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RESEARCH

Fine Boundary Continuity for Degenerate Double-Phase Diffusion

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Abstract
We study the boundary behavior of solutions to parabolic double-phase equations through the celebrated Wiener's sufficiency criterion. The analysis is conducted for cylindrical domains and the regularity up to the lateral boundary is shown in terms of either its p or q capacity, depending on whether the phase vanishes at the boundary or not. Eventually we obtain a fine boundary estimate that, when considering uniform geometric conditions as density or fatness, leads us to the boundary Hölder continuity of solutions. In particular, the double-phase elicits new questions on the definition of an adapted capacity.

Keywords Double-phase parabolic equations · Boundary regularity · Wiener's criterion

Mathematics Subject Classification (2010) 35B65 · 35B45 · 35K65

1 Introduction and Main Results

1.1 Heuristics

Let $u(x, t)$ be a scalar quantity describing the flow, in a space configuration x and at a time t, of a non-Newtonian fluid in a pipe (see for instance, [4] Ch.4 Section 7.6) which changes the power-law of its stress tensor according to a dramatic switch of the energy density. This change is specified by a law $a(x, t)$, that can be catered by an electromagnetic field or a mechanical device that suddenly obstructs the flow. Some of the fluids just described are addressed as

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electro-rheological and are of promising technological interest (see for instance [65, 66] or the book [68]); their special feature being a heavy change of viscosity in a very short time. As a guiding example, in [5] and [15], the authors consider the stationary flow of a generalized non-Newtonian fluid, modeled after an anisotropic dissipative potential $\varphi(z) = |z|^p + a(x,t)|z|^q$, whose energy is trapped between two power laws. Here we are interested in this description, as opposed to a slower change of rate that happens when $\Phi(z) = |z|^{p(x)}$ and p is a log-continuous function; though, instead of working in the framework of systems, we will be addressing equations. When the case of φ is concerned, the regularity of the solution, if any, is expected to follow some rule dictated by $a(x,t)$ itself, which we call *the phase*. In the present work we propose an analysis of the boundary behavior of solutions to equations that embody these features, and whose prototype is referred to as the *parabolic double-phase equation*, given by

$$\partial_t u - \operatorname{div}\Big(\big(|\nabla u|^{p-2} + a(x,t)|\nabla u|^{q-2}\big)\, \nabla u \Big) = 0 \quad \text{in} \quad \Omega_T = \Omega \times (0,T], \tag{1.1}$$

for $\Omega \subset \mathbb{R}^N$ open and bounded. Given a continuous initial datum f prescribed on the parabolic boundary of Ω_T, we address the question of whether solutions u to the parabolic double-phase Eq. 1.1 reach f in a continuous fashion; and in such case, when the datum is more regular (for instance, Hölder continuous), we would like to describe how fast this happens. Our answer to this question is presented in Theorem 1.1. In particular, we find that the trade-off between the geometry of the lateral boundary $\partial\Omega \times (0,T)$ in terms of the elliptic p or q capacity of $\partial\Omega$ and the behaviour of the phase at these points determines the desired rate.

1.2 Origins and Framing Of The Topic

In recent years, the stationary version of Eq. 1.1 has received a great attention, especially with regard to the regularity theory; we refer, for instance, to the surveys [57, 60, 64], and the extensive lists of references therein. While the local boundedness of solutions was already studied in the 70's in [49] and [50], the non-standard behavior was faced by Zikhov in [83] in the context of averaging of variational problems and the first pioneering analysis of the regularity of the gradient appeared in [58, 79] (see also [20] for a full-anisotropic version). In parallel, a fruitful theory of adapted and generalized energy spaces has seen the light, as Orlicz and Musielak-Orlicz spaces: here we refer, for instance, to the practical survey [23].

Equation 1.1 belongs to a wider class of equations exhibiting the so-called (p,q)-growth, that for its mathematical challenges together with its numerous applications draw a considerable attention for several decades. Regarding the stationary point of view, a non-exhaustive list of contributions is [1–3, 7, 9–11, 13, 25–27, 29, 30, 40, 41, 43, 44, 62, 63, 67, 75] to which we refer for results, references, historical notes and extensive survey of regularity issues, being the literature so wide that it results complicated to track every result in this direction.

On the other hand, the regularity theory for evolutionary double-phase equations has received less attention, most probably because of the merging of the difficulties inherent to the double-phase with the ones of the non-homogeneity of the operator caused by the parabolic term. A study of the local L^∞ norm of the gradient has been brought on in [12, 18, 32] and [71]. Refined quantitative gradient bounds have been addressed in [28], while higher differentiability of the gradient has been investigated in [38].

Our interest specifies towards equations with measurable and bounded coefficients, in the framework of a fine boundary estimate that is irrespective of the higher-order regularity. Within this perspective, the continuity and Hölder continuity for parabolic equations with Orlicz growth (generalizing Eq. 1.1) has been studied in [12, 46, 47, 72, 75] and [76]; while in [21, 69] and [70] the authors proved suitable versions of the Harnack inequality (see also [77, 80] for the variable exponent case).

1.3 Fine Boundary Regularity

A sufficient condition for the regularity of a boundary point for the prototype p-Laplacian elliptic equation has been known since the famous paper of Maz'ya [59], and is named after Wiener, who studied the Dirichlet problem for the linear case from the potential point of view (see [81, 82]). Later, Gariepy and Ziemer in [33] generalized this criterion to the case of quasi-linear elliptic equations. Roughly speaking this sufficiency condition is the following: picking $x_o \in \partial\Omega$ and defining for $p > 1, r > 0$ the number

$$\delta_p(r) = \left(\frac{\mathcal{C}_p(\overline{B_r(x_o)} \setminus \Omega;\, B_{2r}(x_o))}{\mathcal{C}_p(\overline{B_r(x_o)};\, B_{2r}(x_o))} \right)^{\frac{1}{p-1}}, \tag{1.2}$$

where $\mathcal{C}_p(K, B)$ is the elliptic variational p-capacity of the condenser (K, B) (see Eq. 1.8 for details), then, weak solutions of quasi-linear elliptic equations of p-Laplacian type are continuous up to the point x_o if

$$\int_0^1 \delta_p(r) \frac{dr}{r} = \infty\,. \tag{1.3}$$

We will refer, here and in the sequel, to the books [45] and [55] for an account of capacity methods for the fine boundary regularity in the context of elliptic p-Laplacian type equations.

The problem of fine boundary regularity for the diffusive p-Laplacean equation is much more recent. Continuity up to the boundary with monotonicity conditions was proved in [73, 74] under the condition Eq. 1.3. This result was generalized in [34] for more general parabolic evolution equations by using a weak Harnack inequality (see also [35, 36] for the singular super/sub-critical cases).

Finally, a sufficiency criterion of Wiener-type for parabolic equations with double-phase growth conditions is, up to our knowledge, a novelty. The present work is therefore a first step on the understanding of fine boundary regularity for double-phase parabolic operators.

1.4 Setting of the Problem

Let us denote $B_\rho(y)$ the ball in $\mathbb{R}^N$ of center y and radius ρ, and let Ω be a bounded domain in $\mathbb{R}^N$. For $T > 0$, we consider $\Omega_T := \Omega \times (0, T]$ the cylinder with base Ω and length T, and we denote by $S_T := \partial\Omega \times (0, T]$ its lateral boundary. We consider equations

$$\partial_t u - \operatorname{div} \mathbb{A}(x, t, \nabla u) = 0, \quad \text{weakly in } \Omega_T, \tag{1.4}$$

where we assume that the function $\mathbb{A} : \Omega_T \times \mathbb{R}^N \to \mathbb{R}^N$ is Caratheodory, i.e. $\mathbb{A}(\cdot, \cdot, \xi)$ is Lebesgue measurable for all $\xi \in \mathbb{R}^N$, and $\mathbb{A}(x, t, \cdot)$ is continuous for almost all $(x, t) \in \Omega_T$; and that $\mathbb{A}$ satisfies the following structure conditions

$$\begin{aligned} \mathbb{A}(x, t, \xi) \cdot \xi &\geqslant C_1 \left(|\xi|^p + a(x, t)|\xi|^q\right) =: C_1\, \varphi(x, t, |\xi|), \quad 2 < p < q, \\ |\mathbb{A}(x, t, \xi)| &\leqslant C_2 \left(|\xi|^{p-1} + a(x, t)|\xi|^{q-1}\right) =: C_2\, \varphi(x, t, |\xi|)/|\xi|, \end{aligned} \tag{1.5}$$

for C_1, C_2 given positive constants, that we will refer to as structural data. The function $a(x,t) : \mathbb{R}^{N+1} \to [0, \infty)$ is everywhere defined and non-negative. We assume $a(x,t)$ to be locally Hölder continuous around S_T: for any $(x_o, t_o) \in S_T$, we assume that there exist positive numbers R_o, A_o such that, for any $0 < r < R_o$, the following inequality holds true,

$$\operatorname*{osc}_{Q_{r,r^2}(x_o,t_o)} a(x,t) \le A_o\, r^{q-p}, \tag{1.6}$$

being $Q_{r,r^2}(x_o,t_o) = B_r(x_o) \times (t_o - r^2, t_o + r^2)$.

As our estimates are of local nature, the constants R_o and A_o will also be referred to as structural constants. Thence, we are concerned with the boundary behaviour of solutions to the Cauchy-Dirichlet problem

$$\begin{cases} \partial_t u - \operatorname{div} \mathbb{A}(x,t,\nabla u) = 0, & \text{weakly in } \Omega_T, \\ u(x,t) = f(x,t), & \text{on } S_T, \\ u(x,0) = f(x,0), & \text{attained in } L^2_{loc}(\Omega), \end{cases} \tag{1.7}$$

where $\mathbb{A}$ obeys to Eqs. 1.5-1.6 above for $2 < p < q$, and

$$f \in L^q(0,T; W^{1,q}(\Omega)) \cap C(\overline{\Omega_T}).$$

The boundary datum f is taken in the weak sense, i.e. $(u - f)(\cdot, t) \in W_o^{1,q}(\Omega)$ for almost every time $t \in (0, T]$. As typical of parabolic equations, what happens in the future is determined entirely from the past: this motivates the omission of a prescription of the boundary datum at $\Omega \times \{T\}$. In agreement with this principle, for our local estimates we will work with backward parabolic cylinders (See Section 2 for more details).

The well-posedness of this problem has been addressed in [18, 72] and very recently in [6], with slightly different notions of solutions. We refer to Section 3 below for the details of our definitions.

Finally, another important topic concerns global boundedness of solutions, for which there seems not to be a complete picture in the parabolic case for equations such as Eq. 1.4. In general and within an elliptic context, for this generality of choice of exponents $q > p > 2$, local weak solutions to stationary equations with (p,q) growth as Eq. 1.4 above are not meant to be locally bounded, as shown by the two pioneering counter-examples [39, 56]. Nonetheless, these two examples are fully anisotropic, meaning with this that the energy is not a function of the modulus of the gradient, but just of its components. For general non-standard parabolic equations, global boundedness is shown in Theorem 3 of [61] for fully anisotropic parabolic equations; see also [24] for refined local bounds. The condition given may not be sharp in the case of Eq. 1.4, as it is unrelated to the degree of Hölder continuity of $a(x,t)$ (see for instance [72]), or its L^∞ norm. For this reason, in what follows we consider solutions that are globally bounded in Ω_T, thereby admitting a wider set of solutions.

1.5 Main Result and Applications

In order to formulate our boundary estimate, we briefly recall the definition of capacity at hands. Let $s \in (1, N]$, $B \subset \mathbb{R}^N$ be an open set and $K \subset B$ be a compact set. We denote by $\mathcal{C}_s(K; B)$ the Newtonian (or variational) capacity of the condenser $(K; B)$ and defined as

$$\mathcal{C}_s(K;B) = \inf \left\{ \|\nabla f\|^s_{L^s(B)} \;:\; f \in C_o^\infty(B),\;\; f \ge 1 \text{ on } K \right\}. \tag{1.8}$$

This introduced version of capacity pertains to domains of $\mathbb{R}^N$ and it extends, within its elliptic fashion, spontaneously to cylinders $Q = B \times (t_1, t_2)$ in $\mathbb{R}^{N+1}$. Let $\tilde{K} \subset Q$ be a compact subset of such cylinder, and if we define

$$\mathcal{C}_s(\tilde{K}, Q) = \inf\left\{\|\nabla f\|^s_{L^s(Q)} \ :\ f \in C^\infty_o(Q),\ f \geq 1 \text{ on } \tilde{K}\right\}, \quad \text{then} \quad \mathcal{C}_s(\tilde{K}, Q) = \int_{t_1}^{t_2} \mathcal{C}_s(\tilde{K}_\tau, B)\, d\tau,$$

being $\tilde{K}_\tau = K \times \{\tau\}$. The proof of this last equality can be found in [16], while other notions of parabolic capacity are investigated in various other circumstances, see for instance [8] and [84].

With this definition and Eq. 1.2, we are ready to state our main result. We say that a constant *depends only on the data*, if it depends only on the fixed parameters $\{N, p, q, C_1, C_2, A_o, M\}$, where

$$M = \sup_{\Omega_T} u\ .$$

Theorem 1.1 *Let $(x_o, t_o) \in S_T$ and let u be a bounded, weak solution to the Cauchy-Dirichlet problem Eq. 1.7. Depending on the point (x_o, t_o), we assume that either*

$$\int_0^1 \delta_p(r)\frac{dr}{r} = \infty, \quad \text{if} \quad a(x_o, t_o) = 0, \tag{1.9}$$

or

$$\int_0^1 \delta_q(r)\frac{dr}{r} = \infty, \quad \text{if} \quad a(x_o, t_o) > 0. \tag{1.10}$$

Then, in each case respectively, there exist $\{\rho_0(p), \eta_0(p)\}$, $\{\rho_0(q), \eta_0(q)\}$ couples of positive numbers depending only on the data and conditions Eqs. 1.9-1.10, and positive constants $\gamma, \hat{\gamma}, \gamma^*$ depending only on the data, such that, defining

$$Q_0(p) = B_{\rho_0(p)}(x_o) \times (t_o - \eta_0(p),\ t_o],$$

$$Q_0(q) = B_{\rho_0(q)}(x_o) \times (t_o - \eta_0(q),\ t_o], \quad \text{and} \quad \omega_0 = \operatorname*{osc}_{\Omega_T} u,$$

the following inclusions

$$Q_\rho(\omega_0, p) = B_\rho(x_o) \times (t_o - \gamma^*\rho^p\omega_0^{2-p},\ t_o] \subset Q_0(p),$$

$$Q_\rho(\omega_0, q) = B_\rho(x_o) \times (t_o - \gamma^*\rho^q\omega_0^{2-q},\ t_o] \subset Q_0(q),$$

hold true for $\rho = \rho_0(p), \rho_0(q)$, and for all $0 < \rho < \rho_0(p)$ we have the estimate

$$\operatorname*{osc}_{Q_\rho(\omega_0,p)\cap\Omega_T} u \leq \omega_0 \exp\left\{-\frac{1}{\gamma}\int_\rho^{\rho_0(p)} \delta_p(s)\frac{ds}{s}\right\} + \operatorname*{osc}_{Q_0(p)\cap S_T} f + \hat{\gamma}[\rho_0(p)]^{\frac{\epsilon}{p-2}}, \quad \text{if} \quad a(x_o, t_o) = 0,$$

while for all $0 < \rho < \rho_0(q)$ we have

$$\operatorname*{osc}_{Q_\rho(\omega_0,q)\cap\Omega_T} u \leq \omega_0 \exp\left\{-\frac{1}{\gamma}\int_\rho^{\rho_0(q)} \delta_q(s)\frac{ds}{s}\right\} + \operatorname*{osc}_{Q_0(q)\cap S_T} f + \hat{\gamma}[\rho_0(q)]^{\frac{\epsilon}{q-2}}, \quad \text{if} \quad a(x_o, t_o) > 0.$$

Remark 1.2 (Constants dependence) We observe that the geometric construction is dependent on the assumptions Eqs. 1.9-1.10, differently from the isotropic singular case (see for instance [36]). Once Eq. 1.10 is verified, the constants $\rho_0(q), \eta_0(q)$ are determined irrespective of the numerical value of $a(x_o, t_o)$, similarly to [34], Thm 1.1.

In this sense, coherently with the nature of the potential switch, the oscillation estimates above do not deteriorate when $0 < a(x_o, t_o)$ is big or small; on the other hand, when

$a(x_o, t_o) = 0$, the first estimate of Theorem 1.1 holds true, with $\rho_0(p), \eta_0(p)$ found by condition Eq. 1.9, and no further reference on the size of the phase. Finally, the constants $\gamma, \gamma^*, \hat{\gamma}$ are dependent only on the data $\{N, p, q, C_1, C_2, A_o, M\}$ and independent of any value of the phase.

