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SPECTRUM BOUNDS IN GEOMETRY

ANTONELLA GRASSI

To V. V. Shokurov for his 70+2 birthday

Abstract. Filipazzi, Hacon and Svaldi proved that there are only finitely many topological types of elliptically fibered Calabi-Yau threefolds. We explore the implications of their results on the boundedness of the geometric quantities in the massless spectrum of the F-theory Calabi-Yau compactifications. A key ingredient is what we call the geometric anomaly equation, and extension of the gravitational anomaly cancellation in physics, also to singular spaces. We review and extend the dictionary between geometry and physics. We conclude with explicit bounds.

1. Introduction

Recently Filipazzi, Hacon and Svaldi [6] have shown that there are only finitely many types of topologically different elliptically fibered Calabi-Yau threefolds, answering a long standing question from physics. We explore the implications of their results and the previous work of Gross [16] as well as [7] on the boundedness of the geometric quantities in the massless spectrum of the F-theory Calabi-Yau compactifications. The results build also on the work of Prokhorov and Shokurov [25].

An F-theory compactification on a Calabi-Yau variety X is an elliptic fibration $X \to B$. From now on, the threefolds are intended to be complex projective Calabi-Yau varieties. An important question coming from string theory is the boundedness of the massless spectrum of the F-theory compactifications, also in the context of the "Swampland Program" [31]. The geometry of the Calabi-Yau, the structure of the fibration and the massless particle spectrum in physics are closely related. In Result from Physics 2.7, (1)-(4) we review the descriptions of the spectrum for F-theory compactifications. The dictionary between the quantities (1)-(2)-(3) in the physics spectrum and the geometry is immediate. The geometric counterparts of the quantity (4), the charged hypermultiples H_{ch}, are discussed in Section 3.2. In the same section we also outline how to calculate explicitly some of the other geometric quantities in the spectrum, with some relevant examples in the Appendix 6. We extend the dictionary first to Calabi-Yau varieties with $\mathbb{Q}$-factorial terminal singularities with Definition 3.2–(3') and then to singular bases B with isolated multiple fibers over the singularities of B, by adding (5) to the (modified) spectrum in Definition 3.11 ; see also [3], [13]. These additions to the massless

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spectrum are consistent with the findings in physics; we show also that they are natural from the geometry of the Calabi-Yau. In Corollary 2.9, Corollary 3.3 and Corollary 3.14 we show the boundedness of the quantities in (1)-(2)-(3)-(3′)-(5) in the spectrum.

To prove the boundedness of the quantity (4) in the spectrum 2.7, the charged multiples H_{ch}, we use the Gravity Anomaly Cancellations and its extensions (Corollaries 2.11, 3.8 and 3.15). (In the physics theory the Gravitational Anomaly Cancellation Formulas must in fact be satisfied by the quantities in the spectrum for the theory to be consistent.) However, the Gravitational Anomaly Cancellation 2.10 was originally derived in physics for fibrations with section between manifolds. We extend it in Proposition 3.6 and Theorem 3.13 to Calabi-Yau with $\mathbb{Q}$-factorial terminal singularities and then to equidimensional fibrations with isolated multiple fibers; see also [13].

In Corollary 4.1 and Corollary 4.2 we apply the boundedness of the spectrum and a geometric formulation of the anomaly cancellation to deduce boundedness of the type of singular fibers (and of the matter representations) and the components of the discriminant. We conclude with explicit bounds in Section 5.

2. Elliptic Fibrations and F-theory

All the varieties are assumed to be complex projective algebraic.

Definition 2.1. A Calabi-Yau threefold X is a variety with $\mathbb{Q}$-factorial terminal singularities, $h^i(X, \mathcal{O}_X) = 0$, $i = 1, 2$, and $K_X \sim 0$.

Definition 2.2. An elliptic fibration is a morphism $\pi : X \to B$ whose fibers over a dense set in B are elliptic curves. The complement of this dense locus in B, the discriminant of the fibration, is denoted by $\Sigma_{X/B}$. Let $\{\Sigma_j\}$ be the irreducible codimension one components of $\Sigma_{X/B}$. Let $\Sigma_m \stackrel{def}{=} \{Q \in B \text{ such that } \pi^{-1}(Q) \text{ is a multiple fiber}\}$

Theorem 2.3. *Let Y be a birational Calabi-Yau threefold and $Y \to S$ be an elliptic fibration. Then there exists a birational equivalent elliptic fibration $\pi : X \to B$ such that X is minimal, with $\mathbb{Q}$-factorial terminal singularities and*

(1) (B, Δ) has $\mathbb{Q}$-factorial klt singularities, $K_X \equiv \pi^*(K_B + \Delta)$ and Δ is supported on the discriminant of the fibration.
(2) $\pi : X \to B$ is equidimensional.
(3) The locus Σ_m consists of a set of points.
(4) If B is singular at Q then $Q \in \Sigma_m$ and the singularity is a rational double point of type A_n.
(5) If $\Delta \neq 0$, B is rational.
(6) If $\Delta = 0$ the singularities of B are rational double points, the minimal resolution of B is an Enriques surface, the fibration π is isotrivial.

