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On the Love Numbers of an Andrade Planet

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Key Points:

- The Andrade rheological law is widely employed in planetary studies since it accounts for transient flow in a simple way
- We obtain a first analytical expression of the time domain Love numbers for a homogeneous incompressible sphere with Andrade rheology
- The solutions found provide new insight into the planetary response and its sensitivity upon model parameters

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Abstract The Andrade rheological model is often employed to describe the response of solar system or extra-solar planets to tidal perturbations, especially when their properties are still poorly constrained. While for uniform planets with steady-state Maxwell rheology the analytical form of the Love numbers was established long ago, for the transient Andrade rheology no closed-form solutions have been yet determined, and the planetary response is usually studied either semi-analitically in the frequency domain or numerically in the time domain. Closed-form expressions are potentially important since they could provide insight into the dependence of Love numbers upon the model parameters and the time-scales of the isostatic readjustment of the planet. First, we focus on the Andrade rheological law in 1-D and we obtain a previously unknown explicit form, in the time domain, for the relaxation modulus in terms of the higher Mittag-Leffler transcendental function $E_{\alpha,\beta}(z)$ that generalizes the exponential function. Second, we consider the general response of an incompressible planetary model — often referred to as the "Kelvin sphere" — studying the Laplace domain, the frequency domain and the time domain Love numbers by analytical methods. Through a numerical approach, we assess the effect of compressibility on the Love numbers in the Laplace and frequency domains. Furthermore, exploiting the results obtained in the 1-D case, we establish closed-form — although not elementary — expressions of the time domain Love numbers and we discuss the frequency domain response of the Kelvin sphere with Andrade rheology analytically.

Plain Language Summary Rheology studies how materials flow when subject to stress. To constrain the rheology of planets, it is useful to consider their response to tidal forces or surface loads that allow to probe their interiors. The Andrade rheological model—introduced in 1910—has recently gained popularity in planetary studies. In contrast with the traditional Maxwell model, the Andrade rheology manifests "transient" creep, an initial stage of flow characterized by a time varying rate of strain. Up to now, the response of a planet characterized by an Andrade rheology has been studied numerically, since exact expressions for the Love numbers were only known in the frequency domain. For the first time, we show that in the case of an incompressible homogeneous planet, analytical Love numbers can also be found in the time domain. They are expressed in terms of the Mittag-Leffler function, which generalizes the exponential. Using analytical methods, we study the Love numbers of a planet with Andrade rheology in the Laplace, frequency and time domains, also giving some numerical results.

1. Introduction

In creep phenomena, a *transient* is a stage of deformation that immediately follows the elastic response, and is characterized by a time-dependent strain rate (e.g., Christensen, 1982). Among the possible rheological laws that exhibit a transient phase—also referred to as primary creep—the appealingly simple Andrade law (Andrade, 1910, 1962) has been successfully employed to describe the behavior of several materials, including metals (Cottrell & Aytekin, 1947), silicate rocks (Gribb & Cooper, 1998; Jackson & Faul, 2010; Tan et al., 1997), poly-crystalline ices (McCarthy & Castillo-Rogez, 2013), and glass forming materials (Plazek & Plazek, 2021). First introduced by Andrade (1910) to describe the elongation of some metal wires under a constant tensile stress, the main feature of Andrade law is a transient exhibiting a fractional power function time dependence $\sim t^{\alpha}$. In his empirical stress-strain relationship for the transient "β-flow," Andrade originally suggested the exponent $\alpha = 1/3$. However, subsequent laboratory investigations have indicated that values in the range $0 < \alpha \leq 1$ are indeed possible for some materials (Walterová et al., 2023). Further studies (see Cottrell, 1996; Louchet & Duval, 2009; Mott, 1953) have attempted to justify the Andrade power law theoretically; in particular, Cottrell (2004) has developed a microscopic theory that can explain, under certain conditions, the "$t^{\frac{1}{3}}$ law of flow" suggested by Andrade (1910, 1962).

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The rationale of adopting the Andrade rheology for Earth-like planets relies upon the existence of a power law scaling for dissipation, recognized both in seismic studies and geodetic experiments (Efroimsky, 2012b). Furthermore, an Andrade rheology has been preferred to the traditional non-transient Maxwell model, having the potential of capturing the inelasticity that characterizes high-frequency deformations, such as those caused by solid tides (Bagheri et al., 2019; Castillo-Rogez et al., 2011; Efroimsky, 2012a; Renaud & Henning, 2018; Steinbrügge et al., 2018; Tobie et al., 2019). Although Maxwell rheology is appropriate for describing the relaxation of a planet for forcing time scales of the order of or exceeding the Maxwell time (i.e., the viscosity to shear modulus ratio), it largely underestimates viscosity when it is employed to describe tidal deformations (Bierson & Nimmo, 2016; Tobie et al., 2019; Walterová et al., 2023). For this reason, and especially in view of its transient nature, many recent planetary studies have adopted the Andrade rheology. A comprehensive list is given by Gevorgyan et al. (2020).

Similarly to the four-elements Burgers biviscous model, the Andrade model fits data of torsional oscillation experiments, although the persistence of the transient behavior in the limit of $t \to \infty$ is undesirable (Jackson & Faul, 2010). In addition, the Andrade rheology inherits one of the limitations of the Maxwell model, that is, the inability to distinguish between the relaxed and unrelaxed values of the elastic part of the deformation. To overcome this limitation, more general transient laws have been proposed, such as the one by Sundberg and Cooper (2010). Nonetheless, the advantage of the Andrade rheology consists in a reduced number of material parameters, which makes it more convenient for modeling the tidal response of inaccessible planets for which few constraints (if any) on the internal structure are available (Padovan et al., 2014). Due to the various successful applications and the versatility demonstrated, the scope of the Andrade model has been reviewed very recently by Walterová et al. (2023), who have emphasized its usefulness in planetary science.