Nonetheless, even if Theorem 1.1 is stated for the Cauchy-Dirichlet problem, as soon as a lateral boundary datum is concerned, the oscillation estimates above are of local nature. In this framework, it is a simple consequence that a Wiener-type test is a sufficient condition for a point $(x_o, t_o) \in S_T$ to be a regular point to the parabolic double-phase operator Eqs. 1.4-1.5-1.6.

We recall that a lateral boundary point $(x_o, t_o) \in S_T$ is said to be regular to Eqs. 1.4-1.5-1.6 if, for any weak solution u to Eq. 1.4, satisfying

$$(u(x,t) - f(x,t)) \in V_o^{2,q}(\Omega_T), \tag{1.11}$$

with any $f(x,t) \in C(\overline{\Omega_T})$, the limit

$$\lim_{\Omega_T \ni (x,t) \to (x_o,t_o)} u(x,t) = f(x_o, t_o)$$

is attained. Here and in what follows, we denote with $V_o^{2,q}$ the parabolic space

$$V_o^{2,q}(\Omega_T) = C(0,T; L^2(\Omega)) \cap L^q(0,T; W_o^{1,q}(\Omega)),$$

and the attainment of the datum Eq. 1.11 is understood weakly. The geometric conditions Eqs. 1.9-1.10 are also common in the literature when referring to the set $\mathbb{R}^N \setminus \Omega$ as (p or) q-thick at x_o (e.g. [45]).

Corollary 1.3 *Let u be a bounded, weak solution to Eqs. 1.4-1.5, and let Eq. 1.6 be satisfied in $(x_o, t_o) \in S_T$. If moreover*

- $a(x_o, t_o) = 0$, *then Eq. 1.9 is a sufficient condition for (x_o, t_o) to be regular to Eqs. 1.4-1.5-1.6;*

otherwise, if

- $a(x_o, t_o) > 0$, *then Eq. 1.10 is a sufficient condition for (x_o, t_o) to be regular to Eqs. 1.4-1.5-1.6.*

Classically, in the case $p = q = 2$, when at the point $x_o \in \Omega$ further requirements are satisfied, as the logarithmic Wiener condition (see [14]), the solutions attain a Hölder continuous datum in a Hölder continuous fashion. For ease of exposition here we ask Ω to enjoy a uniform geometrical property; which is ensured, for instance, by the classic corkscrew condition (see [45] Thm 6.31). We briefly recall it here below.

Let $X \subset \mathbb{R}^N$ be a closed set, $Y \subseteq X$ and $s \in (1, N]$. The set X is uniformly s-fat in Y if there exist positive constants λ_s, R_s such that, for all $y \in Y$ and $0 < \rho < R_s$,

$$\mathcal{C}_s(\overline{B_\rho(y)} \cap X; B_{2\rho}(y)) \geq \lambda_s \rho^{N-s}.$$

When $X = Y$ we just say that X is uniformly s-fat. With this definition at hands, we can present a notion of fatness that suits the parabolic double-phase problem under consideration.

Definition 1.4 Given a continuous function $a : \mathbb{R}^{N+1} \to [0, \infty)$, we say that a closed set $X \subset \mathbb{R}^N$ is uniformly (p,q)-fat with phase $a(x,t)$ if X is uniformly p-fat at those points $x_o \in \partial X$ such that $a(x_o, t_o) = 0$ for some $t_o \in \mathbb{R}$, and it is uniformly q-fat at those points $x_o \in \partial X$ such that $a(x_o, t) > 0$ for all $t \in \mathbb{R}$.

Remark 1.5 We observe that in the above definition if X is uniformly p-fat and the function a vanishes on ∂X for all times, then trivially X is uniformly (p,q)-fat with phase $a(x,t)$ with any q. Moreover, when $q \geq p$, a uniformly p-fat set is also a uniformly q-fat set, by a simple application of Hölder's inequality. Hence the introduced definition is weaker than the usual p-fatness. The property of a set of being uniformly p-fat is an open-end condition (see for instance [53]) and it is equivalent to a point-wise Hardy inequality (see [51]). The definition of fatness obliges $q < N$: in the cases where $p > N$ condition Eq. 1.9 is satisfied and when $q > N$ condition Eq. 1.10 is satisfied, because in such cases the capacities of point and ball are comparable with a uniform constant. This remark further implies that, when $p < N < q$, if for these times $t \in \mathbb{R}$ such that the set $\{X \times \{t\}\} \cap a(\cdot,t)^{-1}(\{0\})$ is not empty, it is also uniformly p-fat, then X is uniformly (p,q)-fat with phase $a(x,t)$. See Subsection 1.6 below for a comparison with more general notions of fatness and capacity.

Finally, when the complement of Ω is uniformly (p,q)-fat with phase $a(x,t)$, then both integrals Eqs. 1.9-1.10 diverge at every boundary point $x_o \in \partial\Omega$. Together with an Hölder continuous datum f, the oscillation estimates of Theorem 1.1 show that the oscillation of the solution is comparable with powers of the radii: as in (Corollary 1.2, [34]) we obtain that solutions are Hölder continuous up to the boundary.

Corollary 1.6 *Let u be a bounded, weak solution to Eqs. 1.4-1.5-1.6, with an Hölder continuous boundary datum $f \in C^{0,\alpha}(\overline{\Omega_T})$. Suppose furthermore that $\mathbb{R}^N \setminus \Omega$ is (p,q)-fat with phase $a(x,t)$. Then, the solution u is Hölder continuous up to S_T.*

Classically (for instance in [22, 52, 78] and for the parabolic p-Laplacean in [31]) the Hölder continuity up to the boundary was obtained for domains $\Omega \subset\subset \mathbb{R}^N$ satisfying the density condition

$$\exists \alpha, R_D > 0 \, : \, \forall x_o \in \partial\Omega \quad \forall\, 0 < \rho < R_D \quad |\Omega \cap B_\rho(x_o)| \leq (1-\alpha)|B_\rho|. \tag{1.12}$$

By a simple application of the definition of the s-capacity together with the Poincaré inequality, condition Eq. 1.12 implies that $\mathbb{R}^N \setminus \Omega$ is uniformly s-fat; however, the converse statement is not true in general, as already seen by the case of points when $s > N$, or by the fact that sets of zero s-capacity do not separate the space $\mathbb{R}^N$. Nonetheless, when dealing with a global problem and for the purpose of precise integral estimates, these two conditions meet when a zero-extension is available; see for instance [19], Prop. 5.9 in the context of Campanato theory. Finally, we refer to Corollary 11.25 of [17] for more geometrical notions implying boundary regularity: among these examples, the p-fatness of the complement is the weakest assumption.

1.6 Comparison with Known Results

In the elliptic framework, the problem of the regularity of solutions is posed for equations whose prototype is the stationary counterpart of Eq. 1.1, with a condition of Hölder continuity on the phase

$$a(x) \in C^{0,\alpha}(\Omega), \qquad \frac{q}{p} < 1 + \frac{\alpha}{N}, \tag{1.13}$$

that avoids the so-called Lavrentyev phenomenon, in the context of minima of calculus of variations. In our treatment here we chose $\alpha = q - p$ (coherently with condition (A1-n) of [42]), and we are allowed to choose $q > p > 2$ arbitrary.

No additional constraints on the ratio q/p are imposed, because we assume that our solutions are bounded. Finally, as an Hölder continuous function with exponent greater than one is constant, when $q - p > 1$, condition Eq. 1.6 trivializes to a constant phase $a(x,t) = a(x_o, t_o)$; and we find known results of [34] when the phase is constantly zero, while the other case is closer to the treatment of [54], whose parabolic counterpart has been examined in [12] with an approach involving the comparison principle and the construction of a barrier. Continuity up to the boundary for a wider set of functions, in a formulation which is closer to ours, is proved in [75]-[76]. In all these last non-standard contexts, an oscillation estimate as Eqs. 1.9-1.10 is new.

Finally we briefly comment on Definition 1.4 and our use of p and q capacity, still taking the example of elliptic equations. In the context of generalized Orlicz spaces $W^{1,\varphi}(\Omega)$, the notion of capacity

$$\mathcal{C}_{\varphi}(K,\Omega)=\inf\left\{\int_{\Omega}\varphi(x,|\nabla f|)\,dx\;:\; f\in W_o^{1,\varphi}(\Omega),\quad f\geq 1\ \text{ on some open set }\ B\supset K\right\},$$

is usually more convenient to treat $p(x)$, double-phase and various non-power like growth operators (see for instance [42, 48]). In our framework, the Wiener's conditions Eqs. 1.9, 1.10 and the p or q capacity of the sets that we will estimate, are intimately intertwined with the choice of an appropriate time-length (see the choices of η^* and η_* in the section of the geometric construction 4.2, and the choices Eqs. 5.9-5.10 for the initial cylinders of Theorem 1.1). For this reason, we find Definition 1.4 more convenient and tailored on the potential switch of the operator.

Structure of the Paper

In Section 2, we collect the notation used in the overall paper. Then, in Section 3, we define local weak solutions and we describe various Lemmata concerning Energy (Caccioppoli) estimates, a measure-theoretical maximum principle, negative-powers Energy estimates, a Reverse Hölder's inequality and finally the weak Harnack inequality for nonnegative local weak supersolutions to Eqs. 1.4-1.5-1.3. In Section 4, we draw the geometric setting of the proof and we use the results of Section 3 to prove a reduction of oscillation of the solution near the boundary by means of the capacity of $\partial\Omega$ at the point considered. Finally in Section 5, we prove the main result, Theorem 1.1, and in Section 6, we collect the proof of the Energy Estimates of Section 3, in order to leave space in the main text to what is really new.

2 Notation

- *Constants dependency.*
  We refer to the parameters N, p, q, C_1, C_2, A_o and $M := \sup_{\Omega_T} |u|$ as our structural data, and we say that a constant γ depends only on the data if it can be quantitatively determined a priori only in terms of the above quantities. The generic constant γ may change from line to line.
- *Geometry.*
  We denote by O the origin in $\mathbb{R}^N$. Let $r, \eta > 0$. We denote with $B_r(x)$ the ball of radius r centered in $x \in \mathbb{R}^N$. Then we write

$$\begin{cases} Q^+_{r,\eta}(\bar{x},\bar{t}) = B_r(\bar{x}) \times (\bar{t},\ \bar{t}+\eta), \\ Q^-_{r,\eta}(\bar{x},\bar{t}) = B_r(\bar{x}) \times (\bar{t}-\eta,\ \bar{t}), \\ Q_{r,\eta}(\bar{x},\bar{t}) = B_r(\bar{x}) \times (\bar{t}-\eta,\ \bar{t}+\eta), \end{cases}$$

respectively, for the forward, backward and full cylinders centered at $(\bar{x}, \bar{t})$ of radius r and length η (or 2η). When writing

$$Q_r^{\pm} = Q_{r,r^2}^{\pm},$$

we denote the cylinder centered at O and whose time interval has length r^2; being

$$Q_r = Q_{r,r^2} = Q_r^- \cup Q_r^+ .$$

- *Levels.*
  For any level $k \in \mathbb{R}$, $(\bar{x}, \bar{t}) \in \Omega_T$, r, η as before such that the inclusion $Q_{r,\eta}^+(\bar{x}, \bar{t}) \subset \Omega_T$ is satisfied, we denote by:

$$A_{k,r,\eta}^- = Q_{r,\eta}^+(\bar{x}, \bar{t}) \cap \{u \leqslant k\}$$

  the sub-level sets of u in $Q_{r,\eta}^+(\bar{x}, \bar{t})$ and by

$$\varphi^{\pm}_{Q^+_{r,\eta}(\bar{x},\bar{\eta})}\left(\frac{k}{r}\right) = \left(\frac{k}{r}\right)^p + a^{\pm}_{Q^+_{r,\eta}(\bar{x},\bar{t})}\left(\frac{k}{r}\right)^q,$$

  where $a : Q \subset \mathbb{R}^{N+1} \to \mathbb{R}_0^+ = [0, \infty)$, $a_Q^+ = \max_Q a$ and $a_Q^- = \min_Q a$.

3 Preliminaries

3.1 Definition of Solution

We say that a function

$$u \in V_{loc}^{2,q}(\Omega_T) := C_{\text{loc}}(0, T; L^2_{\text{loc}}(\Omega)) \cap L^q_{\text{loc}}(0, T; W^{1,q}_{\text{loc}}(\Omega)),$$

is a local weak super(sub)-solution to Eq. 1.4 if for any compact set $E \subset \Omega$ and every sub-interval $[t_1, t_2] \subset (0, T]$ there holds

$$\int_E u\zeta \, dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_E \{-u\partial_\tau\zeta + \mathbb{A}(x, \tau, \nabla u)\nabla\zeta\} \, dxd\tau \geqslant 0, \qquad (\leqslant 0), \tag{3.1}$$

for any nonnegative test function $\zeta \in W^{1,2}_{\text{loc}}(0, T; L^2(E)) \cap L^q_{\text{loc}}(0, T; W_o^{1,q}(E))$.

A function

$$u \in C(0, T; L^2(\Omega)) \cap L^q(0, T; W^{1,q}(\Omega)),$$

such that

$$(u - f) \in W_o^{1,q}(\Omega) \quad \text{for a.e.} \quad t \in (0, T],$$

is a weak super(sub)-solution to the Cauchy-Dirichlet problem Eq. 1.7, if for all $t \subset (0, T]$ it satisfies

$$\int_\Omega u\zeta(x, t) \, dx + \iint_{\Omega_T} \{-u\partial_\tau\zeta + \mathbb{A}(x, \tau, \nabla u)\nabla\zeta\} \, dxd\tau \geqslant \int_\Omega f\zeta(x, 0) \, dx, \qquad (\leqslant 0), \tag{3.2}$$

for any nonnegative test function $\zeta \in W^{1,2}(0, T; L^2(\Omega)) \cap L^q(0, T; W_o^{1,q}(\Omega))$.

To the aim of our computations, it is technically convenient to have a formulation of weak super(sub)-solution that involves the weak derivative of an approximant of u. Let $\rho(x) \in C_o^\infty(\mathbb{R}^N)$, $\rho(x) \geqslant 0$ in $\mathbb{R}^N$, $\rho(x) \equiv 0$ for $|x| > 1$ and $\int_{\mathbb{R}^N} \rho(x)\,dx = 1$, and set

$$\rho_h(x) := h^{-N}\rho\,(x/h)\,, \quad u_h(x,t) := h^{-1}\int_t^{t+h}\int_{\mathbb{R}^N} u(y,\tau)\rho_h(x-y)\,dyd\tau.$$

We fix $t \in (0,T)$ and let $h > 0$ be so small that $0 < t < t+h < T$. In Eq. 3.1 we take $t_1 = t$, $t_2 = t+h$ and replace ζ by $\int_{\mathbb{R}^n} \zeta(y,t)\rho_h(x-y)\,dy$. Dividing by h, since the testing function does not depend on τ, we obtain

$$\int_{E\times\{t\}} \left(\frac{\partial u_h}{\partial t}\,\zeta + [\mathbb{A}(x,t,\nabla u)]_h \nabla\zeta\right) dx \geqslant 0\,(\leqslant 0), \tag{3.3}$$

for all $t \in (0\,,\,T-h)$ and for all $\zeta \in W_o^{1,q}(E)$, $\zeta \geqslant 0$.