Proof. (1)-(2) and (5)-(6) are proved in previous works of the author [7], [8] and [9]. (3) follows from Kodaira's canonical divisor formula for surfaces and $K_X \simeq \mathcal{O}_X$. (4) follows form [8], [7] and [16]. □

The converse of the statement (4) does not hold: The analysis of the Example B in Appendix B of [24] gives a counterexample, namely there is an elliptic fibration $\pi : X \to B$, X and B smooth, $Q \in \Sigma_m$ and in a divisorial component of Σ.

2.1. F-theory, Massless spectrum, Gravitational Anomaly, for manifolds.

F-theory compactifications were first constructed on elliptic Calabi-Yau manifolds with a section[1] [21, 22, 30]. When there is a section there are no multiple fibers and the base of the fibration B can be assumed to be birationally smooth 2.3. The physics theory with the presence of multiple fibers is still being understood. We discuss the multiple fibers in Section 3.1.

Result (Physics) 2.4 (Gauge algebra)**.** *Let $X \to B$ be an equidimensional elliptic fibration with X a smooth Calabi-Yau threefold. The gauge algebra of the F-theory compactified to $X \to B$ is a naturally associated algebra $\mathfrak{g} = \bigoplus_j \mathfrak{g}(\Sigma_j) \oplus \mathfrak{u}(1)^{\oplus r}$ where*

- *$\mathfrak{g}(\Sigma_j)$ are either semisimple Lie algebras or "twisted algebras", which are quotients of the extended (affine) semisimple Lie algebras,*
- *the sum is taken over the irreducible divisorial components $\{\Sigma_j\}$ of the discriminant locus,*
- *r is the rank of the Mordell-Weil group of sections of the elliptic fibration.*

$\mathfrak{u}(1)^{\oplus r}$ is called the abelian component and $\bigoplus_j \mathfrak{g}(\Sigma_j)$ the non-abelian component of $\mathfrak{g}$. We write $\mathfrak{g}_j \overset{def}{=} \mathfrak{g}(\Sigma_j)$.

Remark 2.5. **(Geometry)** There are several proposed methods to determine the algebras $\mathfrak{g}(\Sigma_j)$ with mathematical constructions. If the fiber over the general point of Σ_j is a cusp or a node $\mathfrak{g}(\Sigma_j) = \{e\}$. If the fibration has a section $\mathfrak{g}(\Sigma_j)$ is either the simply laced Lie algebra associated to the Kodaira fiber over the general point of Σ_j, or the non-simply laced ones, a quotient by an outer automorphism of the simply laced ones if the rational curves in the general fiber of Σ_j have a relation in the Mori cone $\mathrm{NE}(X)$ of effective curves. If the fibration does not have a section, *twisted* algebras might also occur in the presence of multiple fibers, again when the rational curves in the general fiber of Σ_j are dependent in $\mathrm{NE}(X)$. The twisted algebra are then counted with multiplicity, [4] and [18]. We present and review different constructions and properties in [13].

[1] In the physics literature an elliptic fibration is assumed to be an elliptic fibration with a section, otherwise it called a genus one fibration.

By duality, the construction implies that the following always hold:

Result (Physics) 2.6. $\sum_j \mathrm{rk}\,\mathfrak{g}(\Sigma_j) < \mathrm{rk}(\mathrm{Pic}(X))$.

Result (Physics) 2.7 (Gauge algebra and spectrum)**.** *Let $X \to B$ be an equidimensional elliptic fibration with X a smooth Calabi-Yau threefold and B smooth. The massless physical spectrum of the gauge algebra $\mathfrak{g}$ of the F-theory compactified on X consists of*

(1) $V = h^{1,1}(X) - h^{1,1}(B) - 1 + \dim\mathfrak{g} - \mathrm{rk}\,\mathfrak{g}$ *vector multiplets,*
(2) $T = h^{1,1}(B) - 1$ *tensor multiplets,*
(3) $H = H_{unch} + H_{ch}$ *hypermultiplets, where* $H_{unch} = h^{2,1}(X) + 1 = \frac{1}{2}b_3(X)$ *is the number of uncharged multiplets and* H_{ch} *the number of hypermultiplets charged under* $\mathfrak{g}$*,*
(4) one universal gravity multiplet.

