in \mathbf{C}} \mathcal{M}(c)$, the *span* $\text{span}(S)$ of S is the minimal \mathbf{C} -submodule of \mathcal{M} containing each element of S . We are primarily interested in finitely generated modules, which can be defined as follows.

Definition 1.1. A \mathbf{C} -module \mathcal{M} is *finitely generated* if there exists a finite set of elements $m_1, \dots, m_k \in \bigoplus_{c \in \mathbf{C}} \mathcal{M}(c)$, such that $\text{span}(m_1, \dots, m_k) = \mathcal{M}$.

We can also characterise finitely generated modules in terms of simpler modules. For each object c of \mathbf{C} , define a *principal projective* \mathbf{C} -module \mathcal{P}_c , as follows. The functor \mathcal{P}_c is defined on an object c' of \mathbf{C} by setting

$$\mathcal{P}_c(c') := R \langle \text{Hom}_{\mathbf{C}}(c, c') \rangle,$$

that is, the free R -module with basis $\text{Hom}_{\mathbf{C}}(c, c')$; \mathcal{P}_c is then extended to morphisms accordingly, by compositions. For a morphism $\gamma : c \rightarrow c'$, we denote by e_γ the corresponding element in $\mathcal{P}_c(c')$. Observe that, for any \mathbf{C} -module \mathcal{M} , we have a natural isomorphism

$$\text{Hom}_{\mathbf{Rep}_R(\mathbf{C})}(\mathcal{P}_c, \mathcal{M}) \xrightarrow{\cong} \mathcal{M}(c),$$

which is given by $\phi \mapsto \phi(c)(e_{\text{id}_c})$. Therefore, for each short exact sequence of \mathbf{C} -modules

$$0 \rightarrow \mathcal{M}_1 \xrightarrow{f} \mathcal{M}_2 \xrightarrow{g} \mathcal{M}_3 \rightarrow 0,$$

application of the functor $\text{Hom}_{\mathbf{Rep}_R(\mathbf{C})}(\mathcal{P}_c, -) : \mathbf{Rep}_R(\mathbf{C}) \rightarrow R\text{-Mod}$ yields a sequence which is naturally equivalent to

$$0 \rightarrow \mathcal{M}_1(c) \xrightarrow{f(c)} \mathcal{M}_2(c) \xrightarrow{g(c)} \mathcal{M}_3(c) \rightarrow 0$$

and hence exact. It follows that the module \mathcal{P}_c is a projective object in $\mathbf{Rep}_R(\mathbf{C})$ — which justifies the name *principal projective*. Finitely generated \mathbf{C} -modules can be characterised in terms of principal projectives, cf. [8, Proposition 2.3].

Lemma 1.2. A \mathbf{C} -module \mathcal{M} is finitely generated if and only if there exists a surjection

$$\bigoplus_{i=1}^n \mathcal{P}_{c_i} \rightarrow \mathcal{M}$$

for some objects c_1, \dots, c_n of \mathbf{C} .

For a finitely generated \mathbf{C} -module \mathcal{M} , we will also refer to the objects c_1, \dots, c_n of \mathbf{C} in the lemma as *generators* of \mathcal{M} . We now recall the notion of Noetherianity in the context of modules over an arbitrary category, which is central for this work.

Definition 1.3. A \mathbf{C} -module is *Noetherian* if all its submodules are finitely generated. The category $\mathbf{Rep}_R(\mathbf{C})$ is Noetherian if all its finitely generated \mathbf{C} -modules are Noetherian.

Observe that, in discussing properties related to finite generation, it is often possible to restrict to principal projective modules. Indeed, by [31, Proposition 3.1.1], the category $\mathbf{Rep}_R(\mathbf{C})$ is Noetherian if and only if every principal projective module \mathcal{P}_c is Noetherian.

Example 1.4. Denote by **FI** the category of finite sets and injective functions. Then, the category $\mathbf{Rep}_R(\mathbf{FI})$ is Noetherian by [8, Theorem A] for any Noetherian ring R .

The category **FI** is an example of an *EI*-category, that is, a category in which every endomorphism is an isomorphism. It was proven that $\mathbf{Rep}_R(\mathbf{C})$ is Noetherian, for any finite *EI*-category **C** and Noetherian ring R [27, Lemma 16.10]. This result was further extended to infinite *EI*-categories (satisfying some mild combinatorial conditions) in [9, Theorem 3.7].

Example 1.5. Let **FA** be the category of finite sets and all functions. Then, the category $\mathbf{Rep}_R(\mathbf{FA})$ is Noetherian [31, Corollary 7.3.5] for any Noetherian commutative unital ring R .

Following the seminal paper [7], Noetherian properties of various other categories have been extensively investigated, in particular thanks to the techniques developed by Sam and Snowden in [31]. One of the main results in the latter paper is that, for a given category **C** (with some combinatorial assumptions), and a (possibly non-commutative) left Noetherian ring R , the associated category of representations is Noetherian as well.

1.2 | Graph categories

The aim of this section is to discuss the combinatorial properties of certain categories of graphs, from the viewpoint of [29].