3.2 Local Energy Estimates and Critical Mass Lemma

Let u be a weak non-negative super-solution to Eq. 1.4 in Ω_T, and suppose that for $(\bar{x},\bar{t}) \in \Omega_T$ and $\eta, r > 0$ the following inclusion holds true

$$Q^+_{r,\eta}(\bar{x},\bar{t}) := B_r(\bar{x}) \times (\bar{t}\,,\,\bar{t}+\eta) \subset \Omega_T.$$

Lemma 3.1 *(Energy Estimates) Let u be a non-negative, local weak super-solution to Eq. 1.4 in Ω_T, and let $\eta, r > 0$ and $(\bar{x},\bar{t}) \in \Omega_T$ be as above. For any $\sigma \in (0,1)$, let $\zeta(x,t) = (\zeta_1(x)\zeta_2(t))^q$, for $0 \le \zeta_i \le 1$, be a cut-off function such that*

$$\begin{cases} \zeta_1 \in C_o^\infty(B_r(\bar{x})): \quad \zeta_1(x) = 1 \quad in \quad B_{r(1-\sigma)}(\bar{x}), \ and \ \ \|\nabla\zeta\|_\infty \le \|\nabla\zeta_1\|_\infty \leqslant \gamma(\sigma r)^{-1}; \\ \\ \zeta_2 \in C^1(\mathbb{R}^+_0): \quad \begin{cases} \zeta_2(t) = 1, & t \leqslant \bar{t}+\eta(1-\sigma), \\ \zeta_2(t) = 0, & t \geqslant \bar{t}+\eta, \end{cases} \quad and \ \ \|\partial_t\zeta\|_\infty \le \|\zeta_2'\|_\infty \leqslant \gamma(\sigma\eta)^{-1}. \end{cases}$$

where the ∞-norm is taken in $Q^+_{r,\eta}(\bar{x},\bar{t})$. Let k be any positive constant. Then, if we define

$$[\varphi^\pm_{k,r}] = \varphi^\pm_{Q^+_{r,\eta}(\bar{x},\bar{\eta})}\left(\frac{k}{r}\right) = \left(\frac{k}{r}\right)^p + a^\pm_{Q^+_{r,\eta}(\bar{x},\bar{t})}\left(\frac{k}{r}\right)^q,$$

there exists a positive constant γ, depending only on the data, such that

$$\sup_{\bar{t}<t<\bar{t}+\eta} \int_{B_r(\bar{x})} \zeta(u-k)_-^2\,dx + \left(\frac{r}{k}\right)^p \frac{[\varphi^-_{k,r}]}{\gamma} \iint\limits_{Q^+_{r,\eta}(\bar{x},\bar{t})} |\nabla[\zeta(u-k)_-]|^p\,dxdt$$
$$\leqslant \gamma\sigma^{-q}[\varphi^+_{k,r}]\left(1+\frac{k^2}{\eta[\varphi^+_{k,r}]}\right)|A^-_{k,r,\eta}|, \tag{3.4}$$

$$\sup_{\bar{t}<t<\bar{t}+\eta} \int_{B_r(\bar{x})} \zeta_1^q(u-k)_-^2\,dx + \left(\frac{r}{k}\right)^p \frac{[\varphi^-_{k,r}]}{\gamma} \iint\limits_{Q^+_{r,\eta}(\bar{x},\bar{t})} |\nabla[\zeta_1^q(u-k)_-]|^p\,dxdt$$
$$\leqslant \int_{B_r(\bar{x})\times\{\bar{t}\}} \zeta_1^q(u-k)_-^2 dx + \gamma\sigma^{-q}\,[\varphi^+_{k,r}]\,|A^-_{k,r,\eta}|, \tag{3.5}$$

where $A^-_{k,r,\eta}$ are the k sub-level sets of u in $Q^+_{r,\eta}(\bar{x},\bar{t})$ (see Section 2).

Classically, for most parabolic differential equations it is possible to show that the energy estimates, chained with a proper Sobolev-Poincaré inequality, imply some sort of measure-theoretical maximum property (see [31] for instance). The double-phase Eq. 1.4 is no exception, provided that a particular geometry is chosen; here we specialize to super-solutions in Q^+_{4r} (see Section 2).

Lemma 3.2 *(Initial-values Critical Mass) Let u be a bounded, weak, non-negative super-solution to Eqs. 1.4 in $Q^+_{4r}(\bar{x},\bar{t})$, with $0 \le u \le M$. Assume also that for some $0 < k < M$*

$$u(x,\bar{t}) \geqslant k, \quad x \in B_r(\bar{x}). \tag{3.6}$$

Then there exists $\delta \in (0,1)$, depending only on the data, such that for almost all $(x,t) \in Q^+_{r/2,\eta_k}(\bar{x},\bar{t})$

$$u(x,t) \geqslant \delta\, k\,, \tag{3.7}$$

provided that

$$\eta_k = \frac{k^2}{[\varphi^+_{k,2r}]} \le (4r)^2 \le R_o^2\,, \qquad [\varphi^+_{k,2r}] = \left(\frac{k}{2r}\right)^p + \left(\max_{Q^+_{2r}(\bar{x},\bar{t})} a\right)\left(\frac{k}{2r}\right)^q . \tag{3.8}$$

Proof We consider $(\bar{x},\bar{t}) = (0,0)$, to simplify the notation, and an intermediate level $0 < \bar{k} < k$. For $n \in \mathbb{N}_0$, we construct

$$Q_n := Q^+_{r_n,\eta_k} \subset Q^+_{r,\eta_k} =: Q_o, \quad \text{being} \quad r_n = r(1+2^{-n})/2, \quad \text{and let} \quad k_n = \bar{k}(1+2^{-n})/2.$$

Let us define

$$[\varphi^\pm_n] = \left(\frac{k_n}{r_n}\right)^p + \left(a^\pm_{Q_n}\right)\left(\frac{k_n}{r_n}\right)^q.$$

Function u satisfies Eq. 3.5 for cut-off functions $\zeta_n = \zeta_1^q$ between Q_n and Q_{n+1} independent of time. The assumption Eq. 3.6 simplifies the right-hand side of Eq. 3.5 and provides the estimates

$$\sup_{0<t<\eta_k} \int_{B_n} [\zeta_n(u-k_n)_-]^2\, dx \le \gamma 2^{nq} [\varphi^+_n]\, |[u < k_n] \cap Q_n|, \tag{3.9}$$

and

$$\iint_{Q_n} |\nabla[\zeta_n(u-k_n)_-]|^p\, dxdt \le \gamma 2^{nq} \left(\frac{[\varphi^+_n]}{[\varphi^-_n]}\right)\left(\frac{k_n}{r_n}\right)^p |[u < k_n] \cap Q_n|. \tag{3.10}$$

Hence Sobolev's parabolic embedding theorem applies to $[\zeta(u-k_n)_-]$ and Eqs. 3.9-3.10 imply

$$
\begin{aligned}
&(2^{-(n+1)}\bar{k})^p|[u<k_{n+1}]\cap Q_{n+1}| \le \iint_{Q_{n+1}} (u-k_n)_-^p \,dxdt\\
&\le \iint_{Q_n} [\zeta_n(u-k_n)_-]^p\,dxdt\\
&\le \left(\iint_{Q_n} [\zeta_n(u-k_n)_-]^{\frac{p(N+2)}{N}}\,dxdt\right)^{\frac{N}{N+2}} |[u<k_n]\cap Q_n|^{\frac{2}{N+2}}\\
&\le \gamma\left(\sup_{0<t<\eta_k}\int_{B_{r_n}}[\zeta_n(u-k_n)_-]^2\,dx\right)^{\frac{p}{N+2}}\left(\iint_{Q_n}|\nabla[\zeta_n(u-k_n)_-]|^p\,dxdt\right)^{\frac{N}{N+2}}|[u<k_n]\cap Q_n|^{\frac{2}{N+2}}\\
&\le \gamma 2^{\frac{nq(N+p)}{N+2}}\left([\varphi_n^+]|[u<k_n]\cap Q_n|\right)^{\frac{p}{N+2}}\left(\left(\frac{k_n}{r_n}\right)^p\frac{[\varphi_n^+]}{[\varphi_n^-]}|[u<k_n]\cap Q_n|\right)^{\frac{N}{N+2}}|[u<k_n]\cap Q_n|^{\frac{2}{N+2}}\\
&\quad = \gamma 2^{\frac{nq(N+p)}{N+2}}\left(\frac{k_n}{r_n}\right)^{\frac{pN}{N+2}}[\varphi_n^+]^{\frac{p}{N+2}}\left(\frac{[\varphi_n^+]}{[\varphi_n^-]}\right)^{\frac{N}{N+2}}|[u<k_n]\cap Q_n|^{1+\frac{p}{N+2}}.
\end{aligned}
$$

Now we employ condition Eq. 1.6, under the assumption $\eta_k \le (4r)^2 \le R_o^2$, therefore we can estimate the ratio $([\varphi_n^+]/[\varphi_n^-])$ with

$$
[\varphi_n^+] \le [\varphi_n^-] + A_o r_n^{q-p}\left(\frac{k_n}{r_n}\right)^q \le [\varphi_n^-]\left(1+\frac{A_o k_n^q r_n^{-p}}{(\frac{k_n}{r_n})^p + a_{Q_n}^-(\frac{k_n}{r_n})^q}\right) \le [\varphi_n^-]\left(1+A_o M^{q-p}\right),
$$

having used also that $k_n \le k \le M$. Hence, letting

$$
Y_n = \frac{[u<k_n]\cap Q_n|}{|Q_n|},
$$

and using that $|Q_n| \ge \gamma|Q_{n+1}|$ we obtain

$$
Y_{n+1} \le \gamma 2^{\frac{nq(N+p)}{N+2}}\left(\frac{[\varphi_n^+]\eta_k}{k_n^2}\right)^{\frac{p}{N+2}} Y_n^{1+\frac{p}{N+2}} \le \gamma 2^{\frac{nq(N+p)}{N+2}}\left(\frac{2^q[\varphi_{k,2r}^+]\eta_k}{(k/2)^2}\right)^{\frac{p}{N+2}} Y_n^{1+\frac{p}{N+2}}. \tag{3.11}
$$

We recall here that both A_o and M are structural data. For $0<\delta<1$ to determined, let $\bar{k}=\delta k$. The fast convergence Lemma (see for instance [31], Chap I, Lemma 4.1) gives $Y_n \to 0$ as $n\to\infty$, provided

$$
Y_0 = \frac{|[u<\delta k]\cap Q_0|}{|Q_0|} \le \gamma\left(\frac{k^2}{[\varphi_{k,2r}^+]\eta_k}\right) =: \nu. \tag{3.12}
$$

Observe that with our definition of η_k, the number $\nu\in(0,1)$ depends only on the data. In order to prove $Y_0\le\nu$, we use again the energy estimates Eq. 3.5 to get for $\delta\in(0,1/2)$ the bound

$$
\sup_{0<t<\eta_k}(\delta k)^2|[u(\cdot,t)<\delta k]\cap B_{2r}| \le \sup_{0<t<\eta_k}\int_{B_{2r}}(u-2\delta k)_-^2\,dx \le \gamma\,[\varphi_{2\delta k,2r}^+]\,|Q_{2r,\eta}^+| \le \gamma\delta^p[\varphi_{k,2r}^+]\,|Q_{2r,\eta}^+|,
$$

where we have used the property $[\varphi^{\pm}_{ck,r}] \le c^p\,[\varphi^{\pm}_{k,r}]$ for $c \in (0,1)$. Hence

$$\begin{aligned} Y_0 &= \frac{\int_0^{\eta_k} |[u(\cdot,t) < \delta k] \cap B_{2r}|\,dt}{|Q_0|} \\ &\le \frac{\eta_k\ \sup_{0<t<\eta_k} |[u(\cdot,t) < \delta k] \cap B_{2r}|}{|Q_0|} \\ &\le \gamma \frac{\delta^{p-2}}{k^2}[\varphi^{+}_{k,2r}]\eta_k = \gamma\delta^{p-2}/\nu \quad , \end{aligned}$$

and condition Eq. 3.12 is satisfied by choosing δ according to

$$Y_0 \le \nu \quad \Leftarrow \quad \delta \le (\gamma^{-1}\nu^2)^{\frac{1}{p-2}}.$$

□

Remark 3.3 Smaller radii than the levels ensure the previous necessary restriction on η_k, as

$$\begin{cases} (k/2r) \ge 1, \\ r \le R_o/4, \end{cases} \qquad \Rightarrow \qquad \begin{cases} \eta_k < (4r)^2, \\ \eta_k \le R_o^2. \end{cases} \tag{3.13}$$

Now, we need a tool to prolong the information Eq. 3.8 to indefinite longer times.

Next result roughly states that the estimate Eq. 3.7 is valid for all times that respect the law $|t-\bar{t}| \le (4r)^2$, at the price of a suitable decay of the level k. It is an adaptation of Corollary 3.4 of [37] to our double-phase problem.

Corollary 3.4 *Let the assumptions of Lemma 3.2 be satisfied, and suppose the equation Eq. 1.4 is satisfied in $Q^+_{4r}(\bar{x},\bar{t})$, with $0 < r < R_o$. Let us define the decreasing function*

$$\Psi(s) = \frac{s^2}{s^p + a^+_{Q^+_{4r}(\bar{x},\bar{t})} s^q}, \qquad \text{and} \quad \Psi^{-1} \quad \text{its inverse.}$$

Then for all $\bar{t} \le t \le \bar{t} + (4r)^2$ and δ, η_k as in Eq. 3.8, the following estimate holds true for all $x \in B_{r/2}(\bar{x})$

$$u(x,t) \ge \delta k \Psi^{-1}\left(1 + \frac{(t-\bar{t})}{\eta_k}\right). \tag{3.14}$$

Proof Observe first that, because Eq. 3.6 is preserved by diminishing k, we can take $0 < k < 1$. Consider, in the statement of Lemma 3.2 the alternatives

$$\bar{t} \le t \le \bar{t} + \eta_k \qquad \text{or} \qquad t > \bar{t} + \eta_k.$$

In the first case, the application of the aforementioned Lemma turns the information

$$u(x,\bar{t}) \ge k, \qquad x \in B_r(\bar{x}),$$

into

$$u(x,t) \ge \delta k = \delta k \Psi^{-1}(\Psi(1)) \ge \delta k \Psi^{-1}(1) \ge \delta k \Psi^{-1}(1 + (t-\bar{t})/\eta_k),$$

as both Ψ, Ψ^{-1} are decreasing and $\Psi(1) \le 1$. In the second case, we let

$$\bar{k} = k\Psi^{-1}\left(\frac{t-\bar{t}}{\eta_k}\right) \le k,$$

and the information

$$u(x,\bar{t}) \ge \bar{k} \quad \text{in} \quad B_r(\bar{x})$$

together again with the use of Lemma 3.2 brings us to

$$u(x,t) \geq \delta \bar{k}, \quad \text{in} \quad B_{r/2}(\bar{x}) \times (\bar{t},\, \bar{t}+\eta_{\bar{k}}),$$

with

$$\eta_{\bar{k}} = \Psi\left(k\Psi^{-1}\left(\frac{t-\bar{t}}{\eta_k}\right)\right) \geq \Psi(k)\Psi\left(\Psi^{-1}\left(\frac{t-\bar{t}}{\eta_k}\right)\right) = \eta_k\left(\frac{t-\bar{t}}{\eta_k}\right) = (t-\bar{t}).$$

Here we have used the simple fact that $\Psi(st) \geq \Psi(s)\Psi(t)$ for $s < 1$. □

3.3 Testing with Negative Powers Towards A Reverse Hölder's Inequality

Lemma 3.5 *Let $(\bar{x},\bar{t}) \in \Omega_T$, and $r,\eta > 0$ such that $Q^+_{4r,4\eta}(\bar{x},\bar{t}) \subset \Omega_T$. If u is a non-negative, local weak super-solution to Eq. 1.4 in Ω_T, then there exists a positive constant γ depending only on the data, such that, for any $\delta \geq 0$, and any $\alpha, \sigma \in (0,1)$, the inequality*

$$\begin{aligned}
&\frac{1}{1-\alpha}\sup_{\bar{t}<t<\bar{t}+\eta}\int_{B_r(\bar{x})}(u+\delta)^{1-\alpha}\zeta\,dx + \frac{\alpha}{\gamma}\iint\limits_{Q^+_{r,\eta}(\bar{x},\bar{t})}|\nabla[(u+\delta)^{\frac{p-\alpha-1}{p}}\zeta]|^p\,dxdt+\\
&+\frac{\alpha}{\gamma}\iint\limits_{Q^+_{r,\eta}(\bar{x},\bar{t})} a(x,t)|\nabla[(u+\delta)^{\frac{q-\alpha-1}{q}}\zeta]|^q\,dxdt \leqslant \frac{1}{(1-\alpha)}\|\partial_t\zeta\|_\infty \iint\limits_{Q^+_{r,\eta}(\bar{x},\bar{t})}(u+\delta)^{1-\alpha}dxdt+\\
&+\gamma\alpha^{1-p}\|\nabla\zeta\|^p_\infty \iint\limits_{Q^+_{r,\eta}(\bar{x},\bar{t})}(u+\delta)^{p-\alpha-1}dxdt + \gamma\alpha^{1-q}\|\nabla\zeta\|^q_\infty a^+_{Q^+_{r,\eta}(\bar{x},\bar{t})}\iint\limits_{Q^+_{r,\eta}(\bar{x},\bar{t})}(u+\delta)^{q-\alpha-1}dxdt.
\end{aligned} \tag{3.15}$$

holds true for any ζ_1, ζ_2 as in Lemma 3.1, being $\zeta = (\zeta_1\zeta_2)^q$.

The following Lemma constitutes, for nonnegative super-solutions to Eq. 1.4, the reverse Hölder's inequality that we will need for our purpose.

Lemma 3.6 *Let u be a non-negative, bounded, local weak super-solution to Eq. 1.4 in $Q^+_{r,\eta}(\bar{x},\bar{t}) \subset Q^+_r(\bar{x},\bar{t}) \subset \Omega_T$, with $r < R_o$. Then, for all $m \in (0,1)$ and $\delta \geq 0$, there exists a positive constant $\gamma(m)$, depending on the data and m, such that*

$$\begin{aligned}
&\frac{1}{r^p}\int_{\bar{t}}^{\bar{t}+\eta}⨍_{B_{r/2}(\bar{x})}(u+\delta)^{p-2+\frac{m(p+N)}{N}}\,dxdt + \frac{a^+_{Q_{r,\eta}(\bar{x},\bar{t})}}{r^q}\int_{\bar{t}}^{\bar{t}+\eta}⨍_{B_{r/2}(\bar{x})}(u+\delta)^{q-2+m(\frac{p+N}{N})}\,dxdt\\
&\qquad\leq \gamma(m) I^{m(\frac{p+N}{N})}\left\{1+\eta\left(\frac{I^{p-2}}{r^p} + a^+_{Q_{r,\eta}(\bar{x},\bar{t})}\frac{I^{q-2}}{r^q}\right)\right\},
\end{aligned} \tag{3.16}$$

where

$$I := \sup_{\bar{t}<t<\bar{t}+\eta} ⨍_{B_r(\bar{x})} u(x,t)dx.$$

The constant $\gamma(m)$ degenerates as soon as $m \downarrow 0$ or $m \uparrow 1$.