Remark 2.8. **(Geometry)** After $\mathfrak{g}$ is determined, the quantities in the massless spectrum have immediate geometric counterparts, with the exception of the charged multiplets H_{ch}. They are charged by the gauge algebra $\mathfrak{g}$. In the geometry of F-theory compactifications, the hypermultiplets H_{ch} are associated to the singular fibers of the fibration (see Section 3.2 and [13]). The general definition and construction of H_{ch} in physics and in mathematics is still largely an open question. We refer for example to [13] [32] for a review of different mathematical constructions. We summarize the main points in Section 3.2.

Corollary 2.9 (Bounds, I)**.** *Let $X \to B$ be an equidimensional elliptic fibration with X smooth Calabi-Yau threefold and B smooth. The quantities (1), (2) and H_{unch} in (3) of the massless spectrum are bounded.*

Proof. By [6], the equidimensional model $X \to B$ is isomorphic to a fiber in a bounded family $\mathcal{X}$; $X \simeq X_t$, and $B \simeq S_t$, for some t in a bounded family $\mathbf{S}$. T is bounded and so is V, because the rk(Pic(X)) is bounded [16]. It follows also that H_{unch} is bounded. □

The 6-dimensional effective theory obtained by compactification of F-theory on X is consistent if the anomalies are satisfied. In particular

Result (Physics) 2.10 (Gravitational Anomalies [15])**.** *Let $X \to B$ be an equidimensional elliptic fibration without multiple fibers, with X a smooth Calabi-Yau threefold, B smooth and rational. The Gravitational Anomalies are cancelled by the 6-dimensional Green-Schwarz mechanism if the following equation holds:*

$$H - V + 29T = 273$$

where H, V and T are as in 2.7.

Corollary 2.9 then implies:

Corollary 2.11 (Bounds, I'). *Let $X \to B$ be an equidimensional elliptic fibration without multiple fibers, with X a smooth Calabi-Yau threefold, B smooth and rational. Assume that the Gravitational Anomalies in 2.10 are cancelled. Then H_{ch} is bounded.*

3. Massless spectrum, H_{ch} The geometric anomaly equation and singularities

We turn now our attention to the geometric interpretation and the extension of the anomaly cancellation formula to Calabi-Yau varieties with $\mathbb{Q}$-factorial terminal singularities. The motivation is twofold: the first one is to describe in geometric terms H_{ch}, the second to derive the (extended) anomaly cancellation formula consistent for mathematics and physics when X and B are also singular, as in Theorem 2.3.

In [3] and [13] we show that (3) in the massless spectrum 2.7 must be modified for Calabi-Yau with $\mathbb{Q}$-factorial terminal singularities. Gorenstein terminal singularities are isolated hypersurface singularities and we can define the Milnor number:

Definition 3.1. Let $(\mathcal{U}, 0) \subset \mathbb{C}^{n+1}$ be a neighborhood of an isolated hypersurface singularity $P = 0$, defined by $f = 0$. The Milnor number of P can be defined as

$$m(P) = \dim_{\mathbb{C}}(\mathbb{C}\{x_1, \ldots, x_{n+1}\}/{<f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{n+1}}>}).$$

Also: $\mathrm{CxDef}(X) + 1 = \frac{1}{2}(b_3(X) + \sum_P m(P))$ [23]. Then:

Definition 3.2. [Massless Spectrum with $\mathbb{Q}$-factorial terminal singularities] Let $X \to B$ be an equidimensional elliptic fibration without multiple fibers, with X a Calabi-Yau threefold with $\mathbb{Q}$-factorial terminal singularities and B smooth. (3) in Massless Spectrum 2.7 is replaced by

$$(3') \quad H_{unch} \stackrel{def}{=} \mathrm{CxDef}(X) + 1 = \frac{1}{2}(b_3(X) + \sum_P m(P)),$$

where $m(P)$ is the Milnor number of the hypersurface singularity at $\in X$.

The extended version of Corollary 2.9 holds:

Corollary 3.3 (Bounds, II). *Let $X \to B$ be an equidimensional elliptic fibration without multiple fibers, with X a Calabi-Yau threefold with $\mathbb{Q}$-factorial terminal singularities and B smooth. The quantities (1), (2) and $(3')$ of the extended massless spectrum are bounded.*

Proof. The Milnor number is a topological invariant of the isolated singularities of the minimal Calabi-Yau threefolds and we conclude by [6]. □

We write the results of [3] and [13] as:

Proposition 3.4. *Let $X \to B$ an equidimensional elliptic fibration without multiple fibers, with X a Calabi-Yau threefold with $\mathbb{Q}$-factorial terminal singularities and B smooth. Let the massless spectrum be as in Definition 3.2.*

The Anomaly Cancellation with the modification in Definition 3.2 remains formally unchanged as

$$H - V + 29T = 273.$$

Proof. The dimension of the complex deformations of X, $\mathbb{Q}$-factorial Calabi-Yau is computed by $\mathrm{CxDef}(X) = \frac{1}{2}(b_3(X) - 1 + \sum_m m(P))$ [23]. □

Remark 3.5. In [3] we also explicitly check the consistency in physics of the Gravitational Anomaly Cancellation with the modified definition 3.2.