In the following, by a graph G , we mean a finite, connected and non-empty one-dimensional CW-complex; it has a set of vertices $V(G)$ and a set of edges $E(G)$, which are unordered pairs of vertices, possibly with multiplicities. We recall the intuitive notions of contractions, deletions and minor morphisms; for a more detailed account of these operations, we refer to [28, 30].

Let G be a graph and $e \in E(G)$ be an edge. The *contraction* of G with respect to the edge e is the graph G/e obtained from G by contracting e to a point. The *deletion* of e is the graph $G \setminus e$ obtained from G by removing e from the set of edges of G . Note that the operation of contracting edges does not change the homotopy type of G , unless the edge contracted is a self-loop. We are not going to consider this latter case, and only allow contractions of edges with distinct endpoints. Similarly, when dealing with deletions, we will only allow deletions of graphs for which $G \setminus e$ is connected. A *minor* of a graph G' is a graph G that is isomorphic to a graph obtained from G' by iterative contractions and deletions. More formally, we have the following definition of minor morphism of graphs.

Definition 1.6 [30, Definition 2.1]. A *minor morphism* $\phi : G' \rightarrow G$ is a map of sets

$$\phi : V(G') \sqcup E(G') \sqcup \{\star\} \rightarrow V(G) \sqcup E(G) \sqcup \{\star\},$$

such that:

- $\phi(V(G')) = V(G)$ and $\phi(\star) = \star$;
- if an edge $e \in E(G')$ has endpoints $\{v, w\}$, and $\phi(e) \neq \star$, then either $\phi(e) = \phi(v) = \phi(w)$ is a vertex of G , or $\phi(e)$ is an edge of G with endpoints $\phi(v)$ and $\phi(w)$;
- ϕ maps $\phi^{-1}(E(G))$ bijectively onto $E(G)$;
- for each vertex $v \in G$, the preimage $\phi^{-1}(v)$ (as a subgraph of G') is a tree.

The preimage of \star under ϕ consists of deleted edges, whereas the edges that are mapped to vertices of G represent the contracted ones. Furthermore, the last item in the definition implies that self-loops cannot be contracted, but only deleted.

In the follow-up, we will mainly consider (subcategories of) the category **Graph** of finite non-empty graphs, with minor morphisms of graphs. For example, we consider the subcategory **CGraph** of **Graph** consisting of finite, non-empty, connected, graphs, where the morphisms are given by contractions.

For a graph G , we call the (combinatorial) *genus* of G the quantity $|E(G)| - |V(G)| + 1$. This is often called *circuit rank* but, for consistency, here we prefer to follow the terminology adopted in [29].

Remark 1.7. Consider the full subcategory **CGraph**_{*g*} of **CGraph** given by graphs of genus g . In particular, as contractions do not change the genus of a graph, **CGraph** can be seen as the disjoint union of the categories **CGraph**_{*g*}, for $g \in \mathbb{N}$. We will also denote by **CGraph** _{$\leq g$} the subcategory of **CGraph** spanned by graphs of genus $\leq g$. The category **CGraph**_{*g*} was denoted by \mathcal{G}_g in [29].

We have the following key result.

Theorem 1.8 [29, Theorem 1.1]. *For any $g \geq 0$, the representation category of **CGraph**^{op} _{$\leq g$} is Noetherian over any Noetherian ring.*

Thus, for any commutative unital Noetherian ring R , all submodules of finitely generated **CGraph**^{op} _{$\leq g$} modules are finitely generated.

2 | MAGNITUDE (CO)HOMOLOGY

In this section, we prove the main result of the paper, that is, that magnitude cohomology of graphs is a finitely generated functor on the category **CGraph**^{op} _{$\leq g$} of connected graphs with bounded genus, and contractions. As a consequence, in the next section, we shall recover structural results on its torsion and rank growth.

2.1 | Magnitude homology and cohomology of graphs

We start with recalling the definition of magnitude homology of graphs, and we will then focus on magnitude cohomology. We will mainly follow [1, 14, 17].

Firstly, observe that a connected graph can be seen as a metric space with the path metric — the distance between two vertices of the graph being given by the length of a shortest path in the graph connecting them. To be more precise, the points of the metric space associated to a graph G are its vertices, and edges are declared to have length 1. Concretely, the metric d_G on G is given by

$$d_G(v, w) := \min \left\{ d_G(v, v_1) + \sum_{i=1}^{k-2} d_G(v_i, v_{i+1}) + d_G(v_{k-1}, w) \mid \{v, v_1\}, \{v_{k-1}, w\}, \{v_i, v_{i+1}\} \in E(G), i = 1, \dots, k - 2 \right\}$$

for v, w vertices of G . If the graph is not connected, we set $d_G(v, w) := \infty$ for every $v, w \in G$ not connected by any path; hence, for non-connected graphs, d_G is an extended metric. For a $(k + 1)$ -tuple (v_0, \dots, v_k) of vertices of G , with $v_i \neq v_{i+1}$ and $d_G(v_i, v_{i+1}) < \infty$ for each i , the *length* of (v_0, \dots, v_k) in G is the number

$$\ell(v_0, \dots, v_k) := \sum_{i=0}^{k-1} d_G(v_i, v_{i+1}).$$

We recall the definition of magnitude chain groups. Let R be a commutative unital ring.