Proof Let $(\bar{x},\bar{t})$ be the origin (just to ease the notation) and let us define, for $n \in \mathbb{N}\cup\{0\}$,

$$Q_n = B_n \times (0,\eta), \quad B_n = B_{r_n}, \quad r_n = (r/2)(1+2^{-n}),$$

and $\zeta_n \in C^1_o(B_n)$ a cut-off function such that $\zeta_n \equiv 1$ on B_{n+1}, obliged to satisfy

$$\|\nabla\zeta_n\|_\infty := \|\nabla\zeta_n\|_{L^\infty(B_n)} \leq \gamma 2^n/r. \tag{3.17}$$

We use Hölder's inequality first with exponent N/p and then with exponent $1/m$ to estimate, in Q_n, the quantity

$$\begin{aligned}
&\iint_{Q_n} (u+\delta)^{p-2+m+\frac{mp}{N}} \zeta_n^q \, dxdt \\
&\quad \le \int_0^\eta \left(\int_{B_n} (u+\delta)^m \, dx \right)^{\frac{p}{N}} \left(\int_{B_n} [(u+\delta)^{(p-2+m)} \zeta_n^q]^{\frac{N}{N-p}} \, dx \right)^{\frac{N-p}{N}} dt \\
&\quad \le \int_0^\eta \left[\left(\int_{B_n} (u+\delta) \, dx \right)^m |B_n|^{1-m} \right]^{\frac{p}{N}} \left[\int_{B_n} \left((u+\delta)^{\frac{p-2+m}{p}} \zeta_n^{\frac{q}{p}} \, dx \right)^{\frac{Np}{N-p}} \right]^{\frac{N-p}{N}} dt \\
&\quad \le \int_0^\eta \left[\left(\sup_{0<t<\eta} \fint_{B_n} u \, dx + \delta \right)^m |B_n| \right]^{\frac{p}{N}} \left[\int_{B_n} |\nabla[(u+\delta)^{\frac{p-2+m}{p}} \zeta_n^{\frac{q}{p}}]|^p \, dx \right] dt,
\end{aligned} \tag{3.18}$$

by applying Sobolev-Poincaré embedding in the last inequality. Now, the first factor of the product on the right-hand side of Eq. 3.18 is a power of I, while we estimate the second integral on the right-hand side with Lemma 3.5 with $m = 1-\alpha$ and $\zeta_n = \zeta_1^q$ independent of time, to get from Eq. 3.18 the inequality

$$\begin{aligned}
&\iint_{Q_{n+1}} (u+\delta)^{p-2+\frac{m(p+N)}{N}} \, dxdt \\
&\le \iint_{Q_n} (u+\delta)^{p-2+\frac{m(p+N)}{N}} \zeta_n^q \, dxdt \\
&\le \gamma(m)(2I^m|B_n|)^{\frac{p}{N}} \left\{ \int_{B_n} (u+\delta)^m \, dx + \iint_{Q_n} \|\zeta_n\|_\infty^p (u+\delta)^{p-2+m} + \|\zeta_n\|_\infty^q a_{Q_n}^+ (u+\delta)^{q-2+m} \, dxdt \right\} \\
&\le \gamma |B_n|^{\frac{p}{N}+1} \left\{ I^{m(\frac{p+N}{N})} + I^{\frac{mp}{N}} \int_0^\eta \fint_{B_n} \|\zeta_n\|_\infty^p (u+\delta)^{p-2+m} + \|\zeta_n\|_\infty^q a_{Q_n}^+ (u+\delta)^{q-2+m} \, dxdt \right\} \\
&\qquad =: \gamma |B_n|^{\frac{p}{N}+1} E_n.
\end{aligned} \tag{3.19}$$

We perform a similar estimate for the phase energy: first we use Hölder's inequality with power N/p and then with $(N-p)/(N-q)$ to get

$$\begin{aligned}
&a_{Q_0}^- \iint_{Q_{n+1}} (u+\delta)^{q-2+m(\frac{p+N}{N})} \, dxdt \\
&\qquad \le a_{Q_0}^- \iint_{Q_n} (u+\delta)^{q-2+m(\frac{p+N}{N})} \zeta_n^q \, dxdt \\
&\qquad \le a_{Q_0}^- \int_0^\eta \left(\int_{B_n} (u+\delta)^m dx \right)^{\frac{p}{N}} \left(\int_{B_n} [(u+\delta)^{q-2+m} \zeta_n^q]^{\frac{N}{N-p}} dx \right)^{\frac{N-p}{N}} dt \\
&\qquad \le a_{Q_0}^- (2I^m|B_n|)^{\frac{p}{N}} \int_0^\eta \left(\int_{B_n} \left([(u+\delta)^{q-2+m} \zeta_n^q]^{\frac{N}{N-q}} dx \right)^{\frac{N-q}{N}} |B_n|^{\frac{q-p}{N}} dt \right. \\
&\qquad \le \gamma (I^m|B_n|)^{\frac{p}{N}} |B_n|^{\frac{q-p}{N}} \iint_{Q_n} a(x,t) |\nabla[(u+\delta)^{\frac{q-2+m}{q}} \zeta_n]|^q \, dxdt \\
&\qquad \le \gamma (I^m |B_n|^{\frac{q}{N}+1}) E_n,
\end{aligned} \tag{3.20}$$

where we have used again Sobolev-Poincaré inequality in the fourth inequality and Lemma Eq. 3.5 in the fifth, denoting the averaged right-hand side of Eq. 3.15 in Q_n with E_n, as above. Now we use the assumption

$$Q_0 = Q^+_{r,\eta} \subset Q^+_r,$$

to apply Eq. 1.6 and estimate

$$\begin{aligned}
&\frac{a^+_{Q_0}}{r^q}\int_0^\eta \fint_{B_n}(u+\delta)^{q-2+m(\frac{p+N}{N})}\zeta_n^q\,dxdt\\
&\quad\le \frac{a^-_{Q_0}}{r^q}\int_0^\eta \fint_{B_n}(u+\delta)^{q-2+m(\frac{p+N}{N})}\zeta_n^q\,dxdt+\frac{AM^{q-p}}{r^p}\int_0^\eta \fint_{B_n}(u+\delta)^{p-2+m(\frac{p+N}{N})}\zeta_n^q\,dxdt\\
&\qquad\qquad\le \gamma I^m(1+AM^{q-p})E_n,
\end{aligned}$$

applying Eqs. 3.19-3.20. Finally, we estimate E_n by Young's inequality as

$$\begin{aligned}
E_n \le\ & I^{m(\frac{p+N}{N})}+\int_0^\eta \fint_{B_n}\|\nabla\zeta_n\|^p_\infty(u+\delta)^{p-2}\Big(\epsilon(u+\delta)^{m(\frac{p+N}{N})}+c(\epsilon)I^{\frac{m(p+N)}{N}}\Big)dxdt+\\
&+\int_0^\eta \fint_{B_n}\|\nabla\zeta_n\|^q_\infty a^+_{Q0}(u+\delta)^{q-2}\Big(\epsilon(u+\delta)^{m(\frac{p+N}{N})}+c(\epsilon)I^{\frac{m(p+N)}{N}}\Big)dxdt\\
&\le \gamma\epsilon\int_0^\eta \fint_{B_0}\Big(\frac{(u+\delta)^{p-2+m(\frac{p+N}{N})}}{\|\nabla\zeta_n\|^{-p}_\infty}+a^+_{Q_0}\frac{(u+\delta)^{q-2+m(\frac{p+N}{N})}}{\|\nabla\zeta_n\|^{-q}_\infty}\Big)dxdt+\\
&\qquad+\gamma I^{m(\frac{p+N}{p})}\Big(1+\int_0^\eta \fint_{B_n}\frac{(u+\delta)^{p-2}}{\|\nabla\zeta_n\|^{-p}_\infty}+a^+_{Q_0}\frac{(u+\delta)^{q-2}}{\|\nabla\zeta_n\|^{-q}_\infty}\,dxdt\Big).
\end{aligned} \tag{3.21}$$

Hence, collecting the terms with ϵ on a whole initial energetic term J_0, and specifying the properties Eq. 3.17 of ζ_n, we have

$$\begin{aligned}
&J_{n+1} := \frac{1}{r^p}\int_0^\eta \fint_{B_{n+1}}(u+\delta)^{p-2+\frac{m(p+N)}{N}}\,dxdt+\frac{a^+_{Q_0}}{r^q}\int_0^\eta \fint_{B_{n+1}}(u+\delta)^{q-2+m(\frac{p+N}{N})}\,dxdt\\
&\le \gamma I^m E_n\\
&\le \epsilon J_0+\gamma 2^n I^{m(\frac{p+N}{N})}\Big\{1+r^{-p}\int_0^\eta \fint_{B_n}(u+\delta)^{p-2}\,dxdt+r^{-q}a^+_{Q_0}\int_0^\eta \fint_{B_n}(u+\delta)^{q-2}\,dxdt\Big\}\\
&\le \epsilon J_0+\gamma 2^n\Big\{I^{m(\frac{p+N}{N})}+r^{-p}\int_0^\eta \fint_{B_n}I^{m(\frac{p+N}{N})}\Big((u+\delta)^{p-2}+r^{-q}a^+_{Q_0}(u+\delta)^{q-2}\Big)dxdt\Big\}\\
&\le \epsilon J_0+\gamma 2^n\Big\{I^{m(\frac{p+N}{N})}+\epsilon r^{-p}\int_0^\eta \fint_{B_n}(u+\delta)^{p-2+m(\frac{p+N}{N})}+C(\epsilon)I^{p-2+m(\frac{p+N}{N})}\,dxdt+\\
&\qquad+r^{-q}a^+_{Q_0}\int_0^\eta \fint_{B_n}\frac{\tilde\epsilon}{2\gamma}(u+\delta)^{q-2+m(\frac{p+N}{N})}+C(\tilde\epsilon)I^{q-2+m(\frac{p+N}{N})}\,dxdt\Big\},
\end{aligned} \tag{3.22}$$

through the use of Young's inequality again, on the last estimate with powers $\frac{p-2+m(\frac{p+N}{N})}{p-2}$ and $\frac{q-2+m(\frac{p+N}{N})}{q-2}$ separately on the terms involving powers of $(u+\delta)$ and I. This finally provides, by choosing again appropriately $\epsilon\in(0,1)$ and reabsorbing the terms in J_0, the estimate

$$J_{n+1}\le \epsilon J_0+\gamma\epsilon^{-\gamma}2^{bn}I^{m(\frac{p+N}{N})}\Big\{1+\eta\Big(\frac{I^{p-2}}{r^p}+a^+_{Q_0}\frac{I^{q-2}}{r^q}\Big)\Big\}.$$

Hence a classical iteration provides

$$J_\infty \le \gamma I^{m(\frac{p+N}{N})}\left\{1+\eta\left(\frac{I^{p-2}}{r^p}+a^+_{Q_0}\frac{I^{q-2}}{r^q}\right)\right\}.$$

□

3.4 Weak Harnack's Inequality

We borrow the following result from [70].

Lemma 3.7 *Let u be a non-negative, bounded, weak super-solution to Eq. 1.4 in $Q^+_{16r}(\bar{x},\bar{t})$. Then there exist positive constants $C_{\mathcal{H}}$ and b, depending only on the data, such that*

$$\bar{\mathcal{I}} := \fint_{B_r(\bar{x})} u(x,\bar{t})dx \leqslant C_{\mathcal{H}}\left\{r + r\varphi^{-1}_{Q^+_{12r}(\bar{x},\bar{t})}\left(\frac{r^2}{\eta}\right) + \inf_{B_{2r}(\bar{x})} u(\cdot,t)\right\}, \tag{3.23}$$

for all time levels

$$\bar{t}+\frac{\eta_1}{2} \leqslant t \leqslant \bar{t}+\eta_1, \quad \eta_1 := \min\left(\eta\,, \frac{br^2}{\varphi^+_{Q^+_{12r}(\bar{x},\bar{t})}(\frac{\bar{\mathcal{I}}}{r})}\right). \tag{3.24}$$

Here $\varphi^{-1}_Q(v)$ is the inverse function to the function $\varphi^+_Q(v) := v^{p-2} + a^+_Q v^{q-2}$.

Remark 3.8 Let $f : \mathbb{R} \to \mathbb{R}$ be a function that has an increasing inverse f^{-1} and satisfies $f(\lambda s) \le \lambda^{q-2} f(s)$ for all $\lambda > 1$, $s \in \mathbb{R}$. By applying f^{-1} to the previous property one gets $\lambda s \le f^{-1}(\lambda^{q-2} f(s))$ and choosing $s = f^{-1}(x)$ and $\alpha = \lambda^{q-2}$ results in formula

$$f^{-1}(x) \le \alpha^{\frac{-1}{q-2}} f^{-1}(\alpha x), \quad \forall x \in \mathbb{R}, \quad \alpha > 1.$$

The scaling property above translates to $\varphi^{-1}_Q(cx) \le c^{\frac{1}{q-2}}\varphi^{-1}_Q(x)$ for all $x \in \mathbb{R}, 0 < c < 1$.

4 Geometric Setting and Auxiliary Results

All the estimates of the previous Sections were of local nature. Here we refine the classic approach to parabolic boundary regularity, in the framework of the double-phase operator Eqs. 1.4-1.5.

4.1 Preamble

Let $(x_o, t_o) \in S_T$ be a point of the lateral boundary of Ω_T. As conditions Eq. 1.5 imply $\mathbb{A}(x,t,O) = O$, we extend $\mathbb{A}$ to a vector field $\mathbb{A}(x,t,\xi) : \mathbb{R}^N \times (0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ by defining it zero on those vector fields $\xi(x,t) : \mathbb{R}^N \times (0,\infty) \to \mathbb{R}^N$ such that $\xi(x,t) = O$ in the complement of Ω_T. It is easily seen that this extension preserves Eq. 1.4-1.5 in its local definition Eq. 3.1, that now can be formulated in any cylinder

$$Q_r(x_o,t_o) = Q^-_r(x_o,t_o) \cup Q^+_r(x_o,t_o) \not\subset \Omega_T.$$

In this sense we say that some function v, that vanishes outside Ω_T, is a local weak sub (super)-solution to Eqs. 1.4-1.5 in such a cylinder.

In the previous Sections we mainly only cared about super-solutions: next Lemma motivates this specialization (Fig. 1). Indeed, by extending the equation as above on a cylinder, the truncations $(u-k)_\pm$ are sub-solutions to an equation of the type Eq. 1.4, so that $(c-(u-k)_\pm)$ are non-negative super-solutions (to an equation of the type Eq. 1.4), for an appropriate choice of $c > 0$. We refer to Lemma 2.1 of [36] and Lemma 2.2 of [35] for more details.

Lemma 4.1 *Let u be a local weak solution to Eqs. 1.4-1.5 in Ω_T and assume that for a given function $f \in C(\overline{\Omega_T})$ it holds $(u-f) \in V_o^{2,q}$. Let $(x_o, t_o) \in S_T$ and, for some $r > 0$ and $0 < \eta \leqslant r^2$, construct the cylinder*

$$Q^-_{r,\eta}(x_o, t_o) := B_r(x_o) \times (t_o - \eta, t_o) \subset Q_r(x_o, t_o).$$

We define the zero-extension of the truncations

$$u_k^+ = \begin{cases} (u-k^+)_+ & , \text{ in } \Omega_T \cap Q^-_{r,\eta}(x_o,t_o) \\ 0 & , \text{ in } Q^-_{r,\eta}(x_o,t_o) \setminus \Omega_T \end{cases} , \quad \textit{for levels} \quad k^+ \geq \sup_{S_T \cap Q^-_{r,\eta}(x_o,t_o)} f, \tag{4.1}$$

and

$$u_k^- = \begin{cases} (u-k^-)_- & , \text{ in } \Omega_T \cap Q^-_{r,\eta}(x_o,t_o) \\ 0 & , \text{ in } Q^-_{r,\eta}(x_o,t_o) \setminus \Omega_T \end{cases} , \quad \textit{for levels} \quad k^- \leq \inf_{S_T \cap Q^-_{r,\eta}(x_o,t_o)} f. \tag{4.2}$$

Then $u_k^\pm$ is a weak sub-solution to Eq. 1.4 in $Q^-_{r,\eta}(x_o, t_o)$.