Proposition 3.6. *Let $X \to B$ be an equidimensional elliptic fibration without multiple fibers, with X a Calabi-Yau threefold with $\mathbb{Q}$-factorial terminal singularities and B smooth. Let $m(P)$ be the Milnor number of the point $P \in X$. Then the Anomaly Cancellation Equation 2.10 with the modification of Definition 3.2 becomes:*

$$30K_B^2 + \frac{1}{2}(\chi_{top}(X) - \sum_P m(P)) \quad = \quad H_{ch} - (\dim \mathfrak{g} - \mathrm{rk}\,\mathfrak{g}).$$

Proof. We extend the arguments of [11] and [13]. Note that by hypothesis B can assumed to be smooth. The argument in [11, Section 6] for $\mathrm{rk}(MW(X/B)) \neq \{e\}$ and B rational builds on the relations (1), (2), (3) of the Physics Result 2.10 and on Noether's formula for smooth surfaces. The same relations provide the result when $\mathrm{rk}(MW(X/B)) \neq \{e\}$ and when there is no section, with Shioda-Tate-Wazir formula as modified by [4] for fibration without a section. The statement then follows from (5) and (6) in Theorem 2.3. □

Remark 3.7. The statement also includes the case of when B is an Enriques surface.

Corollary 3.8 (Bounds, II'). *Let $X \to B$ be an equidimensional elliptic fibration, with X a Calabi-Yau threefold with $\mathbb{Q}$-factorial terminal singularities. Then the Gravitational Anomaly Cancellation implies that H_{ch} is bounded.*

3.1. **Multiple fibers.**

The interpretation of the physics of elliptic fibrations in the presence of multiple fibers is still a topic in its initial stages and much investigation is needed. However we have the following:

Result (Physics) 3.9 ([1], [2], [18]). *Let $X \to B$ be an equidimensional elliptic fibration, X a Calabi-Yau threefold with $\mathbb{Q}$-factorial terminal singularities, Σ_m disjoint from the divisorial components of the discriminant and B possibly singular. The spectrum in 3.2 and the Anomaly Cancellation in Results 2.10 must be modified for the theory to be consistent.*

The following Definitions 3.11 and 3.12 are motivated by the findings for Calabi-Yau quotients and Calabi Yau symmetric toroidal orbifolds in [1] [2] and [18]. One example of such fibration is X, a quotient of an elliptic Calabi-Yau threefold Y by $\mathbb{Z}/(m+1)\mathbb{Z}$ such that the group acts without fixed points and it preserves the fibration. X is smooth with isolated multiple fibers over the singular points, of type A_m, of the quotient base. In fact

Remark 3.10. If $X \to B$ is any equidimensional elliptic fibration, X a Calabi-Yau threefold with $\mathbb{Q}$-factorial terminal singularities, then the singularities of B are at worse of type A_{m_i}, because there exists an equidimensional birational model $\bar{X} \to \bar{B}$ such that $\bar{B}$ has at worse A_{m_i} singularities (Theorem 2.3).

We can give the following:

Definition 3.11. [Massless Spectrum, with singularities and multiple fibers] Let $X \to B$ be an equidimensional elliptic fibration, with X a Calabi-Yau threefold with $\mathbb{Q}$-factorial terminal singularities. Assume that Σ_m is disjoint from the divisorial components of Σ, that is $\Sigma_m \not\subset \Sigma_j,\ \forall j$. If $Q_i \in \Sigma_m \subset B$ is a singular point, then the spectrum in 3.2 must also include for each such Q_i, necessarily of type A_{m_i},

(5.a) One neutral hypermultiplet
(5.b) (m_i) tensor multiplets

Definition 3.12. [Anomalies, with singularities and multiple fibers] In the same hypothesis of Definition 3.11, let n_i be the number of such points. The Anomaly Cancellation in Results 2.10 must be modified as follows for the theory to be consistent:

$$H - V + 29T + \sum_i n_i \cdot (m_i) = 273,$$

The Geometric Anomaly Cancellation Formula remains however unchanged:

Theorem 3.13 ([13])**.** *Let $X \to B$ be an equidimensional elliptic fibration, with X a Calabi-Yau threefold with $\mathbb{Q}$-factorial terminal singularities and B possibly singular. Assume that Σ_m is disjoint from the divisorial components of Σ, that is $\Sigma_m \not\subset \Sigma_j,\ \forall j$. Let $m(P)$ be the Milnor number of the point $P \in X$. Then the modified Anomaly Cancellation Equation 2.10 and 3.9 with the modification of Definition 3.2 remains:*

$$30K_B^2 + \frac{1}{2}(\chi_{top}(X) - \sum_P m(P)) \quad = \quad H_{ch} - (\dim \mathfrak{g} - \operatorname{rk} \mathfrak{g}).$$

Note that K_B^2 is an integer because the singularities are at worse rational double points.