Definition 2.1. Let G be a graph, and $l, k \in \mathbb{N}$ natural numbers. We let

$$\text{MC}_{k,l}(G; R) := R\langle (v_0, \dots, v_k) \mid v_0 \neq \dots \neq v_k, \ell(v_0, \dots, v_k) = l \rangle$$

be the free R -module on the paths of length l on $k + 1$ vertices of G . The differential $\partial : \text{MC}_{k,l}(G; R) \rightarrow \text{MC}_{k-1,l}(G; R)$ is defined on $(k + 1)$ -tuples (v_0, \dots, v_k) by

$$\partial(v_0, \dots, v_k) := \sum_{i=1}^{k-1} (-1)^i \partial_i(v_0, \dots, v_k),$$

where $\partial_i(v_0, \dots, v_k) = (v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_k)$ assuming that $\ell(v_0, \dots, v_k) = l = \ell(v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_k)$, and ∂_i is set to be 0 otherwise.

For a given $l \in \mathbb{N}$, the pair $(\text{MC}_{*,l}(G; R), \partial)$ is a chain complex by [17, Lemma 2.11].

Definition 2.2 [17, Definition 2.4]. The *magnitude homology* $\text{MH}_{*,*}(G; R)$ of G is defined as the bigraded R -module $\bigoplus_{k,l \geq 0} \text{MH}_{k,l}(G; R)$, where

$$\text{MH}_{k,l}(G; R) := H_k(\text{MC}_{*,l}(G; R), \partial)$$

is given by the homology of the magnitude chain complex.

We can reinterpret the definition of magnitude homology groups as follows — see also [1, Section 2] for the case of directed graphs. Given a non-negative integer k , define $\Lambda_k(G; R)$ as the R -module freely generated by the $(k + 1)$ -tuples of vertices of G , that is,

$$\Lambda_k(G; R) := R\langle (v_0, \dots, v_k) \mid v_i \in V(G) \rangle.$$

Consider the submodule $I_k(G; R) := R\langle (v_0, \dots, v_k) \mid v_i = v_{i+1} \text{ for some } i \rangle$ of $\Lambda_k(G; R)$, where $I_0(G)$ is set to 0. Note that these modules can be equipped with a differential d , defined analogously to ∂ , which makes them chain complexes. Since $I_k(G; R) \subseteq \Lambda_k(G; R)$, we can form the quotient chain complex with modules $R_k(G; R) := \Lambda_k(G; R)/I_k(G; R)$. Then, confining ourselves to the setting of undirected graphs, we get that the R -module $\text{MC}_{k,l}(G; R)$ can also be defined as the submodule of $R_k(G; R)$ given by $(k + 1)$ -tuples of vertices whose length is precisely l ; this is compatible with the chain complex structure – cf. [1, Lemma 2.14]. In particular, we get the following.

Remark 2.3. The magnitude chain modules $MC_{k,l}(G; R)$ are sub-quotients of the free R -module $\Lambda_k(G; R)$.

It is possible to modify the differential δ so to get a new differential

$$\delta' : MC_{k,l}(G; R) \rightarrow MC_{k-1,l-1}(G; R),$$

in turn inducing an homomorphism $\partial' : MH_{k,l}(G; R) \rightarrow MH_{k-1,l-1}(G; R)$ between the magnitude homology groups. Equipped with this new differential, also magnitude homology can be seen as a chain complex $(MH_{k-*}^{l-*}(G; R), \partial')$. The homology of the resulting chain complex was denoted by $\mathcal{MH}_{k-*}^{l-*}(G)$ in [1, Definition 2.21].

Remark 2.4 [1, Proposition 6.11]. For $k = l$, the homology theory $\mathcal{MH}_{k-*}^{l-*}(G)$, for directed graphs, recovers the (reduced) path homology of directed graphs introduced in [11]. The same proof, in the undirected setting, produces an isomorphism of $\mathcal{MH}_{k-*}^{l-*}(G)$ for undirected graphs, with the reduced path homology of graphs — see, for example, [3, Section 2.2] for the definition.

We take the opportunity to point out here that both magnitude homology and path homology appear as instances of the so-called magnitude-path spectral sequence. The connection described in Remark 2.4, between path homology and magnitude homology, was further investigated in [15], and extended to the whole second page of the spectral sequence. Furthermore, in [15], it was shown that each page of the magnitude-path spectral sequence yields an homology theory of (di)graphs. This is also the case for the ∞ -page, called *reachability homology* — see [6, 16].