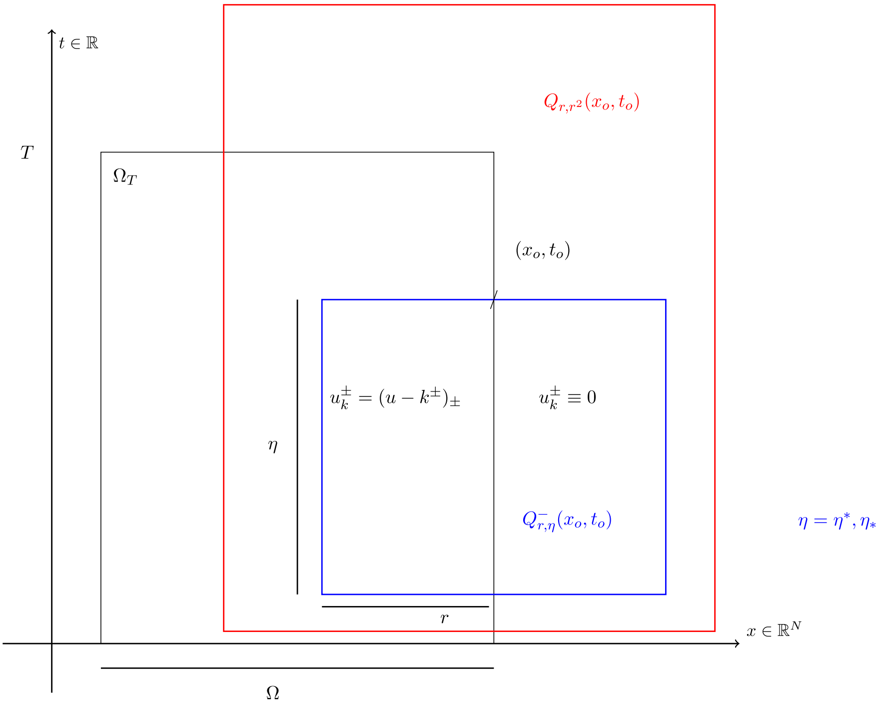

Fig. 1 Scheme of the geometric setting of the proof. For the definition of η^*, η_* see Subsection 4.2 below. Considered a same radius r, when $a(x_o, t_o)$ approaches zero, η_* stretches to infinity while η^* stays unvaried. This motivates the reduction of radii $r < R$ in the former case, according to the size of the phase

Finally, for $k^\pm$ as above, we define $u_k^\pm$ as in Eqs. 4.1-4.2, and we set

$$w^\pm = \mu^\pm - u_k^\pm, \qquad \text{for} \quad \mu^\pm \geq \sup_{Q_r^-(x_o,t_o)} u_k^\pm .$$

Evidently $w^\pm$ is a non-negative weak super-solution to Eq. 1.4 in $Q_r^-(x_o, t_o)$. From here to Section 5, we drop the superscript $\pm$, because all we need is to work with a generic super-solution w.

4.2 Geometric Setting

The definition of the time-length $\eta > 0$ (that must obey $\eta \leq r^2$) needs to distinguish between two different cases: $a(x_o, t_o) = 0$ and $a(x_o, t_o) > 0$.

Case $a(x_o, t_o) = 0$

For a number $\gamma^* > 1$ to be chosen, we let

$$\delta_p(r) := \left(\frac{\mathcal{C}_p\big(\overline{B_r(x_o)} \setminus \Omega; B_{2r}(x_o)\big)}{\mathcal{C}_p(\overline{B_r(x_o)}; B_{2r}(x_o))} \right)^{\frac{1}{p-1}}$$

and consider the time-length

$$\eta^* := \frac{\gamma^* r^p}{(\mu \delta_p(r))^{p-2}}.$$

Case $a(x_o, t_o) > 0$

Here we set a maximal radius

$$R^{q-p} := \frac{a(x_o, t_o)}{2A_o}, \tag{4.3}$$

and further we will assume that $r \leqslant \min\{R, R_o\}/24$. This gives us the control on the phase: indeed, in this case

$$a^+_{Q^-_{r,\eta}(x_o,t_o)} \leqslant 2a^-_{Q^-_{r,\eta}(x_o,t_o)} \tag{4.4}$$

as the simple following computation shows

$$a^+_{Q^-_{r,\eta}(x_o,t_o)} - a^-_{Q^-_{r,\eta}(x_o,t_o)} \leqslant A_o(r)^{q-p} \leqslant \frac{a(x_o, t_o)}{2} \leqslant \frac{1}{2} a^+_{Q^-_{r,\eta}(x_o,t_o)} .$$

Moreover, the time-length η_* here is defined through the q-capacity and the value of $a(x_o, t_o)$, as

$$\delta_q(r) := \left(\frac{\mathcal{C}_q\big(\overline{B_r(x_o)} \setminus \Omega; B_{2r}(x_o)\big)}{\mathcal{C}_q(\overline{B_r(x_o)}; B_{2r}(x_o))} \right)^{\frac{1}{q-1}}, \qquad \eta_* := \frac{\gamma_* r^q}{a(x_o, t_o)(\mu \delta_q(r))^{q-2}},$$

again for a number $\gamma_* > 0$ to be chosen (in Eq. 4.31).

Remark 4.2 The conditions $\eta^*, \eta_* < r^2$ imply the estimates

$$\eta^* < r^2 \iff \mu \delta_p(r) \geq (\gamma^*)^{\frac{1}{p-2}} r, \qquad \text{and} \qquad \eta_* < r^2 \iff \mu \delta_q(r) \geq (\gamma_*)^{\frac{1}{q-2}} r. \tag{4.5}$$

4.3 Capacity Estimates

Now we specialize our estimates towards capacity, considering a test function that vanishes outside a small cylinder. Within the special local geometry chosen, the equation provides a bound for the p or q-capacity of Ω around the point (x_o, t_o), in terms of the averaged L^1-norm of w. Since a distinction between time lengths is due, for any $0 < \eta \leq r^2$ we define

$$\mathcal{I}(\eta, r) = \sup_{t_o-\eta \leqslant t \leqslant t_o - \frac{\eta}{4}} \fint_{B_{2r}(x_o)} w(x,t)\, dx, \quad \text{and} \quad \mathcal{I}_p(r) = \mathcal{I}(\eta^*, r), \quad \mathcal{I}_q(r) = \mathcal{I}(\eta_*, r). \tag{4.6}$$

Lemma 4.3 *Let u be a non-negative, local weak solution of Eqs. 1.4-1.5 in Ω_T and $u = f$ on S_T. Fix $(x_o, t_o) \in S_T$ and construct $Q^-_{r,\eta}(x_o, t_o)$ as above (in Section 4.2). There exists a constant $\hat{\gamma} > 0$, depending only on the data, such that the following is valid. If $a(x_o, t_o) = 0$, then for any $0 < r < R_o/2$ we have that*

$$\mu\, \delta_p(r) \leqslant \hat{\gamma} \mathcal{I}_p(r). \tag{4.7}$$

On the other hand, if $a(x_o, t_o) > 0$, for all $0 < r < \min\{R_o, R/24\}$, then we find the inequality

$$\mu\, \delta_q(r) \leqslant \hat{\gamma} \mathcal{I}_q(r) + \hat{\gamma} \left(\frac{r}{R}\right)^{q-1} \frac{1}{(\mu \delta_q(r))^{q-2}}. \tag{4.8}$$

Proof We divide the argument in two steps: in the first one the special geometry of η^*, η_* plays no role; while the second one specializes toward p or q capacities.

STEP 1 - A common potential estimate For any $0 < r < \min\{R_o/2, R/16\}$, $0 < \eta < r^2$, we construct cylinders $Q_1 \subset Q_2 \subset Q_3$

$$Q_1 = B_r(x_o) \times \left(t_o - \frac{3\eta}{4}, t_o - \frac{5\eta}{8}\right), \; Q_2 = B_{2r}(x_o) \times \left(t_o - \frac{7\eta}{8}, t_o - \frac{3\eta}{8}\right), \; Q_3 = B_{4r}(x_o) \times \left(t_o - \eta,\; t_o - \frac{\eta}{4}\right)$$

and let $\zeta \in C^1_o(Q_2)$, be a cut-off function between Q_1 and Q_2, i.e.

$$\zeta_{|Q_1} \equiv 1, \quad \text{and} \quad 0 \leqslant \zeta \leqslant 1, \quad |\nabla \zeta| \leqslant \frac{2}{r}, \quad |\zeta_t| \leqslant \frac{8}{\eta} \quad \text{in } Q_2.$$

By testing Eq. 3.3 with $u_{k,h}\zeta$, for $t \in (t_o - \frac{7\eta}{8}, t_o - \frac{3\eta}{8} - h)$, using the fact that u_k is a sub-solution of Eq. 1.4 we obtain

$$\int_{B_{2r}(x_o)} \frac{\partial u_{k,h}}{\partial t} u_{k,h} \zeta^q\, dx + \int_{B_{2r}(x_o)} [\mathbb{A}(x,t,\nabla u_k)]_h \nabla (u_{k,h} \zeta^q) dx \leqslant 0,$$

which yields

$$\int_{B_{2r}(x_o)} \frac{\partial w_h}{\partial t} w_h \zeta\, dx + \int_{B_{2r}(x_o)} [\mathbb{A}(x,t,\nabla u_k)]_h \nabla u_{k,h} \zeta\, dx$$
$$\leqslant\leqslant \mu \int_{B_{2r}(x_o)} \frac{\partial w_h}{\partial t} \zeta\, dx + \int_{B_{2r}(x_o)} [\mathbb{A}(x,t,\nabla u_k)]_h u_{k,h} \nabla \zeta\, dx.$$

Now we integrate this inequality over $(t_o-\frac{7\eta}{8}, t_o-\frac{3\eta}{8}-h)$. Then, by performing integration by parts in the parabolic terms and finally letting $h \downarrow 0$, while using conditions Eq. 1.5, we find

$$\begin{aligned} -q\mu \iint_{Q_2} w|\partial_t\zeta|dxdt - \frac{q}{2}\iint_{Q_2} w^2|\partial_t\zeta|\,dxdt + \iint_{Q_2} \mathbb{A}(x,t,\nabla u_k)\nabla u_k\zeta\,dxdt \\ \le \mu \iint_{Q_2} \mathbb{A}(x,t,\nabla u_k)\nabla\zeta\,dxdt. \end{aligned} \tag{4.9}$$

From here, by using the properties of ζ and the structure conditions Eq. 1.5, we get

$$\iint\limits_{Q_1} \varphi(x,t,|\nabla w|)dxdt \leqslant \gamma\mu r^N \mathcal{I}(\eta,r) + \gamma\frac{\mu}{r}\left\{\iint\limits_{Q_2} |\nabla w|^{p-1}dxdt + \iint\limits_{Q_2} a(x,t)|\nabla w|^{q-1}dxdt\right\}, \tag{4.10}$$

for a constant $\gamma > 0$ depending only on the data. We abbreviate $\mathcal{I}(\eta,r) = \mathcal{I}$ to ease notation.

Let us estimate the terms on the right-hand side of Eq. 4.10. By Hölder's inequality and being $\mathcal{I} > 0$, for any $\bar{m} \in (0,1)$ we obtain

$$\begin{aligned} \iint\limits_{Q_2} |\nabla w|^{p-1}dxdt + \iint\limits_{Q_2} a(x,t)|\nabla w|^{q-1}dxdt \leqslant \\ \leqslant \left(\iint\limits_{Q_2} (w+\mathcal{I})^{-1-\bar{m}}|\nabla w|^p dxdt\right)^{\frac{p-1}{p}} \left(\iint\limits_{Q_2} (w+\mathcal{I})^{(1+\bar{m})(p-1)}dxdt\right)^{\frac{1}{p}} + \\ + \left(\iint\limits_{Q_2} a(x,t)(w+\mathcal{I})^{-1-\bar{m}}|\nabla w|^q dxdt\right)^{\frac{q-1}{q}} \left(\iint\limits_{Q_2} a(x,t)(w+\mathcal{I})^{(1+\bar{m})(q-1)}dxdt\right)^{\frac{1}{q}}. \end{aligned} \tag{4.11}$$

Using Lemma 3.6 with $m = N(1+\bar{m}(p-1))/(N+p) < 1$ we obtain

$$\begin{aligned} &\iint\limits_{Q_2} (w+\mathcal{I})^{(1+\bar{m})(p-1)}dxdt \\ &\quad\leqslant \gamma(\bar{m}) r^{N+p}\mathcal{I}^{1+\bar{m}(p-1)}\left\{1+\eta\left(\frac{\mathcal{I}^{p-2}}{r^p} + a^+_{Q_{2r}(x_o,t_o)}\frac{\mathcal{I}^{q-2}}{r^q}\right)\right\} \\ &\quad =: \gamma(\bar{m}) r^{N+p}\mathcal{I}^{1+\bar{m}(p-1)}\,\mathcal{F}(\mathcal{I}). \end{aligned} \tag{4.12}$$

Similarly, by Lemma 3.6 with $m = N(1+\bar{m}(q-1))/(N+p) < 1$, we evaluate

$$\begin{aligned} \iint\limits_{Q_2} a(x,t)(w+\mathcal{I})^{(1+\bar{m})(q-1)}dxdt \leqslant a^+_{Q_{2r}(x_o,t_o)} \iint\limits_{Q_2} (w+\mathcal{I})^{(1+\bar{m})(q-1)}dxdt \\ \leqslant \gamma(\bar{m}) r^{N+q}\mathcal{I}_1^{1+\bar{m}(q-1)}\mathcal{F}(\mathcal{I}). \end{aligned} \tag{4.13}$$

Now we use Lemma 3.5 for the pair of cylinders Q_2 and Q_3, with $\zeta_1 \equiv 1$ on Q_2, to compute

$$\iint\limits_{Q_2} (w+\mathcal{I})^{-1-\bar{m}}|\nabla w|^p dxdt + \iint\limits_{Q_2} a(x,t)(w+\mathcal{I})^{-1-\bar{m}}|\nabla w|^q dxdt$$
$$\leqslant \gamma(\bar{m}) r^N \mathcal{I}^{1-\bar{m}} + \frac{\gamma(\bar{m})}{r^p} \iint\limits_{Q_3} (w+\mathcal{I})^{p-1-\bar{m}} dxdt +$$
$$+ a^+_{Q_{2r}(x_o,t_o)} \frac{\gamma(\bar{m})}{r^q} \iint\limits_{Q_3} (w+\mathcal{I})^{q-1-\bar{m}} dxdt,$$

which by Lemma 3.6 with $1-\bar{m} = m(p+N)/N$ yields the inequality

$$\iint\limits_{Q_2} (w+\mathcal{I})^{-1-\bar{m}}|\nabla w|^p dxdt + \iint\limits_{Q_2} a(x,t)(w+\mathcal{I})^{-1-\bar{m}}|\nabla w|^q dxdt \leqslant \gamma(\bar{m}) r^N \mathcal{I}^{1-\bar{m}} \mathcal{F}(\mathcal{I}). \tag{4.14}$$

Collecting estimates Eqs. 4.10–4.14, while observing that the powers in Eq. 4.11 adjust to 1, we arrive at

$$\iint\limits_{Q_1} \varphi(x,t,|\nabla w|) dxdt \leqslant \gamma \mu r^N \mathcal{I}\mathcal{F}(\mathcal{I}). \tag{4.15}$$

Now, as we aim to a capacity estimate, we estimate

$$\iint_{Q_2} \varphi(x,t,|\nabla(\zeta w)|)\, dxdt \le \iint_{Q_2} \varphi(x,t,|\nabla w|)\, dxdt + \gamma(C_i) \iint_{Q_2} \left\{ (w|\nabla \zeta|)^p + a(x,t)|w\nabla\zeta|^q \right\} dxdt, \tag{4.16}$$

where we have used the structure conditions Eq. 1.5. We take care of the second integral term, using Lemma 3.6 with $\delta = 0$ and $m = N/(p+N)$ to get (as here $\eta < r^2 < R_o^2/4$),

$$\iint_{Q_2} w^p |\nabla\zeta|^p\, dxdt + \iint_{Q_2} a(x,t) w^q |\nabla\zeta|^q\, dxdt$$
$$\le \frac{\mu}{r^p} \iint_{Q_3} w^{p-1}\, dxdt + a^+_{Q_2} \frac{\mu}{r^q} \iint_{Q_3} w^{q-1}\, dxdt$$
$$\le \gamma \mu r^N \mathcal{I}\mathcal{F}(\mathcal{I}). \tag{4.17}$$

Hence finally, joining estimates Eqs. 4.15 and 4.17 into 4.16 we obtain the potential estimate

$$\iint_{Q_2} \varphi(x,t,|\nabla(\zeta w)|)\, dxdt \le \gamma \mu r^N \mathcal{I}(\eta,r)\mathcal{F}(\mathcal{I}(\eta,r)), \tag{4.18}$$

where we recall that φ is defined in Eq. 1.5, $\mathcal{I}(\eta,r)$ is defined in Eq. 4.6, and $\mathcal{F}$ is defined in Eq. 4.12.