As before we conclude:

Corollary 3.14 (Bounds, III)**.** *Let $X \to B$ be an equidimensional elliptic fibration, with X a Calabi-Yau threefold with $\mathbb{Q}$-factorial terminal singularities and B possibly singular. Assume that Σ_m is disjoint from the divisorial components of Σ, that is $\Sigma_m \not\subset \Sigma_j,\ \forall j$. The quantities (1), (2) and H_{unch} in (3), (5) of the massless spectrum are bounded.*

Corollary 3.15 (Bounds, III'). *Let $X \to B$ be an equidimensional elliptic fibration, with X a Calabi-Yau threefold with $\mathbb{Q}$-factorial terminal singularities and B possibly singular. Assume that Σ_m is disjoint from the divisorial components of Σ, that is $\Sigma_m \not\subset \Sigma_j$, $\forall j$. The Anomaly Cancellation formula implies that H_{ch} is also bounded.*

3.2. **The charged hypermultiples H_{ch}.**

The charged multiples in H_{ch} have been computed explicitly in many cases in the physics literature. In [13] we present some of the different methods for the computations and definitions in geometry. We review here the main points with the goal of establishing bounds for the geometric counterparts.

In the following let $X \to B$ be an equidimensional elliptic Calabi-Yau threefold with $\mathbb{Q}$-factorial terminal singularities and reduced discriminant Σ, with divisorial components Σ_j. Assume that $\Sigma_m \not\subset \Sigma_j$, for every j. Let $g_j \stackrel{def}{=} g(\Sigma_j)$ be the geometric genus of Σ_j. If $\mathfrak{g}(\Sigma_j)$ is not simply laced, let Σ'_j be the curve parametrizing the (curves spanning the) divisorial components of $\mathfrak{g}_j$. Recall that $\mathfrak{g}_j \stackrel{def}{=} \mathfrak{g}(\Sigma_j)$.

In [13] we also extend and reformulate the physics constructions and expectations of the charged hypermultiplets:

Charged Hypermultiplets 3.16.

(1) To each irreducible component Σ_j one associates the charged hypermultiplets representation adj_j with multiplicity $g(\Sigma_j)$, where $\mathrm{adj}_j \stackrel{def}{=} \mathrm{adj}(\mathfrak{g}(\Sigma_j))$ is the adjoint representation of the Lie algebra $\mathfrak{g}_j$.

(2) If $\mathfrak{g}(\Sigma_j)$ is not simply laced, there exists a canonically determined representation $\rho_0(\Sigma_j)$, with multiplicity $g(\Sigma'_j) - g(\Sigma_j)$.

(3) There exist (multi)-representations $\{\rho_Q\}_Q$ of $\oplus_j \mathfrak{g}_j$, where $Q \in \Sigma_j$ is in the singular locus of Σ (the matter representations).

(4) If there is a section and $Q \in \Sigma_j$ is in the singular locus of Σ, with possibly $\mathfrak{g}(\Sigma_j) = \{e\}$, there exist $\mathfrak{u}(1)_r$-charged hypermultiplets $\{c_Q\}$.

(5) If there is a multisection of index $m > 1$, and $Q \in \Sigma_j$ is in the singular locus of Σ, possibly $\mathfrak{g}(\Sigma_j) = \{e\}$, there exists hypermultiplets with multiplicity c_Q, charged under the discrete [2] gauge group which is associated with the index of the multisection.

Remark 3.17. If the general singular fibers of the elliptic fibration are irreducible, then $\{Q\}$ are the loci in the discriminant where the reducible fibers are located. c_Q is expected to be generically 1, but it is not always the case, see Example 6.7. Physics dualities relate the quantities $\{c_Q\}$ in (4) and (5) to the BPS states, and the (relative) Gopakumar-Vafa invariants at genus zero. The recent paper [17] computes the (relative) genus zero Gopakumar-Vafa invariants for general elliptic fibration with a multisection of index 5. See also [19] and

[2] See [19] and references therein for an overview.

references therein for the bounds related to the discrete charges. We elaborate on (4) and (5) in [13].

Definition 3.18. Let ρ be a representation of a Lie algebra $\mathfrak{g}$, with Cartan subalgebra $\mathfrak{h}$. The *charged dimension of* ρ is $(\dim\rho)_{ch} = \dim(\rho) - \dim(ker\rho|_{\mathfrak{h}})$.