Magnitude homology is an homology theory of (directed) graphs, and it is functorial with respect to contractions. Recall that a contraction of a graph G with respect to an edge e is the graph obtained from G by contracting e to a point. More specifically, if G and H are graphs, consider maps $\phi : G \rightarrow H$ of vertices that preserve or contract each edge of G . Observe that such maps do not increase the length of tuples of vertices of G : that is, we have $\ell(\phi(v_0), \dots, \phi(v_k)) \leq \ell(v_0, \dots, v_k)$. Every contraction $\phi : G \rightarrow H$ of graphs induces a chain map

$$\phi_{\#} : MC_{*,*}(G; R) \rightarrow MC_{*,*}(H; R),$$

which to a tuple (v_0, \dots, v_k) of G associates the tuple $(\phi(v_0), \dots, \phi(v_k))$ if the length $\ell(\phi(v_0), \dots, \phi(v_k))$ equals the length $\ell(v_0, \dots, v_k)$, and it is set to be 0 otherwise. The map $\phi_{\#}$ is a chain map, as it commutes with the differential δ , and it induces a map in magnitude homology. Recall that we denote by **CGraph** the category of graphs and contractions.

Proposition 2.5 [17, Proposition 3.3]. *Magnitude homology is a functor*

$$MH_{*,*} : \mathbf{CGraph} \rightarrow \mathbf{BiGrMod}_R$$

from the contraction category of graphs to the category of bigraded R -modules.

By dualising the definition of magnitude homology, as customary, we get the definition of magnitude cohomology — see [14] — which we now recall:

Definition 2.6. *Magnitude cohomology* MH_*^* is the cohomology of the complex

$$MC_l^k(G; R) = \text{Hom}(MC_{l,k}(G; R); R),$$

equipped with the dual differential.

This defines a functor with respect to the dual maps inducing functoriality in magnitude cohomology — cf. [14, Definition 2.2]. In particular, the dualisation defines a functor

$$MH_*^* : \mathbf{CGraph}^{\text{op}} \rightarrow \mathbf{BiGrMod}_R.$$

We recall that in $\mathbf{CGraph}^{\text{op}}$, there is a morphism $G \rightarrow G'$ if, and only if, the graph G is obtained from the graph G' by a sequence of contractions. We conclude the section observing that magnitude homology and magnitude cohomology are related by a universal coefficients short exact sequence by [14, Remark 2.5]. Thanks to this short exact sequence, results on magnitude homology of graphs can be derived from results on magnitude cohomology, as customary. In the next subsections, we will restrict ourselves to the case of magnitude cohomology.

2.2 | Magnitude cohomology is finitely generated

In order to prove that magnitude cohomology (in a fixed k -degree) is finitely generated, we exhibit a finitely generated module such that magnitude cohomology is a subquotient of it.

Recall that \mathbf{Graph} denotes the category of graphs and minor morphisms and let R be a commutative unital Noetherian ring.

Definition 2.7. The *vertex module* is the functor $\mathcal{V} : \mathbf{Graph} \rightarrow R\text{-Mod}$, which assigns to each graph G the R -module

$$\mathcal{V}(G) = R\langle v \mid v \in V(G) \rangle$$

freely generated by the vertices of G . To each minor morphism $\phi : G \rightarrow G'$, that is, if G' is obtained from G via contractions and deletions, it assigns the map $\mathcal{V}(\phi) : \mathcal{V}(G) \rightarrow \mathcal{V}(G')$ given by $v \mapsto \phi(v)$.

Consider the restriction of the module $\mathcal{V}^{\otimes k}$ to the category \mathbf{CGraph} of graphs and contractions. Denote by $MH_{k,*}(-; R)$ the sum $\bigoplus_l MH_{k,l}(-; R)$.

Lemma 2.8. *For each k , the magnitude homology $MH_{k,*}(-; R)$ is a sub-quotient of the module $\mathcal{V}^{\otimes k+1}$.*

Proof. It is sufficient to notice that $\Lambda_k(G)$ can be identified with $\mathcal{V}^{\otimes k+1}(G)$, by identifying (v_0, \dots, v_k) with the elementary tensor $v_0 \otimes \dots \otimes v_k$. The maps which induce the functoriality in magnitude homology are induced by the identification of the vertices of the contracted edges. Taking the directed sum over all lengths l , we get the result. \square

Magnitude cohomology MH_*^* is by definition the cohomology of the dual complex of the magnitude homology of graphs. If a module M is a subquotient of a module N , then it is not generally

true that $\text{Hom}(M, R)$ is a subquotient of $\text{Hom}(N, R)$. However, this is the case for magnitude cohomology; in fact, we have the following.