STEP 2 - Geometry enters into play

Here we divide the study in two cases depending on the value of the phase at the boundary point.

If $a(x_o,t_o)=0$, we fix $\eta=\eta^*$ as above (Section 4.2) and proceed by contradiction: we assume that for any $\varepsilon\in(0,1)$ (to be determined and depending only on the data) the converse inequality

$$\mathcal{I}_p\leqslant\varepsilon\mu\delta_p(r) \tag{4.19}$$

holds true, because otherwise inequality Eq. 4.7 is found. Now, by the definition, the scaling properties of the p-capacity $\mathcal{C}_p(B_r(x_o);B_{2r}(x_o))=\gamma r^{N-p}$ for a positive constant γ depending only on N and p, and our choice of η^*, we have

$$\iint_{Q_2}\varphi(x,t,|\nabla(\zeta w)|)dxdt\geqslant\frac{3}{4}\mu^p\eta^*\mathcal{C}_p(B_r(x_o)\setminus\Omega;B_{2r}(x_o))\geq\gamma\gamma^*\mu^2\delta_p(r)r^N. \tag{4.20}$$

Moreover, since $0<\eta<r^2<R_o^2$ condition Eq. 1.6 is in force and

$$a^+_{Q_{2r}(x_o,t_o)}\frac{\mathcal{I}_p^{q-2}}{r^q}\leqslant A_o(2r)^{q-p}\frac{\mathcal{I}_p^{q-2}}{r^q}\leqslant\gamma\frac{\mathcal{I}_p^{p-2}}{r^p},\quad\text{with}\quad\gamma=A_o(2M)^{q-p},$$

so that the inequalities Eq. 4.18-4.20, chained, can be rewritten as

$$\mu\delta_p(r)r^N\leqslant\frac{\gamma}{\gamma^*}r^N\mathcal{I}_p+\frac{\gamma}{\gamma^*}r^{N-p}\eta^*\mathcal{I}_p^{p-1}\leqslant\gamma(\varepsilon+\varepsilon^{p-1})\mu\delta_p(r)r^N. \tag{4.21}$$

Choosing ε small enough, such that $\gamma(\varepsilon+\varepsilon^{p-1})=\frac{1}{2}$, a contradiction to Eq. 4.19 is reached. This proves inequality Eq. 4.7.

Now if $a(x_o,t_o)>0$, we fix $\eta=\eta_*$ for $0<r<R/16$ (still referring to Section 4.2) and we assume that for any $\epsilon\in(0,1)$, the estimate

$$\mathcal{I}_q+\left(\frac{r}{R}\right)^{q-1}\frac{1}{(\mu\delta_q(r))^{q-2}}\leqslant\varepsilon\mu\delta_q(r) \tag{4.22}$$

holds true; otherwise, inequality Eq. 4.8 would be in force. By the assumption $0<r<R/16$ the estimate Eq. 4.4 is valid, while the definition of q-capacity and the choice of η_* imply

$$\iint_{Q_2}\varphi(x,t,|\nabla(\zeta w)|)dxdt\geqslant a(x_o,t_o)\mu^q\ \frac{3}{4}\eta_*\,\mathcal{C}_q(B_r(x_o)\setminus\Omega;B_{2r}(x_o))\geq\gamma\gamma_*\,\mu^2\delta_q(r)r^N, \tag{4.23}$$

and Eqs. 4.18-4.23, chained, can be rewritten as

$$\begin{aligned}\mu\delta_q(r)r^N&\leqslant\frac{\gamma}{\gamma_*}r^N\mathcal{I}_q+\frac{\gamma}{\gamma_*}a(x_o,t_o)\eta_*r^{N-q}\mathcal{I}_q^{q-1}+\frac{\gamma}{\gamma_*}\eta_*r^{N-p}\mathcal{I}_q^{p-1}\\&\leqslant\gamma(\varepsilon+\varepsilon^{q-1})\mu\delta_q(r)r^N+\frac{\gamma}{\gamma_*}\varepsilon^{p-1}\mu^{p-1}\eta_*\delta_q(r)^{p-1}r^{N-p}\\&\leqslant\gamma(\varepsilon+\varepsilon^{p-1}+\varepsilon^{q-1})\mu\delta_q(r)r^N+\gamma\varepsilon^{p-1}\frac{r^{N+q-1}}{a(x_o,t_o)^{\frac{q-1}{q-p}}(\mu\delta_q(r))^{q-2}}\\&=\gamma(\varepsilon+\varepsilon^{p-1}+\varepsilon^{q-1})\mu\delta_q(r)r^N+\gamma\varepsilon^{p-1}\left(\frac{r}{R}\right)^{q-1}\frac{r^N}{(\mu\delta_q(r))^{q-2}},\end{aligned} \tag{4.24}$$

where in the second inequality we have used Eq. 4.22 as $\mathcal{I}\leq\varepsilon\mu\delta_q(r)$ and the definition of η_*, while in the third inequality we have used Young's inequality with $(q-1)/(p-1)$ and its conjugate $(q-1)/(q-p)$ weighted on $\mu\delta_q(r)r^N$, separating the term ε^{p-1} from the remainder. To arrive to the wanted contradiction, it is enough to choose ε such that $\gamma(\varepsilon+2\varepsilon^{p-1}+\varepsilon^{q-1})=1/2$. This completes the proof of Lemma 4.3. □

Lemma 4.4 *Let the assumptions of Lemma 4.3 be valid. Then, in the case* $a(x_o, t_o) = 0$, *there exists a constant* $C_p > 0$, *depending only on the data, such that, either*

$$\mu\delta_p(r) \leq 2C_p r, \tag{4.25}$$

or

$$\sup_{Q^-_{\frac{r}{2},\frac{\eta}{8}}(x_o,t_o)} u_k \leqslant \mu\left(1 - \frac{1}{2C_p}\delta_p(r)\right), \tag{4.26}$$

In the case $a(x_o, t_o) > 0$, *there exists a constant* $C_q > 0$, *depending only on the data, such that either*

$$\mu\delta_q(r) \leq 4C_q r + (4C_q)^{\frac{1}{q-1}}\left(\frac{r}{R}\right), \tag{4.27}$$

or

$$\sup_{Q^-_{\frac{r}{2},\frac{\eta}{8}}(x_o,t_o)} u_k \leqslant \mu\left(1 - \frac{1}{2C_q}\delta_q(r)\right). \tag{4.28}$$

Proof Referring Section 4.2 and Lemma 4.3, we let $\eta = \eta^*/2, \eta_*/2$ in the two cases, and considering the continuity of the function

$$[t_o - \eta,\ t_o] \ni t \rightarrow \fint_{B_{2r}(x_o)} w\, dx,$$

we let $t_1 \in [t_o - \eta, t_o - \eta/2]$ be the point such that inequality Eqs. 4.7 (or 4.8)) is achieved, i.e.

$$\mathcal{I}_1 = \sup_{t_o-\eta\leqslant t\leqslant t_o-\eta/2} \fint_{B_{2r}(x_o)} w\, dx = \fint_{B_{2r}(x_o)} w(x, t_1)\, dx, \tag{4.29}$$

depending on the choice of η. Now we apply the weak Harnack inequality 3.23 with

$$\eta_1 = \min\left(\frac{\eta}{4},\ \frac{4br^2}{\varphi^+_{Q^+_{24r}(x_o,t_1)}(\frac{\mathcal{I}_1}{2r})}\right),$$

which yields

$$\mathcal{I}_1 \leqslant \gamma\left\{r + r\varphi^{-1}_{Q^+_{24r}(x_o,t_1)}\left(\frac{2r^2}{\eta}\right) + \inf_{B_{4r}(x_o)} w(\cdot, t_2)\right\}, \qquad \text{at} \quad t_2 = t_1 + \eta_1/2 < t_o - \eta/4. \tag{4.30}$$

We want to estimate the second term on the right-hand side of Eq. 4.30: to this aim, we apply Remark 3.8. If $a(x_o, t_o) = 0$, then we evaluate

$$r\varphi^{-1}_{Q^+_{24r}(x_o,t_1)}\left(\frac{2r^2}{\eta^*}\right) \leqslant \gamma(\gamma^*)^{\frac{-1}{q-2}} r\varphi^{-1}_{Q^+_{24r}(x_o,t_1)}\left(\left(\frac{\mu\delta_p(r)}{r}\right)^{p-2}\right) \leqslant$$
$$\leqslant \gamma(\gamma^*)^{\frac{-1}{q-2}} r\varphi^{-1}_{Q^+_{24r}(x_o,t_1)}\left(\varphi_{Q^+_{24r}(x_o,t_1)}\left(\frac{\mu\delta_p(r)}{r}\right)\right) = \gamma(\gamma^*)^{\frac{-1}{q-2}}\mu\delta_p(r).$$

Similarly, if $a(x_o, t_o) > 0$, we use the condition $\eta_* < r^2$ to get $|t_o - t_1| \leq \eta_* < r^2/2$ and similarly to Eq. 4.4 we can estimate

$$a^+_{Q^+_r(x_o,t_1)} \leq 2a^-_{Q_r(x_o,t_1)} \leq 2a(x_o, t_o) \leq 2a^+_{Q^+_r(x_o,t_1)},$$

because

$$(x_o, t_o) \in Q^+_r(x_o, t_1) \subseteq Q_{2r}(x_o, t_o).$$

Hence using this fact in the second inequality, a similar computation yields

$$r\varphi^{-1}_{Q^+_{24r(x_o,t_1)}}\left(\frac{2r^2}{\eta_*}\right) \leqslant \gamma\gamma_*^{-\frac{1}{q-2}} r\varphi^{-1}_{Q^+_{24r}(x_o,t_1)}\left(a(x_o,t_o)\left(\frac{\mu\delta_q(r)}{r}\right)^{q-2}\right) \leqslant$$
$$\leqslant \gamma\gamma_*^{-\frac{1}{q-2}} r\varphi^{-1}_{Q^+_{24r}(x_o,t_1)}\left(\varphi_{Q^+_{24r}(x_o,t_1)}\left(\frac{\mu\delta_q(r)}{r}\right)\right) \leqslant \gamma\gamma_*^{-\frac{1}{q-2}}\mu\delta_q(r).$$

From this and Eq. 4.30, using Lemma 4.3 and choosing γ_*, γ^* by the conditions

$$\gamma_* = \gamma^* = (2\gamma)^{q-2}, \tag{4.31}$$

we arrive at

$$\mu\delta_p(r) \leqslant \gamma\big\{r + \inf_{B_{4r}(x_o)} w(\cdot,t_2)\big\}, \quad \text{if} \quad a(x_o,t_o) = 0, \tag{4.32}$$

and

$$\mu\delta_q(r) \leqslant \gamma\big\{r + \left(\frac{r}{R}\right)^{q-1}\frac{1}{(\mu\delta_q(r))^{q-2}} + \inf_{B_{4r}(x_o)} w(\cdot,t_2)\big\}, \quad \text{if} \quad a(x_o,t_o) > 0, \tag{4.33}$$

for $t_2 = t_1 + \eta_1/2 < t_o - \eta/4$ by our choice of η_1. We observe that at the moment the quantitative location of t_2 is undetermined, because of the unknown t_1 (see Fig. 2). To complete the proof, we use Corollary 3.4 for the function w, as

$$\begin{cases} w(x,t_2) \ge k_p = \gamma^{-1}\mu\delta_p(r) - r, & x \in B_{2r}(x_o), \quad \text{if} \quad a(x_o,t_o)=0,\\ w(x,t_2) \ge k_q = \gamma^{-1}\mu\delta_q(r) - r - (r/R)^{q-1}\frac{1}{(\mu\delta_q(r))^{q-2}}, & x \in B_{2r}(x_o), \quad \text{if} \quad a(x_o,t_o)>0, \end{cases}$$

where k_p, k_q are positive by assumptions Eqs. 4.25-4.27. As we have

$$t_1 < t_o - \eta/2, \quad \text{for} \quad \eta = \eta^*, \eta_*,$$

then Corollary 3.4 implies that

$$\begin{cases} w(x,t) \ge \delta^*(t)k_p, & (x,t) \in B_r(x_o) \times (t_2, t_2 + \eta_{k_p}), \quad \text{if} \quad a(x_o,t_o) = 0,\\ w(x,t) \ge \delta_*(t)k_q, & (x,t) \in B_r(x_o) \times (t_2, t_2 + \eta_{k_q}), \quad \text{if} \quad a(x_o,t_o) > 0, \end{cases}$$

for η_{k_p}, η_{k_q} referred to levels k_p, k_q as given in Eq. 3.8, and

$$\delta^*(t) = \delta\Psi^{-1}\left(1 + \frac{t - t_2}{\eta_{k_p}}\right), \qquad \delta_*(t) = \delta\Psi^{-1}\left(1 + \frac{t - t_2}{\eta_{k_q}}\right).$$

So we move the point-wise information on w from t_1 to t_o, hence travelling a distance smaller than η:

$$\delta^*(t) \ge \delta\Psi^{-1}\left(1 + \frac{\eta}{\eta_{k_p}}\right) = \Psi^{-1}\left(1 + \frac{\gamma^*[\mu(\delta_p(r)]^{2-p}r^p}{\Psi(\hat\gamma^{-1}\mu\delta_p(r) - r)}\right) \ge \Psi^{-1}\left(1 + \frac{\Psi(\gamma^*\mu(\delta_p(r))}{\Psi(\hat\gamma^{-1}\mu\delta_p(r))}\right) =: C_p^*,$$

$$\delta_*(t) \ge \delta\Psi^{-1}\left(1 + \frac{\eta}{\eta_{k_q}}\right) = \Psi^{-1}\left(1 + \frac{\gamma_*[\mu(\delta_q(r)]^{2-q}r^q}{\Psi(\hat\gamma^{-1}\mu\delta_q(r) - r)}\right) \ge \Psi^{-1}\left(1 + \frac{\Psi(\gamma_*\mu(\delta_q(r))}{\Psi(\hat\gamma^{-1}\mu\delta_q(r))}\right) =: C_q^*.$$

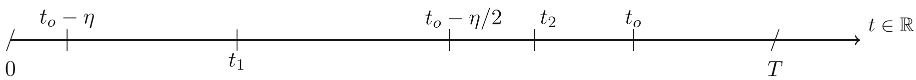

Fig. 2 Comparing time lengths in proof of Lemma 4.4

This implies, in the case $a(x_o, t_o) = 0$ and Eq. 4.25 violated, the estimate

$$\mu\delta_p(r) \leqslant C_p^*\Big(\mu - \sup_{Q^-_{\frac{r}{2},\frac{\eta}{8}}(x_o,t_o)} u_k\Big) + C_p^* r. \tag{4.34}$$

Similarly, in the case $a(x_o, t_o) > 0$ and Eq. 4.27 violated, the above procedure ensures

$$\mu\delta_q(r) \leqslant C_q^*\Big(\mu - \sup_{Q^-_{\frac{r}{2},\frac{\eta}{8}}(x_o,t_o)} u_k\Big) + C_q^* r + C_q\Big(\frac{r}{R}\Big)^{q-1}\frac{1}{(\mu\delta_q(r))^{q-2}}. \tag{4.35}$$

The conclusion follows therefore by implementing the assumption that Eqs. 4.25-4.27 are violated into the estimates Eqs. 4.34-4.35 above. □

5 Proof of Theorem 1.1

We begin with a preliminary consideration. The divergence of Wiener's integral at (x_o, t_o) implies that there exists a suitable sequence of radii that allows to apply Lemma 4.4 iteratively.