Expectation (Physics) 3.19. *H_{ch} is a combination of the charged dimensions of the quantities in 3.16.*

Although it is an open question how to compute explicitly the representations and their combinations in 3.16, they have been computed under general conditions and in many examples. We present some of these results in the Appendix 6). Accordingly, in [13] we make the following:

Conjecture 3.20. *Let $X \to B$ be an equidimensional elliptic Calabi-Yau threefold with $\mathbb{Q}$-factorial terminal singularities and reduced discriminant Σ, with divisorial components Σ_j. Assume that $\Sigma_m \not\subset \Sigma_j$, for every j. Then the charged multiplets H_{ch} in the massless spectrum are:*

$$\boxed{H_{ch} = \sum_j g(\Sigma_j)(\dim \mathrm{adj}_j)_{ch} + \sum_j (g'_j - g_j)(\dim\rho_{0,j})_{ch} + \sum_Q (\dim\rho_Q)_{ch} + \sum_Q c_Q}$$

with the quantities defined as in 3.18 and 3.16.

Proposition 3.21. *Let $X \to B$ be an equidimensional elliptic Calabi-Yau threefold with $\mathbb{Q}$-factorial terminal singularities and reduced discriminant Σ, with divisorial components Σ_j. Assume that $\Sigma_m \not\subset \Sigma_j$, for every j and that 3.20 holds.*

The Geometric Anomaly Equation then becomes

$$\boxed{\begin{aligned} 30K_B^2 + \tfrac{1}{2}(\chi_{top}(X) - \textstyle\sum_P m(P)) \quad &+ \quad (\dim\mathfrak{g} - \mathrm{rk}\,\mathfrak{g}) = \\ = \textstyle\sum_j (g(\Sigma_j)(\dim \mathrm{adj}_j)_{ch} \quad &+ \quad \textstyle\sum_j (g'_j - g_j)(\dim\rho_{0,j})_{ch} + \sum_Q (\dim\rho_Q)_{ch} + \sum_Q c_Q \end{aligned}}$$

4. Geometric Anomaly and other bounds in geometry

The Geometric Anomaly Equation in the boxed formula of Proposition 3.21 is verified under general conditions and in many cases; we present a few in Appendix 6.

Corollary 4.1 (Bounds, IV)**.** *Let $X \to B$ be an equidimensional elliptic Calabi-Yau threefold with $\mathbb{Q}$-factorial terminal singularities and reduced discriminant Σ, with divisorial components Σ_j. Assume that $\Sigma_m \not\subset \Sigma_j$, for every j.*

Conjecture 3.20 and the Geometric Anomaly Cancellation in Proposition 3.21 imply that charged hypermultiplets $(\rho_{0,j})_{ch}$, $(\rho_Q)_{ch}$ and $\{c_Q\}$, the genus zero relative Gopakumar-Vafa invariants in the boxed formula 3.20, the divisorial components Σ_j and the singular points of the discriminants are bounded.

In particular the matter representations are bounded.

Proof. The quantities on right hand side in the boxed formula of Conjecture 3.20 and Proposition 3.21 are non-negative and Corollary 3.15 implies the statement. □

In [13] we show that the Geometric Anomaly Equation implies that $\mathfrak{g}$ and the charged hypermultiplets $(\rho_{0,j})_{ch}$, $(\rho_Q)_{ch}$ and $\{c_Q, c'_Q\}$ (relative genus zero Gopakumar-Vafa invariants) in the boxed formula 3.20 are birational invariants of the relatively minimal elliptic fibration. These invariants which we call *stringy Kodaira* are a natural generalization to higher dimension of Kodaira's classification of singular fibers of elliptic surfaces.

Corollary 4.2 (Bounds, V). *There are finitely many types of stringy Kodaira fibers for elliptically fibered Calabi-Yau threefolds.*

4.1. **Multiple fibers, indices of multisections and bounds.**

The physics of elliptic fibrations with multiple fibers is still not well understood. The multiple fibers over the points $Q \in \Sigma_m$ but $Q \notin \Sigma_j$, $\forall j$ do not contribute to the massless spectrum. Their multiplicity is however associated to *discrete symmetries* (see for example [19] and reference wherein). The multiplicity of the fibers, the associated discrete symmetries, and the index of the multi-sections are related.

It is expected that an elliptic Calabi-Yau $X \to B$ without sections can be transformed, with a generalization of the conifold transition, to a different elliptic Calabi-Yau $Y \to B'$ with section, $\mathbb{Q}$-factorial terminal singularities, in the philosophy of Reid's [26], and that the index of the multisection is bounded by $\operatorname{rk} MW(Y/B')$. There is an explicit prediction for the bounds, based on different reasoning (Conjecture 5.4). The results of Gross [16], Filipazzi-Hacon-Svaldi [6] imply that the index of the multisections and $\operatorname{rk} MW$ are bounded.

By contrast, [6] shows that the index of the multisection of elliptically fibered threefolds of Kodaira dimension 2 is not necessarily bounded. We conjecture that boundedness for Calabi-Yau and Reid's philosophy of [26] are related.