Proposition 2.9. *For each non-negative integer k , magnitude cohomology $\text{MH}_*^k(-; R) = \bigoplus_l \text{MH}_l^k(-; R)$ is a sub-quotient of $\text{Hom}_R(\mathcal{V}^{\otimes k+1}, R)$.*

Proof. Fix l a natural number and consider the magnitude chain group $\text{MC}_{k,l}(G; R)$ of a graph G . This is a sub-quotient of $\mathcal{V}^{\otimes k+1}(G)$ by Lemma 2.8. Observe that the modules $\Lambda_k(G; R), I_k(G; R)$ and $R_k(G; R)$ are free R -modules. Thus, $\text{Hom}_R(R_k(G; R), R)$ can be seen as a submodule of $\text{Hom}_R(\mathcal{V}^{\otimes k+1}(G; R), R)$ — since $\Lambda_k(G; R) \cong \mathcal{V}^{\otimes k+1}(G)$ canonically. By definition, $R_k(G; R) := \Lambda_k(G; R)/I_k(G; R)$ consists of tuples of subsequently distinct elements of $V(G)$, and $\text{MC}_{k,l}(G; R)$ is given by the tuples (v_0, \dots, v_k) in $R_k(G; R)$ of length l . Using the parameter l , we can filter the free module $R_k(G; R)$. Using such filtration, also the magnitude homology groups $\text{MC}_{k,l}(G; R)$ are spanned by elements of a basis of $R_k(G; R)$.

We have a short exact sequence

$$0 \rightarrow \text{MC}_{k,l}(G; R) \rightarrow R_k(G; R) \rightarrow Q_k(G; R) \rightarrow 0, \tag{1}$$

where $Q_k(G; R)$ is the associated quotient, which is also a free R -module. In particular, the group $\text{Ext}_R^i(Q_k(G; R), M)$ are zero for all $i \geq 1$ and R -module M . Hence, dualising the short exact sequence in Equation (1), we get the short exact sequence

$$0 \rightarrow \text{Hom}_R(Q_k(G; R), R) \rightarrow \text{Hom}_R(R_k(G; R), R) \rightarrow \underbrace{\text{Hom}_R(\text{MC}_{k,l}(G; R), R)}_{=\text{MC}_l^k(G; R)} \rightarrow 0$$

from which the statement follows. □

Our goal is now to show that the module $\text{Hom}_R(\mathcal{V}^{\otimes k}, R)$ is finitely generated, for all $k > 0$. As we do not know whether the full category of graphs and minor morphisms, or simply contractions, is Noetherian, we shall restrict to the subcategory of graphs with bounded genus.

For any fixed graph G , the R -module $\text{Hom}_R(\mathcal{V}^{\otimes k}(G), R)$ is freely generated (as R -module) by the functions $\delta_{v_1 \otimes \dots \otimes v_k}^G : \mathcal{V}^{\otimes k}(G) \rightarrow R$, defined by

$$\delta_{v_1 \otimes \dots \otimes v_k}^G(w_1 \otimes \dots \otimes w_k) = \begin{cases} 1 & \text{if } w_i = v_i \text{ for all } i, \\ 0 & \text{otherwise.} \end{cases}$$

Denote by

$$M_k(G) = \text{span}_R(\psi_c(\delta_{w_1 \otimes \dots \otimes w_k}^G) \mid c \in \text{Hom}_{\mathbf{CGraph}^{\text{op}}}(G', G), G' \neq G, w_i \in V(G') \forall i),$$

where $\psi_c : \text{Hom}_R(\mathcal{V}^{\otimes k}(G'), R) \rightarrow \text{Hom}_R(\mathcal{V}^{\otimes k}(G), R)$ denotes the map induced by the contraction $c : G \rightarrow G'$. More precisely, ψ_c is just the pre-composition with the map $\mathcal{V}(c) : \mathcal{V}(G) \rightarrow \mathcal{V}(G')$. We start by proving the following.

Proposition 2.10. *Given a connected graph G with more than $k + 1$ vertices, we have that $\text{Hom}_R(\mathcal{V}^{\otimes k}(G), R) = M_k(G)$.*

Proof. We start by first establishing some notation. For $v_1, \dots, v_k \in V(G)$, define

$$a = a(v_1, \dots, v_k) = k - |\{v_1, \dots, v_k\}|.$$

That is, a is k minus the number of distinct v_i s. Furthermore, define

$$l = l(v_1, \dots, v_k) = \min\{d_G(v, w) \mid v, w \in V(G) \setminus \{v_1, \dots, v_k\}, v \neq w\} \geq 1,$$

where d_G is the shortest path metric. Note that l is well defined because there are at least $k + 2$ vertices. With this notation in place, we can associate to each k -tuple (v_1, \dots, v_k) a pair of (non-negative) integers (a, l) . The set of all possible values of (a, l) is totally ordered by the lexicographic order $<_{\text{lex}}$. We shall prove by recursion that

$$\delta_{v_1 \otimes \dots \otimes v_k}^G \in M_k(G), \tag{2}$$

for all choices of $v_1, \dots, v_k \in V(G)$. As $M_k(G)$ is by definition a submodule of the R -module $\text{Hom}_R(\mathcal{V}^{\otimes k}(G), R)$, the statement will follow.

Firstly, we prove the claim for all (v_1, \dots, v_k) such that $l(v_1, \dots, v_k) = 1$ (and for any value of a). In this case, there are two vertices $v, w \in V(G) \setminus \{v_1, \dots, v_k\}$ which are the endpoints of an edge e . Then, if c denotes the contraction of e , we have

$$\psi_c \left(\delta_{v_1 \otimes \dots \otimes \bar{v}_k}^{G'} \right) = \delta_{v_1 \otimes \dots \otimes v_k}^G,$$

where $\bar{v}_k := c(v_k)$. This proves (2) in this case.