Lemma 5.1 *Let $p > 1$, $\mu_o > 0$ and $\bar{C}, C_1 > 1$ be given numbers. Assume that for a certain $\rho_o > 0$ it holds both*

$$\int_0^{\rho_o} \delta_p(\rho)\frac{d\rho}{\rho} = \infty, \tag{5.1}$$

and

$$\mu_o\delta_p(\rho_o) \geq \bar{C}\rho_o.$$

Then, for any $\tilde{\gamma} > 0$ fixed, there exists a decreasing sequence of radii $\{\rho_j\}_{j\in\mathbb{N}}$ such that, by defining

$$\mu_j = (1 - 1/(2C_1))\mu_{j-1}, \qquad \mu_0 = \mu_o, \quad \rho_0 = \rho_o\,,$$

it has the following properties for all $j \in \mathbb{N} \cup \{0\}$:

$$\mu_j\delta_p(\rho_j) \geq \bar{C}\rho_j; \tag{5.2}$$

$$2\eta_{j+1} := 2\tilde{\gamma}\rho_{j+1}^p(\mu_{j+1}\delta_p(\rho_{j+1}))^{2-p} \leq \tilde{\gamma}\rho_j^p(\mu_j\delta_p(\rho_j))^{2-p} = \eta_j; \tag{5.3}$$

$$\forall l \in \mathbb{N}\ \exists n(l) \in \mathbb{N}: \quad \sum_{j=0}^{l-1}\delta_p(\rho_j) \geq \frac{1}{\gamma_3}\sum_{i=0}^{n(l)}\delta_p(\sigma^i\rho_0) \geq \frac{1}{\gamma_4}\int_{\rho_l}^{\rho_0}\frac{\delta_p(\rho)\,d\rho}{\rho}. \tag{5.4}$$

Remark 5.2 If in the previous Lemma we choose $\bar{C} = \tilde{C}(1 + R^{-1} + R^{\frac{p-q}{q-2}})$, with the choice $\rho_o \leq R = (a(x_o, t_o)/2A_o)^{\frac{1}{q-p}}$, then Eqs. 5.2-5.3-5.4 hold true for the exponent q instead of p and condition Eq. 5.2 is replaced by

$$\mu_j\delta_q(\rho_j) \geq \tilde{C}(1 + R^{-1} + R^{\frac{p-q}{q-2}})\rho_j\ . \tag{5.5}$$

Regarding the three terms on the right-hand side of Eq. 5.5: the first and second one are linked to the requirement of Eq. 4.27; while the third one is given to free the choice of $\tilde{C}$ from $a(x_o, t_o)$ when requiring in Remark 4.2,

$$Q_{\rho_o,\eta^*} \subseteq Q_{\rho_o,\rho_o^2}, \quad \text{with} \quad \eta^* = \frac{\gamma_*\rho_0^q}{a(x_o, t_o)(\mu(\rho_0)\delta_q(\rho_o))^{q-2}}\ .$$

Lemma 5.1 is an adaptation to our framework of an argument of the capacitary estimate between integral and sum of [36], while the extraction of the sequence is modeled after [73]. The novelty is that we assume a priori that ρ_o satisfies Eq. 5.2, and we extract the sequence $\{\rho_j\}_{j\in\mathbb{N}}$ starting from ρ_o.

Proof Let $\eta_0 = \tilde{\gamma}\rho_0^p(\mu_o\delta_p(\rho_0))^{2-p}$, and if

$$\delta_p(\rho_0(1-1/(2C_1))/2) \geq 1/2\,\delta_p(\rho_0), \quad \Rightarrow \quad \text{and set} \quad \rho_1 = \sigma\rho_0, \qquad \sigma := (1-1/(2C_1))/2;$$

while if otherwise

$$\delta_p(\rho_0(1-1/(2C_1))/2) < 1/2\,\delta_p(\rho_0),$$

let $i_1 \in \mathbb{N}$ be the smallest number such that

$$\delta_p(\sigma^{i_1}\rho_0) \geq 2^{-i_1}\delta_p(\rho_0), \quad \Rightarrow \quad \text{and set} \quad \rho_1 = \sigma^{i_1}\rho_0. \tag{5.6}$$

The choice of i_1 is possible, being otherwise

$$\delta_p(\rho_0(1-1/(2C_1))/2) < 2^i\delta_p(\rho_0) \quad \forall i \in \mathbb{N} \qquad \Rightarrow \quad \sum_{i\in\mathbb{N}} \frac{\delta_p(\sigma^i\rho_0)}{2^i} < \infty,$$

that is a contradiction with Eq. 5.1, because of the property

$$\left[\frac{\mathcal{C}_p(B(x_o, 2^{-(k+1)}\rho_0);\, B(x_o,\rho_0))}{\gamma 2^{-k(N-p)}\rho_0}\right]^{\frac{1}{p-1}} \ln(2) \leq \int_{2^{-(k+1)}\rho_0}^{2^{-k}\rho_0} \left[\frac{\mathcal{C}_p(B(x_o,t);\, B(x_o,\rho_0))}{t^{(N-p)}}\right]^{\frac{1}{p-1}} \frac{dt}{t}$$
$$\leq \left[\frac{\mathcal{C}_p(B(x_o, 2^{-k}\rho_0);\, B(x_o,\rho_0))}{\gamma 2^{-(k+1)(N-p)}\rho_0}\right]^{\frac{1}{p-1}} \ln(2).$$

Henceforth, we have Eq. 5.2 for $i = 1$, which is

$$\mu_1\delta_p(\rho_1) = \mu_o(1-1/(2C_1))\delta_p(\rho_1) \geq \frac{\mu_o}{2^{i_1}}\delta_p(\rho_0) \geq \frac{\bar{C}\rho_0}{2^{i_1}} \geq \bar{C}\rho_1.$$

Point Eq. 5.3 follows from Eq. 5.6 and the definition of η_1, η_0, with a simple computation. Finally, the choice of i_1 to be the smallest number is finally useful to have Eq. 5.4, as

$$\sum_{i=0}^{i_1} \delta_p(\sigma^i\rho_0) \leq \delta_p(\rho_0) + 2\delta_p(\rho_0) + \delta_p(\sigma^{i_1}\rho_0) \leq 3\delta_p(\rho_0) + \delta_p(\rho_1) \leq 3\sum_{i=1,2}\delta_p(\rho_i),$$

where (in the first inequality) we have contradicted condition Eq. 5.6 for all previous $0 < i < i_1$ to get

$$2\delta_p(\rho_0) \geq \sum_{i=1}^{i_1-1}\delta_p(\sigma^i\rho_0).$$

By induction, we suppose the statement of the Lemma to be valid until step n and we prove it for $(n+1)$. Thus, we choose $\rho_{n+1} = \sigma^{i_{n+1}-i_n}\rho_0$ for i_{n+1} being the smallest number in $\{i_n, i_n+1, \dots\}$ satisfying as before Eq. 5.6 with i_{n+1} instead. Then all the argument flows in the same style until we arrive to condition Eq. 5.4: here we contradict assumption Eq. 5.6

for all previous $i \in \{i_n, \dots, i_{n+1}\}$, and use the inductive hypothesis to obtain

$$\begin{aligned}\sum_{i=0}^{i_{n+1}} \delta_p(\sigma^i \rho_0) &\le \sum_{i=0}^{i_{n-1}} \delta_p(\sigma^i \rho_0) + \delta_p(\rho_n) + \sum_{i=i_n+1}^{i_{n+1}-1} \delta_p(\sigma^i \rho_0) + \delta_p(\rho_{n+1}) \\ &\le 3\sum_{j=0}^{n-1} \delta_p(\rho_j) + \delta_p(\rho_n) + \delta_p(\rho_n) \sum_{i=i_n+1}^{i_{n+1}-1} 2^{i_n - i} + \delta_p(\rho_{n+1}) \\ &\le 3\sum_{j=0}^{n+1} \delta_p(\rho_j).\end{aligned} \tag{5.7}$$

This ensures that the first inequality of Eq. 5.4 occurs, with $\gamma_3 = 3$ and $n(l) = i_l$.

To prove the second one, we follow a reasoning similar to [36]. For a condenser (K, B_{2r}), Lemma 2.16 of [45] states that for $p > 1$ and when $0 < r \le s \le 2r$, then there exits $\gamma(s, N) > 0$ such that

$$\frac{1}{\gamma} \mathcal{C}_p(K; B_{2r}) \le \mathcal{C}_p(K, B_{2s}) \le \gamma \mathcal{C}_p(K; B_{2r}).$$

Hence, by the previous consideration and the monotonicity of the capacity in the first argument, we obtain

$$\begin{aligned}\int_y^{2y} \frac{\delta_p(s)\, ds}{s} &= \int_y^{2y} \left[\frac{\mathcal{C}_p(\overline{B_s} \setminus \Omega; B_{2s)}}{\mathcal{C}_p(\overline{B_s}, B_{2s})}\right]^{\frac{1}{p-1}} \frac{ds}{s} \le \gamma \int_y^{2y} \left[\frac{\mathcal{C}_p(\overline{B_s} \setminus \Omega; B_{4y)}}{\mathcal{C}_1 s^{N-p}}\right]^{\frac{1}{p-1}} \frac{ds}{s} \\ &\le (2^{N-p}\gamma/\mathcal{C}_1)^{\frac{1}{p-1}} \int_y^{2y} \left[\frac{\mathcal{C}_p(\overline{B_{2y}} \setminus \Omega; B_{4y)}}{\mathcal{C}_p(\overline{B_{2r}}, B_{4r})}\right]^{\frac{1}{p-1}} \frac{ds}{s} = \gamma \delta_p(2y).\end{aligned} \tag{5.8}$$

Hence for all $\mathbb{N} \ni m \ge i_l$ we have

$$\int_{\sigma^{i_l}\rho_0}^{\rho_0} \frac{\delta_p(s)\, ds}{s} \le \sum_{j=0}^{m-1} \int_{2^{-(j+1)}\rho_0}^{2^{-j}\rho_0} \frac{\delta_p(s)\, ds}{s} \le \gamma \sum_{j=0}^{m-1} \delta_p(2^{-j}\rho_0).$$

The considerations done until this point are valid for all $p > 1$, provided that condition Eq. 5.1 is satisfied with such exponent. □

5.1 Conclusion of the Proof of Theorem 1.1

The proof of Theorem 1.1 hinges upon the possibility of finding a family of nested backward cylinders $\{Q_n\}_{n\in\mathbb{N}}$ centered at $(x_o, t_o) \in S_T$, where we can iteratively and quantitatively reduce the oscillation of the solution, truncated from above and below by the boundary datum. At this stage, the major difficulty of this double-phase parabolic problem is to deal with the method of the accommodation of its degeneracy. This is because of the double requirement due both to the intrinsic geometry and the restriction of the radii obliged by the phase, see Remark 4.2 and condition Eq. 1.6.

5.2 Accommodation of the Degeneracy

Let $(x_o, t_o) \in S_T = \partial\Omega \times (0, T]$ and choose

$$k_0^+ = \sup_{S_T} f\,, \quad \text{and} \quad k_0^- = \inf_{S_T} f\,,$$

$$\mu_0^\pm = \sup_{\Omega_T}(u - k_0^\pm)_\pm, \quad \text{and} \quad w_0^\pm = \mu_0^\pm - (u - k_0^\pm)_\pm.$$

$$\omega_0 = \operatorname*{osc}_{\Omega_T} u\,.$$

We assume $\mu_0^\pm > 0$, because otherwise there is nothing to prove. Now, for some $\epsilon \in (0, 1)$ to be determined later, let us define for $s \in (0, 1)$ the numbers

$$\tilde{\eta}_0(p, s) = 3\gamma^*[\delta_p(s)]^{2-p} s^{p-\epsilon}, \qquad \text{if} \qquad a(x_o, t_o) = 0, \tag{5.9}$$

$$\tilde{\eta}_0(q, s) = 3\gamma^*[\delta_q(s)]^{2-q} s^{q-\epsilon}, \qquad \text{if} \qquad a(x_o, t_o) > 0, \tag{5.10}$$

with $\gamma^* > 0$ the geometric constant of Section 4.2, necessary for the application of Lemma 4.4.

Let us choose $\rho_0(p) \in (0, R_o)$ and $\rho_0(q) \in (0, \min\{R_o, R\}/24)$, with R_o the number for which Eq. 1.6 is valid and R the maximal radius Eq. 4.3, to be numbers that satisfy

$$\begin{cases} \tilde{\eta}_0(p) := \tilde{\eta}_0(p, \rho_0(p)) < \min\{t_o,\ R_o^2\}, \\ \mu_0^\pm \delta_p(\rho_0(p)) > 2C_p \rho_0(p) \end{cases} \quad \text{and} \quad \begin{cases} \tilde{\eta}_0(q) := \tilde{\eta}_0(q, \rho_0(q)) < \min\{t_o,\ R_o^2\}, \\ \mu_0^\pm \delta_q(\rho_0(q)) > 4C_q \rho_0(q) + (4C_q)^{\frac{1}{q-1}} (\rho_0(q)/R), \end{cases}$$

where $\mathcal{C}_p, \mathcal{C}_q > 0$ are the constants provided by Lemma 4.4. The existence of such $\rho_0(p)$, $\rho_0(q)$ is guaranteed by assumptions Eqs. 1.9-1.10 in each case.

Indeed, let us show for instance the case of positive phase: we suppose, by contradiction, that for all $s \in (0,\ \min\{R_o, R\}/24)$ we have the alternative

$$\tilde{\eta}_o(q, s) \geq \min\{t_o,\ R_o^2\} \quad \vee \quad \mu_0^\pm \delta_q(s) \leq 4C_q s + (4C_q)^{\frac{1}{q-1}} (s/R).$$

Then, for every such s we can estimate $\delta_q(s)$ from above

$$\left(\frac{3\gamma^*}{\min\{t_o,\ R_o^2\}} \right)^{\frac{1}{q-2}} s^{\frac{q-\epsilon}{q-2}} + (4C_q s + (4C_q)^{\frac{1}{q-1}} (s/R))/\mu_0^\pm \geq [\delta_q(s)]\,.$$

Hence

$$\int_0^{R_o} \delta_q(s) \frac{ds}{s} \leq \int_0^{R_o} \left(\frac{3\gamma^*}{\min\{t_o,\ R_o^2\}} \right)^{\frac{1}{q-2}} s^{\frac{2-\epsilon}{q-2}} ds + (4C_q/\mu_0^\pm)(1 + 1/R) \int_0^{R_o} ds < \infty,$$

contradicting Eq. 1.10. With such numbers $\{\tilde{\eta}_0(p), \rho_0(p)\}$ and $\{\tilde{\eta}_0(q), \rho(q)\}$ we define the cylinders

$$Q_0(p, \pm) = B_{\rho_0(p)}(x_o) \times \left(t_o - [\mu_0^\pm]^{2-p} \tilde{\eta}_0(p)\,,\ t_o \right), \tag{5.11}$$

$$Q_0(q, \pm) = B_{\rho_0(q)}(x_o) \times \left(t_o - [\mu_0^\pm]^{2-q} \tilde{\eta}_0(q)\,,\ t_o \right). \tag{5.12}$$

From now on the proof is standard, we repeat it here for the sake of completeness, within a compact notation.

Let us indicate with an index $i \in \{p, q\}$ the radii $\rho_0(p), \rho_0(q)$ and the time-lengths $\tilde\eta_0(p), \tilde\eta_0(q)$ and the next quantities that we are going to define. Let

$$Q_0(i) = B_{\rho_0(i)}(x_o) \times \Big(t_o - \tilde\eta_0(i)\,,\ t_o\Big), \qquad \omega_0(i) = \operatorname*{osc}_{Q_0(i)} u\ .$$

5.3 The Iteration, First Step

For $i = p, q$, we choose levels

$$k_0^+(i) = \sup_{Q_0(i,+)\cap S_T} f\ , \quad \text{and} \quad k_0^-(i) = \inf_{Q_0(i,-)\cap S_T} f\ .$$

We define

$$\mu_0^\pm(i) = \sup_{Q_0(i,\pm)} (u - k^\pm)_\pm\ , \quad \text{and} \quad w_0^\pm(i) = \mu_0^\pm(i) - (u - k_0^\pm(i))\ .$$

Now we observe that for both $i = p, q$ we can always assume

$$(\mu_0^\pm(i))^{2-i}\ \rho_0^i \le \rho_0^{i-\epsilon}. \tag{5.13}$$

because otherwise the quantities $\mu_0^\pm(i)$ are smaller than a power of the radius $\rho_0(i)$, for $i = p, q$ respectively, and we are done. This means that

$$Q_0(i, \pm) \subseteq B_{\rho_0(i)}(x_o) \times \Big(t_o - \tilde\eta_0(i),\ t_o\Big), \qquad i = p, q,$$

and the special choice of $\rho_0(i)$ allows us to apply Lemma 4.4 to get

$$\sup_{Q_{1,i}(\pm)} (u - k^\pm(i))_\pm \le \mu_1^\pm(i),$$

for

$$\mu_1^\pm(i) = (1 - 1/(2C_i))\mu_0^\pm(i),$$

where for $i = p, q$ we have defined

$$Q_{1,i}(\pm) = B_{\rho_0(i)/2}(x_o) \times \left(t_o - \gamma^*[\rho_0(i)]^i \Big(\mu_0^\pm(i)\delta_i(\rho_0(i))\Big)^{2-i}/8,\ t_o\right)\ .$$

5.4 The Iteration, *n*-th Step

Now, we consider now Lemma 5.1 with $\rho_o = \rho_0(i)$, $\mu_o = \mu_0^\pm(i)$, and $C_1 = C_i$ for $i = p, q$ respectively, depending on the case the phase vanishes at (x_o, t_o) or not.