5. Explicit bounds

It is an important question in string theory to find explicit bounds of the spectrum, also in the context of the "Swampland Program" [31]:

Proposition 5.1 ([29]). *Let $X \to B$ be an elliptic fibration with section, X a Calabi-Yau manifold and B rational, then* $\operatorname{CxDef}(X) \leq 491$.

Proof. The starting point of argument is that $X \to B$ is birational to an elliptic fibration $X \to \bar{B}$, where B is either $\mathbb{P}^2$ or $\mathbb{F}_n$ a Hirzebruch surface with $n \leq 12$ [7]. Then an explicit computation gives that the general Jacobian fibration over such bases B have dimension of complex deformation ≤ 491. Note also the $\mathrm{CxDef}(X)$ is a birational invariant of the minimal model of X. □

Corollary 5.2. *Let $\pi : X \to B$ be an equidimensional elliptic fibration without a section, X a Calabi-Yau threefold, B smooth and rational. Assume that the fibration is general, that is the fibers over the general points of Σ are nodal (type I_1), the singular locus of Σ consists of nodal points Q and the fibers $\pi^{-1}(Q)$ are of type I_2. Then* $\mathrm{CxDef}(X) \leq 491$.

Proof. The Jacobian fibration $\pi^J : Jac(X) \to B$ has fibers of type I_1 over the points Q and $\mathbb{Q}$-factorial terminal ordinary double point singularities at the nodal point of each fiber $(\pi^J)^{-1}(Q)$. By comparing the topological Euler characteristics and the complex deformations as in 3.2 we obtain $\mathrm{CxDef}(X) = \mathrm{CxDef}\, Jac(X) - \sum_Q 1$. $J(X)$ can be deformed to a smooth elliptically fibered Weierstrass model over B and we then conclude by Proposition 5.1. □

Corollary 5.3. *Let $X \to B$ be an elliptic fibration, X Calabi-Yau. Assume that the mirror of X is an elliptic fibration with section. Then $h^{1,1}(X) \leq 491$.*

Conjecture 5.4 (Index of a fibration, Physics)**.** *Let $X \to B$ be an elliptic fibration, X, Calabi-Yau, then $n \leq 6$, where n is the index of a multi-section.*

The highest known multisection index for a Calabi-Yau threefold is $n = 5$ [17].

The conjecture is derived from a duality argument based on the following:

Result (Physics) 5.5 ([19] and references therein)**.** *Let $X \to B$ be an elliptic fibration between manifolds with section, X, Calabi-Yau, then* $\mathrm{tor}\,\mathrm{MW}(X/B) = \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z}$, $(n_1, n_2) = (2,2), (3,3), (2,4)$ *or* $\mathrm{tor}\,\mathrm{MW}(X/B) = \mathbb{Z}/n\mathbb{Z}$, $n \leq 6$.

These extend the work of Miranda, Persson and Shimada on K3 surfaces [28].

Result (Physics) 5.6 ([20])**.** *Let $X \to B$ be an elliptic fibration between manifolds, X, Calabi-Yau then* $\mathrm{rk}(\mathrm{MW}(X)) \leq 20$ *if* $B \neq \mathbb{P}^2$ *and* $\mathrm{rk}(\mathrm{MW}(X)) \leq 24$ *if* $B = \mathbb{P}^2$

The above bounds are not believed not to be sharp. The highest known Mordell-Weil rank for a Calabi-Yau threefold is $\mathrm{rk}(\mathrm{MW}(X)) = 10$ [14]; see also [5].

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6. Appendix: Examples

The Anomaly Cancellation and the Geometric Anomaly Equation have been proven for the many examples and the general conditions which have motivated our constructions. Here we describe a few.

Example 6.1. Let $X \to B$ an equidimensional elliptic fibration, with X a smooth Calabi-Yau threefold and B a smooth Enriques surface. Then Conjecture 3.20 and the Anomaly Cancellation Equation in 2.10 and Proposition 3.21 always hold.

Proof. Since the fibration is isotrivial $\mathfrak{g} = \{e\}$, $\chi_{top}(X) = H_{ch} = 0$. □

Example 6.2. ([10, Theorem 2.2], following [27].) Let $X \to B$ an equidimensional elliptic fibration, with X a smooth Calabi-Yau Weierstrass threefold and B smooth. Then the Anomaly Equation in Proposition 2.10 holds.

Proof. Any smooth Calabi-Yau Weierstrass model $X = W$ satisfies the geometric anomaly, because $H_{ch} - (\dim \mathfrak{g} - \mathrm{rk}\, \mathfrak{g}) = 0$ and $\chi_{top}(X) = 60K_B^2$. □

Definition 6.3. $X \to B$ is a general elliptic fibration if it is relatively minimal, equidimensional (as in Theorem 2.3) and the discriminant is of the form $\Sigma = \Sigma_1 \cup \Sigma_0$, where Σ_1 is a smooth curve, Σ_0 denotes the locus where the general fiber is a nodal elliptic curve.