We now recursively assume that (2) holds for all (w_1, \dots, w_k) such that

$$(a(w_1, \dots, w_k), l(w_1, \dots, w_k)) <_{\text{lex}} (a_0, l_0).$$

Consider $v_1, \dots, v_k \in V(G)$ such that: $a(v_1, \dots, v_k) = a_0$ and $l(v_1, \dots, v_k) = l_0$. Let $v, w \in V(G) \setminus \{v_1, \dots, v_k\}$ be such that $d_G(v, w) = l_0$. We can assume $l_0 > 1$, since we already dealt with the case $l_0 = 1$. In this case, any minimal path between v and w contains at least one other vertex. Furthermore, by path-minimality, all vertices but the endpoints on such a path belong to $\{v_1, \dots, v_k\}$. Let v_s be the first vertex after v on such a minimal path. Consider the set $I_s = \{j \mid v_j = v_s\}$. Then, denoted by $c : G \rightarrow G'$ the contraction of an edge between v and v_s , by definition of ψ_c , we have

$$\psi_c \left(\delta_{v_1 \otimes \dots \otimes \bar{v}_k}^{G'} \right) = \sum_{S \subseteq I_s} \delta_{x_1^S \otimes \dots \otimes x_k^S}^G, \tag{3}$$

where x_j^S equals v if $j \in S$, and v_j otherwise. Clearly, $x_1^\emptyset \otimes \dots \otimes x_k^\emptyset = v_1 \otimes \dots \otimes v_k$. Therefore, we can rewrite (3) as:

$$\delta_{v_1 \otimes \dots \otimes v_k}^G = \psi_c \left(\delta_{v_1 \otimes \dots \otimes \bar{v}_k}^{G'} \right) - \sum_{\substack{S \subseteq I_s \\ S \neq \emptyset}} \delta_{x_1^S \otimes \dots \otimes x_k^S}^G. \tag{4}$$

For each $S \subseteq I_s$ non-empty, we have either $a(x_1^S, \dots, x_k^S) < a_0$, or $S = I_s$. In the former case, we are done by recursion. If $S = I_s$, then we replaced, in the tensor product, each occurrence of v_s with

v . It follows that, $a(x_1^S, \dots, x_k^S) = a_0$. Furthermore, we have

$$l(x_1^S, \dots, x_k^S) \leq d_G(v_s, w) = d_G(v, w) - 1 = l(v_1, \dots, v_k) - 1 < l_0.$$

Consequently, by recursive hypothesis, we obtain also in this case that

$$\sum_{\substack{S \subseteq I_s \\ S \neq \emptyset}} \delta_{x_1^S \otimes \dots \otimes x_k^S}^G \in M_k(G),$$

concluding the proof. □

We are now ready to prove the main theorem of this section. Fix an integer $k \geq 1$.

Theorem 2.11. *The $\mathbf{CGraph}_{\leq g}^{\text{op}}$ -module $\text{Hom}_R(\mathcal{V}^{\otimes k}, R)$ is finitely generated. Moreover, a set of generators is given by all graphs of genus at most g with at most $k + 1$ vertices.*

Proof. Let N_k be the sub-module of the $\mathbf{CGraph}_{\leq g}^{\text{op}}$ -module $\text{Hom}_R(\mathcal{V}^{\otimes k}(-); R)$ spanned by the set $S = \{\delta_{v_1^G \otimes \dots \otimes v_k^G}^G\}$, where G ranges among all graphs of genus at most g with at most $k + 1$ vertices, and v_1, \dots, v_k range among the vertices of G . These graphs are a finite number since the genus is bounded, and hence, S is a finite set.

Clearly, $N_k(G) = \text{Hom}_R(\mathcal{V}^{\otimes k}(G); R)$ for all graphs with at most $k + 1$ vertices.

We proceed by induction on the number of vertices to cover the remaining cases. Suppose that, whenever G' has less than $r > k + 1$ vertices $N_k(G') = \text{Hom}_R(\mathcal{V}^{\otimes k}(G'); R)$. Then, by Proposition 2.10, every $\phi \in \text{Hom}(\mathcal{V}^{\otimes k}(G); R)$ can be written as

$$\phi = \sum_{i=1}^n \alpha_i \psi_{c_i} \left(\delta_{v_1^i \otimes \dots \otimes v_k^i}^{G_i} \right),$$

for some contractions $c_i : G \rightarrow G_i$, and some $v_1^i, \dots, v_k^i \in V(G_i)$. Since the c_i s are contractions, G_i has at most $r - 1$ vertices for each i . Hence, $\delta_{v_1^i \otimes \dots \otimes v_k^i}^{G_i} \in N_k(G_i)$ by inductive hypothesis. Since $\psi_{c_i}(N_k(G_i)) \subseteq N_k(G)$ for each i , the first part of the statement follows. To conclude, note that the generators of the module N_k are all graphs of genus at most g with at most $k + 1$ vertices, in view of Lemma 1.2. □