With these stipulations, we can find two sequences of radii $\{\rho_{j,p}\}_{j\in\mathbb{N}}$, $\{\rho_{j,q}\}_{j\in\mathbb{N}}$ with $\rho_o(i) = \rho_0(i)$, satisfying to Eqs. 5.2-5.3-5.4 and Remark 5.2. We define

$$\begin{cases} \eta_{n,p}^\pm = \gamma^* \rho_{n,p}^p \Big(\mu_0^\pm(p)\delta_p(\rho_{n,p})\Big)^{2-p}, & \text{if} \quad a(x_o, t_o) = 0, \\ \eta_{n,q}^\pm = \gamma^* \rho_{n,q}^q \left(\frac{\mu_0^\pm(q)}{a(x_o,t_o)}\delta_q(\rho_{n,q})\right)^{2-q}, & \text{if} \quad a(x_o, t_o) > 0, \end{cases} \tag{5.14}$$

and cylinders

$$Q_{n,i}(\pm) = B_{\rho_{n,i}}(x_o) \times (t_o - \eta_{n,i}^\pm\ ,\ t_o), \qquad \text{for} \quad i = p, q\ . \tag{5.15}$$

Let us suppose the assertion valid until step $(n-1)$ and let us prove it for step n. Within conditions Eqs. 5.2-5.3 for $j = n-1$ we can apply Lemma 4.4 and obtain, for

$$\mu_n^\pm(i) = (1 - 1/(2C_i))\mu_{n-1}^\pm(i), \qquad \text{for} \quad i = p, q,$$

and

$$\sup_{Q_{n,i}} (u - k^\pm(i))_\pm \le \mu_n^\pm(i),$$

where for $i = p, q$ we have defined

$$Q_{n,i}(\pm) = B_{\rho_{n,i}}(x_o) \times \left(t_o - \gamma^* \rho_{n,i}^i \left(\mu_n^\pm(i)\delta_i(\rho_{n,i}) \right)^{2-i} /2, \ t_o \right),$$

and once observed that

$$Q_{n,i}(\pm) \subseteq \ B_{\rho_{n,i}/2}(x_o) \times \left(t_o - \gamma^* \rho_{n,i}^i [\mu_n^\pm(i)\delta_i(\rho_{n,i})]^{2-i}/8, \ t_o \right).$$

Hence at the $n+1$-th step, the application of Lemma 4.4 provides for $i = p, q$ the estimates

$$\begin{aligned} \sup_{Q_{n+1,i}(\pm)} (u - k^\pm(i))_\pm &\le \mu_n^\pm(i) \left(1 - \frac{1}{C_i} \delta(\rho_{n,i}) \right) \\ &\le \mu_0^\pm(i) \ \exp\left\{ -\frac{1}{C_i} \sum_{j=1}^{n} \delta_i(\rho_{j,i}) \right\} \le \mu_0^\pm(i) \ \exp\left\{ -\frac{1}{\gamma_4} \int_{\rho_{n+1,i}}^{\rho_0(i)} \delta_i(s) \frac{ds}{s} \right\}, \end{aligned}$$

with $\gamma_4 = \gamma_4(C_i)$, using Bernoulli's inequality and Eq. 5.4. Taking into consideration Eq. 5.13, this yields

$$\sup_{Q_{n+1,i}(\pm)} (u - k^\pm(i))_\pm \le \mu_0^\pm(i) \exp\left\{ -\frac{1}{\gamma_3} \int_{\rho_{n+1,i}}^{\rho_0(i)} \delta_i(s) \frac{ds}{s} \right\} + \gamma_3 [\rho_0(i)]^{\frac{\epsilon}{q-2}}, \qquad (5.16)$$

and considering as usual for any $\rho \in (0, \ \ \rho_0(i))$ an integer $n \ge 0$ such that $\rho_{n+1,i} \le \rho \le \rho_{n,i}$, we obtain

$$\sup_{Q_\rho(\mu_0^\pm(i))} (u - k^\pm(i))_\pm \le \mu_0^\pm(i) \exp\left\{ -\frac{1}{\gamma_3} \int_{\rho}^{\rho_0(i)} \delta_i(s) \frac{ds}{s} \right\} + \gamma_3 [\rho_0(i)]^{\frac{\epsilon}{q-2}}, \qquad (5.17)$$

being for our choice of n,

$$Q_\rho(\omega_0) \subseteq Q_\rho(\mu_0^\pm(i)) = B_\rho(x_o) \times \left(t_o - (\mu_0^\pm(i))^{2-p} \rho^p \ , \ t_o \right) \subset Q_{n+1,i}(\pm),$$

where the first set inclusion is due to the degenerate exponent $q, \ p > 2$ and the choice

$$\max\{\mu_0^+(i), \mu_0^-(i)\} \le \omega_0(i) - \operatorname*{osc}_{S_T \cap Q_0(i)} f \le \omega_0.$$

Finally we combine the two aforementioned estimates for $k^\pm(i)$ to obtain

$$\operatorname*{osc}_{Q_\rho(\omega_0)} u \le \omega_0 \exp\left\{ -\frac{1}{\gamma} \int_{\rho}^{\rho_0(i)} \delta_i(s) \frac{ds}{s} \right\} + \operatorname*{osc}_{S_T \cap Q_0(i)} f + 2\gamma_3 [\rho_0(i)]^{\frac{\epsilon}{i-2}}, \qquad (5.18)$$

and the proof is concluded.

Appendix

Proof of Lemma 3.1

Without loss of generality, let $(\bar{x}, \bar{t})$ be the origin in $\mathbb{R}^{N+1}$. We test Eq. 3.3 by $\zeta(u_h - k)_-$ and integrate over (h, τ), for $0 < h < \tau < \eta - h$. Using conditions Eq. 1.5 and the continuity of u as a map $[\tau, \eta] \to L^2(B_r)$, we let $h \downarrow 0$. This provides

$$\int_h^\tau \int_{B_r} \partial_t \zeta (u_h - k)_- \, dxdt$$
$$\xrightarrow[h\downarrow 0]{} -\frac{1}{2}\int_{B_r} \zeta (u-k)_-^2 \, dx\Big|_0^\tau + q\int_0^\tau \int_{B_r} (u-k)_-^2 (\partial_t \zeta_2)(\zeta/\zeta_2)\, dxdt := \mathcal{I}_p, \tag{6.1}$$

for all such $0 < \tau < \eta$, and

$$\int_h^\tau \int_{B_r} [\mathbb{A}(x,t,\nabla u)]_h \Big(\zeta \nabla (u_h - k)_- + q(u_h-k)_-(\nabla\zeta)(\zeta/\zeta_1)\Big) dxdt$$
$$\xrightarrow[h\downarrow 0]{} \int_0^\tau \int_{B_r} \mathbb{A}(x,t,u,\nabla u)\Big(-\zeta\nabla u + q(u-k)_-(\nabla \zeta)(\zeta/\zeta_1)\Big)\chi_{[u<k]}\, dxdt := \mathcal{I}_e,$$

with $\mathcal{I}_p + \mathcal{I}_e \geq 0$. Manipulating the sign of this inequality together with the signs of its various terms, while using conditions Eq. 1.5, we estimate the following energy term as

$$\begin{aligned}
\mathcal{I} = &\sup_{0<t<\eta} \int_{B_r} \zeta (u-k)_-^2(x,t)\,dx + \gamma^{-1}\iint_{Q^+_{r,\eta}} \Big(|\nabla[\zeta(u-k)_-]|^p + a(x,t)|\nabla[\zeta(u-k)_-]|^q\Big) dxdt\\
&\leq \sup_{0<t<\eta}\int_{B_r} \zeta(u-k)_-^2(x,t)\,dx + \gamma^{-1}\iint_{Q^+_{r,\eta}} \Big(|\nabla(u-k)_-|^p\zeta + a(x,t)|\nabla(u-k)_-|^q\zeta\Big) dxdt+\\
&\qquad + \gamma^{-1}\iint_{Q^+_{r,\eta}} \varphi\Big(x,t,|\nabla\zeta|(u-k)_-\Big)\,dxdt =: E + \Phi\\
&\leqslant 2q\int_0^\eta\int_{B_r}(u-k)_-^2|\partial_t\zeta|\,dxdt + \gamma\iint_{Q^+_{r,\eta}} (u-k)_-|\nabla\zeta|\Big(|\nabla u|^{p-1} + a(x,t)|\nabla u|^{q-1}\Big)\,dxdt+\\
&\qquad + \iint_{Q^+_{r,\eta}} \varphi\Big(x,t,|\nabla\zeta_1|(u-k)_-\Big)\,dxdt.
\end{aligned}$$

Using Young's inequality we notice that

$$\iint_{Q^+_{r,\eta}} (u-k)_-|\nabla\zeta|\Big(|\nabla u|^{p-1} + a(x,t)|\nabla u|^{q-1}\Big)\,dxdt$$
$$\leq C(\epsilon)\Phi + \epsilon\iint_{Q^+_{r,\eta}} [|\nabla(u-k)_-|^p\zeta + a(x,t)|\nabla(u-k)_-|^q\zeta]\,dxdt.$$

Hence, reabsorbing on the right-hand side the last terms, using the properties of ζ and the monotonicity of the function $\xi \to \phi(x,t,\xi)$ in the last variable, we get

$$\mathcal{I} \leq \gamma\sigma^{-1}\frac{k^2}{\eta}|A^-_{k,r,\eta}| + \gamma\sigma^{-q}\iint_{A^-_{k,r,\eta}} \varphi(x,t,k/r)dxdt \leqslant \gamma\sigma^{-q}\Big(\frac{k^2}{\eta} + [\varphi^+_{r,k}]\Big)|A^-_{k,r,\eta}|.$$

In order to conclude, we observe that Young's inequality again can be used on the left-hand side as

$$
\begin{aligned}
&\left(1+a^-\left(\frac{k}{r}\right)^{q-p}\right)\iint_{Q^+_{r,\eta}}|\nabla[\zeta(u-k)_-]^p\,dxdt\\
&\leq\iint_{Q^+_{r,\eta}}|\nabla[\zeta(u-k)_-]^p\,dxdt+\iint_{Q^+_{r,\eta}}a(x,t)\left(\frac{k}{r}\right)^{q-p}|\nabla[\zeta(u-k)_-]|^p\,dxdt\\
&\leq\iint_{Q^+_{r,\eta}}|\nabla[\zeta(u-k)_-]^p\,dxdt+2^{-1}a^+(x,t)\left(\frac{k}{r}\right)^q|A_{k,r,\eta}|^-+2^{-1}\iint_{Q^+_{r,\eta}}a(x,t)|\nabla[\zeta(u-k)_-]|^q\,dxdt\\
&\leq\gamma\left(\mathcal{I}+a^+(x,t)\left(\frac{k}{r}\right)^q|A_{k,r,\eta}|^-\right).
\end{aligned}
$$

Inequality Eq. 3.5 centered at the origin is found by putting all the pieces of the puzzle together; then, the usual transformation of coordinates $y=\bar{x}+x$, $s=\bar{t}+t$ finishes the job.

The estimate Eq. 3.15 is proven similarly: choosing $(u_h-k)_-\zeta_1(x)$ as a test function, integral $\mathcal{I}_p$ gets simplified.

Proof of Lemma 3.5

Firstly, we assume $\delta>0$. We test Eqs. 3.3 by $(u_h+\delta)^{-\alpha}\zeta$, $t\in(\bar{t},\,\bar{t}+\tau-h)$, $0<\tau\leq\eta$, and integrate over $(\bar{t},\,\bar{t}+\tau-h)$,

$$
\begin{aligned}
0\leq I_{p,h}+I_{e,h}=\int_0^{\tau-h}\int_{B_r\times\{t\}}&\Big\{\partial_t u_h(u_h+\delta)^{-\alpha}\zeta+\\
&+[\mathbb{A}(x,u,\nabla u)]_h\Big[-\alpha(u_h+\delta)^{-(1+\alpha)}(\nabla u_h)\zeta+q(u_h+\delta)^{-\alpha}\zeta_2^q\zeta_1^{q-1}(\nabla\zeta_1)\Big]\Big\}dxdt
\end{aligned}\tag{6.2}
$$

Here we first notice that, by using Fubini-Tonelli theorem and chain rule for the weak time derivative we can rewrite $I_{p,h}$ as follows

$$
\begin{aligned}
I_{p,h}=&\frac{1}{1-\alpha}\int_{B_r}\int_o^{\tau-h}\partial_t[(u_h+\delta)^{1-\alpha}\zeta]-(u_h+\delta)^{-\alpha}\partial_t\zeta\,dxdt\\
=&\frac{1}{1-\alpha}\int_{B_r}[(u_h+\delta)^{1-\alpha}\zeta](x,t)\,dx\Big|_{t=0}^{t=\tau-h}-\frac{1}{1-\alpha}\int_{B_r}\int_o^{\tau-h}(u_h+\delta)^{-\alpha}\partial_t\zeta\,dxdt.
\end{aligned}\tag{6.3}
$$

Now, in order to let $h\downarrow 0$ we refer to the properties of Steklov approximation (see for instance [31], Lemma 3.2 page 11): we use the fact that $u(t,\cdot):[0,\eta]\to L^{1-\alpha}(B_r)$ is continuous, and use the structure conditions Eq. 1.5 with Young's inequality, in order to apply the dominated convergence theorem (and Fatou's one, on the left hand side) and get, by the generality of $0<\tau\leq\eta$,

$$
\begin{aligned}
&\sup_{\bar t<t<\bar t+\eta}\int_{B_r}[(u+\delta)^{1-\alpha}\zeta](x,t)\,dx \le\\
&\frac{1}{1-\alpha}\int_0^{\eta}\int_{B_r}(u+\delta)^{1-\epsilon}\partial_t\zeta+\mathbb{A}(x,u,\nabla u)\Big[q(u+\delta)^{-\alpha}(\nabla\zeta)(\zeta/\zeta_1)-\alpha(u+\delta)^{-(1+\alpha)}\zeta\nabla u\Big]dxdt\\
&\le\frac{\|\partial_t\zeta\|_\infty}{1-\alpha}\int_0^{\eta}\int_{B_r}(u+\delta)^{1-\epsilon}+\int_0^{\eta}\int_{B_r}qB_2\Big[(u+\delta)^{-\alpha}(\nabla\zeta)(\zeta/\zeta_1)\Big]\Big(|\nabla u|^{p-1}+a(x,t)|\nabla u|^{q-1}\Big)dxdt\\
&\qquad-\alpha K_1\int_0^{\eta}\int_{B_r}\Big[(u+\delta)^{-(1+\alpha)}\zeta\Big]\Big(|\nabla u|^{p}+a(x,t)|\nabla u|^{q}\Big)\Big\}\,dxdt
\end{aligned}
\tag{6.4}
$$

In the second integral term, we use Young's inequality (weighted on $(u+\delta)^{-1-\alpha}\zeta_2^q$) for $(u+\delta)(\nabla\zeta)\zeta_1^{q-p}$ and $\epsilon|\nabla u|^{p-1}\zeta_1^{(p-1)}$ to conjugate powers p and $p/(p-1)$, on the third integral term we use Young's inequality with the same weight for $(u+\delta)(\nabla\zeta)$ and $\epsilon|\nabla u|^{q-1}\zeta_1^{q-1}$ to conjugate powers q and $q/(q-1)$. Choosing ϵ small enough to reabsorb these quantities on the fourth and fifth negative integral terms, we obtain

$$
\begin{aligned}
&\frac{1}{1-\alpha}\sup_{\bar t<t<\bar t+\eta}\int_{B_r(\bar x)}(u+\delta)^{1-\alpha}\zeta\,dx+\frac{\alpha}{\gamma}\iint\limits_{Q^+_{r,\eta}(\bar x,\bar t)}(u+\delta)^{-\alpha-1}|\nabla u|^p\zeta\,dxdt+\\
&+\frac{\alpha}{\gamma}\iint\limits_{Q^+_{r,\eta}(\bar x,\bar t)}a(x,t)(u+\delta)^{-\alpha-1}|\nabla u|^q\zeta\,dxdt\leqslant\frac{1}{(1-\alpha)}\|\partial_t\zeta\|_\infty\iint\limits_{Q^+_{r,\eta}(\bar x,\bar t)}(u+\delta)^{1-\alpha}dxdt+\\
&+\gamma\alpha^{1-p}\|\nabla\zeta\|^p_\infty\iint\limits_{Q^+_{r,\eta}(\bar x,\bar t)}(u+\delta)^{p-\alpha-1}dxdt+\gamma\alpha^{1-q}\|\nabla\zeta\|^q_\infty a^+_{Q^+_{r,\eta}(\bar x,\bar t)}\iint\limits_{Q^+_{r,\eta}(\bar x,\bar t)}(u+\delta)^{q-\alpha-1}dxdt.
\end{aligned}
$$

The desired inequality is therefore obtained by noticing that

$$
\iint\limits_{Q^+_{r,\eta}(\bar x,\bar t)}(u+\delta)^{-\alpha-1}|\nabla u|^p\zeta\,dxdt=\iint\limits_{Q^+_{r,\eta}(\bar x,\bar t)}\left\{|\nabla[(u+\delta)^{\frac{p-\alpha-1}{p}}\zeta^{\frac{1}{p}}]|^p-\frac{1}{p}(u+\delta)^{p-\alpha-1}|\nabla\zeta|^p(\zeta/\zeta_1)^p\right\}dxdt,
$$

and similarly with the third term on the left-hand side of Eq. 3.15.

To include the case $\delta=0$, we let $\delta\downarrow 0$ in the obtained estimates Eq. 3.15, concluding with the help of the Dominated Convergence Theorem.

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Declarations

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