Example 6.4 (General elliptic fibration with 5-section [17])**.** Let $X \to \mathbb{P}^2$ be a general elliptic fibration with a 5-section, and X Calabi-Yau. Assume that $\Sigma_1 = \emptyset$. Then Conjecture 3.20 holds and the Anomaly Cancellation Equation in 2.10 and in Proposition 3.21 hold as well.

Theorem 6.5 (General elliptic fibration with section, [13] [10])**.** *Let $X \to B$ be a general elliptic fibration with section, and X Calabi-Yau. Assume that $\mathrm{rk}\, MW(X) = 0$. Then Conjecture 3.20 holds with ρ_0, ρ_Q and the Milnor numbers are as in Table A.*

The Anomaly Cancellation Equation in 2.10 and in Proposition 3.21 hold as well.

Proof. Here $\mathfrak{g} = \mathfrak{g}(\Sigma_1)$. □

Example 6.6. [12] Let $X \to B$ be a general elliptic fibration with section, and X Calabi-Yau. Assume that $\mathrm{rk}\, MW(X) = 1$. Then Conjecture 3.20 holds. The table with the representations and the hypermultiplets H_{ch}, which extends Table A, includes a column with $\{c_Q = 1\}$.

Example 6.7. [14] Let $X \to B$ be the Schoen-Namikawa-Rossi Calabi-Yau threefold, with $\mathrm{rk}\, MW(X/B) = 10$. Then Proposition 3.21 holds with $\mathfrak{g} = \mathfrak{u}(1)^{10}$ and $\{c_Q > 1\}$.

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Type	$\mathfrak{g}$	ρ_0	$\rho_{Q_1^\ell}$	$\rho_{Q_2^\ell}$	$(\dim \mathrm{adj})_{ch}$	$(\dim \rho_0)_{ch}$	$\dim (\rho_{Q_1^\ell})_{ch}$	$\dim (\rho_{Q_2^\ell})_{ch}$	$m(P_1)$	$m(P_2)$
I_1	$\{e\}$		–	–	0	0	0	*0*	*0*	*1*
I_2	su(2)		–	fund	2	0	0	2		
III	su(2)		$2 \times$ fund		2	0	4			
I_3	su(3)		–	fund	6	0	0	3		
$I_{2k},\ k \geq 2$	$\mathfrak{sp}(k)$	Λ^2_0	–	fund	$2k^2$	$2k^2 - 2k$	0	$2k$		
$I_{2k+1},\ k \geq 1$	$\mathfrak{sp}(k)$	$\Lambda^2 + 2 \times$ fund	$\frac{1}{2}$ fund	fund	$2k^2$	$2k^2 + 2k$	k	$2k$	*1*	*0*
$I_n,\ n \geq 4$	su(n)		Λ^2	fund	$n^2 - n$	0	$\frac{1}{2}(n^2 - n)$	n		
IV	$\mathfrak{sp}(1)$	$\Lambda^2 + 2 \times$ fund	$\frac{1}{2}$ fund 2	4	1					
IV	su(3)		$3 \times$ fund		6	0	9			
I_0^*	$\mathfrak{g}_2$	**7**	–		12	6	0			
I_0^*	$\mathfrak{so}(7)$		–	spin	18	6	0	8		
I_0^*	$\mathfrak{so}(8)$		vect	$\mathrm{spin}_\pm$	24	0	8	8		
I_1^*	$\mathfrak{so}(9)$	vect	–	spin	32	8	0	16		
I_1^*	$\mathfrak{so}(10)$		vect	$\mathrm{spin}_\pm$	40	0	10	16		
I_2^*	$\mathfrak{so}(11)$	vect	–	$\frac{1}{2}$ spin	50	10	0	16		
I_2^*	$\mathfrak{so}(12)$		vect	$\frac{1}{2}\,\mathrm{spin}_\pm$	60	0	12	16		
$I_n^*,\ n \geq 3$	$\mathfrak{so}(2n+7)$	vect	–	–	$2(n+3)^2$	$2n+6$	0	–		
$I_n^*,\ n \geq 3$	$\mathfrak{so}(2n+8)$		vect	–	$2(n+3)(n+4)$	0	$2n+8$	–		
IV^*	$\mathfrak{f}_4$	**26**	–		48	24	0			
IV^*	$\mathfrak{e}_6$		**27**		72	0	27			
III^*	$\mathfrak{e}_7$	$\frac{1}{2}$**56**		126	0	28				
II^*	$\mathfrak{e}_8$		–		240	0	–			

Table A.

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Dipartimento di Matematica, Università di Bologna, Italy
Email address: `antonella.grassi3@unibo.it`

INFN, Sezione di Bologna, Italy

Department of Mathematics, University of Pennsylvania, Philadelphia, PA, USA
Email address: `grassi@math.upenn.edu`