Corollary 2.12. *The $\mathbf{CGraph}_{\leq g}^{\text{op}}$ -module $\text{MH}_l^k(-; R)$ is finitely generated.*

Proof. In view of Theorem 1.8, sub-quotients of finitely generated $\mathbf{CGraph}_{\leq g}^{\text{op}}$ -modules are finitely generated. By Theorem 2.11, the $\mathbf{CGraph}_{\leq g}^{\text{op}}$ -module $\text{Hom}_R(\mathcal{V}^{\otimes k+1}, R)$ is finitely generated. Therefore, the statement follows from Proposition 2.9. □

3 | APPLICATIONS

A first consequence of the finite generation property described in Corollary 2.12 is a bound on the magnitude (co)homology ranks that depends on the number of edges of the considered graphs.

Our first corollary is a general result for graphs of bounded genus, and the second is related to subsequent subdivisions of the edges.

Corollary 3.1. *Let \mathbb{K} be a field, and $g \geq 0$. Then, there exists a polynomial $f \in \mathbb{Z}[t]$ of degree at most $g + k + 1$, such that, for all G of genus at most g , we have*

$$\dim_{\mathbb{K}} \text{MH}_*^k(G; \mathbb{K}) \leq f(\#E(G)),$$

where $\#E(G)$ is the number of edges of G .

Proof. The statement directly follows from [29, Proposition 4.3] after noticing that magnitude cohomology, in cohomological degree k and independently on the length degree l , is a subquotient of a module which is finitely generated in degree $\leq g + k + 1$ — since it is generated by graphs of genus $\leq g$ and at most $k + 2$ vertices, cf. Theorem 2.11. \square

Let G be a graph of genus g , $\underline{e} = (e_1, \dots, e_r)$ a tuple of distinct edges of G which are not self-loops. We fix a direction on e_1, \dots, e_r . This extra data is auxiliary, that is, the choice of the direction is immaterial, but it is needed to explicitly write down the functor below. For a tuple $\underline{m} = (m_1, \dots, m_r)$ of non-negative integers, we let $G(\underline{e}, \underline{m})$ be the graph obtained from G by subdividing each edge e_i a number of m_i times. If $m_i = 0$, then ‘subdivision’ means ‘contraction’ of the edge e_i . Denote by \mathbf{OI} the category of linearly ordered finite sets and ordered inclusions. Consider the product category \mathbf{OI}^r . The directions on the edges e_i have been chosen in order to construct a subdivision functor

$$\Phi_{G, \underline{e}} : \mathbf{OI}^r \rightarrow \mathbf{CGraph}_{\leq g}^{\text{op}},$$

which associates to a linearly ordered set $[\underline{m}] \in \mathbf{OI}^r$ the graph $G(\underline{e}, \underline{m})$, and to a morphism $[\underline{m}] \rightarrow [\underline{n}]$ in \mathbf{OI}^r a contraction $G(\underline{e}, \underline{n}) \rightarrow G(\underline{e}, \underline{m})$ — see [29, Section 4.2] for the details. Then, [29, Proposition 4.4] implies that, if \mathcal{M} is a $\mathbf{CGraph}_{\leq g}^{\text{op}}$ -module which is a subquotient of a module $\mathbf{CGraph}_{\leq g}^{\text{op}} \rightarrow R\text{-Mod}$ (with R here a field) that is finitely generated in degrees $\leq d$, that is, by graphs with at most d edges, then the dimension of $\mathcal{M}(G(\underline{e}, \underline{m}))$ is bounded by a polynomial in \underline{m} of degree $\leq d$ — cf. [29, Corollary 4.5]. As a consequence, we have the following.

Corollary 3.2. *Let \mathbb{K} be a field and G be a graph of genus g . Then, there exists a polynomial $f_{G, \underline{e}}(x_1, \dots, x_r)$ of total degree at most $g + k + 1$ such that*

$$\dim_{\mathbb{K}} \text{MH}_l^k(G(\underline{e}, \underline{m}); \mathbb{K}) = f_{G, \underline{e}}(m_1, \dots, m_r),$$

provided that \underline{m} is large enough in each entry.

Proof. The statement directly follows from [29, Corollary 4.5] since magnitude cohomology is a subquotient of a module which is finitely generated in degree $\leq g + k + 1$. \square

Likewise, a similar result is obtained by considering the functor $\text{MH}_*^k = \bigoplus_l \text{MH}_l^k$, as the finite generation of the magnitude cohomology functor only depends on k and not on l .

Example 3.3. Let C_m be a cycle graph with m edges — see Figure 1. Subdivision of any edge of C_m yields again a cycle. The magnitude homology groups of cycles have been computed in

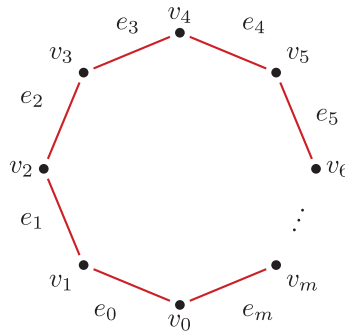


FIGURE 1 The cycle graph C_m .

[12, Theorems 4.6 & 4.8], proving a conjecture of [17, Appendix A.1]. For a fixed k , it can be shown that, in their notation, the dimension satisfies:

$$T_{k,l}^m = \text{rank MH}_{k,l}(C_m) = a(k, l)m + b(k, l). \tag{5}$$

Note that $a(k, l), b(k, l) \neq 0$ for finitely many values of l , whose number is dependent on k . Hence, for each k , we can set $A(k) = \sum_l a(k, l)$ and $B(k) = \sum_l b(k, l)$. It follows that $T_{k,l}^m \leq A(k)m + B(k)$.

Example 3.3 implies that Corollary 3.1 and Corollary 3.2 do not provide sharp results. However, they generalise what happens in the aforementioned example to graphs of fixed or bounded genus. More precisely, Corollary 3.1 is the analogue of the inequality $T_{k,l}^m \leq A(k)m + B(k)$ in Example 3.3. Corollary 3.2 is the analogue of the formula in Equation (5), for families of graphs obtained via subdivision of edges. By applying [29, Corollary 4.7], we get a similar statement when considering the operation of ‘gluing’ trees to G . We refer to [29] for more applications.

A second application, of main interest to us, concerns the behaviour of torsion in magnitude (co)homology. It was shown in [32, Theorem 3.14] that any finitely generated Abelian group may appear as a subgroup of the magnitude homology of a graph, and that there are infinitely many such graphs. More precisely, Sadzanovic and Summers proved the following.

Theorem 3.4 [32, Theorem 3.13]. *Let p be a prime and $n, m \geq 1$ integers. There exist infinitely many distinct isomorphism classes of graphs whose magnitude homology contains \mathbb{Z}_{p^m} torsion in bigrading $(3, 2n + 3)$.*

The proof of [32, Theorem 3.13] is based on Kaneta–Yoshinaga construction [19]. Graphs whose integral magnitude homology has p^r -torsion are obtained from triangulations of (generalised) lens spaces and iterated subdivisions. However, there is no structural theorem concerning the complexity of graphs having given torsion in integral magnitude homology. The next result suggests that, in order to find more torsion in magnitude (co)homology, one needs to increase the combinatorial complexity of the graphs.

Theorem 3.5. *For every pair of integers $k, g \geq 0$, there exists $m = m(g, k) \in \mathbb{Z}$ which annihilates the torsion subgroup of $\text{MH}_*^k(G; \mathbb{Z})$, for each graph G of genus at most g .*

Proof. Firstly, note that magnitude homology and magnitude cohomology are related by a universal coefficients short exact sequence by [14, Remark 2.5]; hence, we can restrict to magnitude cohomology (and results for magnitude homology will be derived by application of such short exact sequence).

Fix a degree k , and take $R = \mathbb{Z}$ the integers. By Corollary 2.12, the $\mathbf{CGraph}_{\leq g}^{\text{op}}$ -module $\text{MH}_l^k(-; \mathbb{Z})$ is finitely generated. Let G be a graph of bounded genus $\leq g$, and τ a torsion class in $\text{MH}_l^k(G; \mathbb{Z})$. For $\phi: H \rightarrow G$ a contraction of graphs, we get by functoriality a map in magnitude cohomology that preserves the torsion class. Therefore, we can consider the submodule $\mathcal{T} \subseteq \text{MH}_l^k(-; \mathbb{Z})$ which sends a graph G to the \mathbb{Z} -module $\text{MH}_l^k(G; \mathbb{Z})$. Then, by Corollary 1.8, the $\mathbf{CGraph}_{\leq g}^{\text{op}}$ -module \mathcal{T} is also finitely generated. But, by definition, this means that there exist graphs $G_1, \dots, G_{m(k)}$ of genus bounded by g , and a surjection $\bigoplus_{i=1}^{m(k)} \mathcal{P}_{G_i} \rightarrow \mathcal{T}$ from the associated principal projectives. Now, choose N to be the least common multiple of the annihilators of $\mathcal{T}(G_1), \dots, \mathcal{T}(G_{m(k)})$; then, for any graph G in $\mathbf{CGraph}_{\leq g}^{\text{op}}$, the torsion part of $\text{MH}_l^k(G; \mathbb{Z})$ has exponent at most N . This concludes the proof. \square

From Remark 2.4 and [1] follows that (reduced) path homology, as introduced in [11], appears as the diagonal in the second page of a spectral sequence whose 0th page features magnitude chain groups. Shifting to later pages in the spectral sequence is obtained by subsequent subquotients. Therefore, after restricting to undirected graphs, Theorem 2.11 yields, with the same proof of Theorem 3.5, the following.

Corollary 3.6. *For each g, k positive integers, there exists a $d = d(g, k) \in \mathbb{Z}$ such that, for each graph G of genus g , the torsion part of the path cohomology $\text{PH}^k(G, \mathbb{Z})$ has exponent at most d .*

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ORCID

Carlo Collari  <https://orcid.org/0000-0003-0034-0702>

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