The response of a rheologically layered, self-gravitating planet to surface or tidal loads is traditionally studied by means of Love numbers, named after Love (1909). For a perturbation of harmonic degree ℓ, by the Love numbers formalism it is possible to describe the gravitational potential variation and vertical and horizontal displacements in terms of the model parameters. For a homogeneous elastic sphere, a well known analytical expression of Love numbers exists (see e.g., Jeffreys, 1976) that is still commonly employed in geodynamics and planetary science (see e.g., Bolmont et al., 2020). For linear visco-elastic rheologies, the analytical Love numbers are derived from the elastic expressions by virtue of the Correspondence Principle (Christensen, 1982; Churkin, 1998; Findley & Davis, 2013) through an application of the visco-elastic normal modes theory introduced by Peltier (1974). However, the inverse Laplace transform needed to retrieve the time domain Love numbers may be not straightforward, especially for multi-layered models or non-Maxwellian, transient rheologies (Spada & Boschi, 2006). Although this issue can be overcome adopting alternative inversion techniques and numerical methods (e.g., Mitrovica & Peltier, 1992; Melini et al., 2022), analytical solutions still have the advantage of showing directly the role and the weight of model parameters. Furthermore, analytical approaches are generally more efficient computationally, which is paramount for large-scale Monte Carlo inversion efforts (e.g., Briaud et al., 2023; Petricca et al., 2024).

Although analytical forms of the Andrade Love numbers can be found in the Laplace or Fourier transformed domain, as far as we know no time domain solutions have been ever published. Herein, for the first time we show that that their derivation is indeed possible, and that the time domain Love numbers for an Andrade planet can be expressed through the Mittag-Leffler function (Mittag-Leffler, 1903). Sometimes considered the "Queen function" of fractional calculus (Gorenflo et al., 2020; Mainardi, 2020), this special function has now a well established role in the field of linear visco-elasticity. Here we recognize that the Andrade rheology constitutes a generalization of the "fractional Maxwell model" studied by Mainardi and Spada (2011) and we reconsider its properties in a geophysical context aiming at the evaluation of Love numbers.

The paper is organized as follows: In Section 2 we discuss the creep compliance (Section 2.1) and we derive the relaxation modulus (Section 2.2) for the Andrade rheology in 1-D; Section 3 is devoted to Love numbers and their representation in the Laplace domain (Section 3.1), in the frequency domain (Section 3.2), and in the time domain (Section 3.3). Finally, our conclusions are drawn in Section 4.

2. Material Functions of the 1-D Andrade Model

In the two subsections that follow we study the material functions of the 1-D Andrade model, namely the creep compliance and the relaxation modulus, respectively.

2.1. Creep Compliance

In his fundamental works, Andrade (1910, 1962) found that for some metal wires loaded by a constant stress, the instantaneous elastic response is followed by transient creep governed by a power law ("β-flow"). Using a modern language, for such experiments the resulting creep compliance (i.e., the strain per unit stress) would be expressed by

$$J_a(t) = \frac{1}{\mu} + \beta t^{\alpha}, \quad t \geq 0, \tag{1}$$

where μ is the elastic shear modulus of the material, β is a parameter depending upon the sample properties and laboratory conditions and, according to experimental evidence, the time exponent is $\alpha = 1/3$. In subsequent laboratory investigations, values ranging between $\alpha = 0.2$ and $\alpha = 0.5$ have been suggested even for some non-metallic substances (see Walterová et al., 2023). Accordingly, in what follows we shall assume that the α exponent in Equation 1 can take a priori any value in the range $0 < \alpha \leq 1$.

In his treatise, Mainardi (2022) refers to Equation 1 as a "fractional Maxwell model," since its mechanical analogue stems from the connection (in series) of a Hookean elastic element (a *spring*) of shear modulus μ with a Scott–Blair creep element characterized by a fractional power-law (sometimes referred to as a *pot*; see Scott-Blair, 1951, 1970). For $\alpha = 1$, the fractional Maxwell model reduces to the traditional steady state Maxwell model since the Scott-Blair element reduces to a Newtonian element. The basic properties of the fractional Maxwell model, that is, the creep compliance, the relaxation modulus and the effective viscosity, have been studied by Mainardi and Spada (2011), who also have analyzed the response of more complex fractional rheological models.

In modern applications to planets, the pure Andrade creep law expressed by Equation 1 is generalized to account for a long-term steady state behavior. This extension is simply performed by connecting a Newtonian element (a *dashpot*) in series with the elastic spring and the transient pot (see Walterová et al., 2023). Accordingly, the complete form of the Andrade creep compliance reads:

$$J_a(t) = \frac{1}{\mu} + \beta t^{\alpha} + \frac{t}{\eta}, \quad t \geq 0, \tag{2}$$

where η is the Newtonian viscosity. As pointed out by Walterová et al. (2023), the physical meaning of the β-factor in Equation 2 is hindered by its fractional dimensions. Therefore, it seems more convenient to rewrite the creep compliance in the form

$$J_a(t) = \frac{1}{\mu}\left(1 + \left(\frac{t}{\tau_A}\right)^{\alpha} + \frac{t}{\tau_M}\right), \quad t \geq 0, \tag{3}$$

where

$$\tau_M = \frac{\eta}{\mu} \tag{4}$$

is the "Maxwell time," that is, the characteristic timescale on which steady-state behavior occurs, while

$$\tau_A = (\beta\mu)^{-\frac{1}{\alpha}} \tag{5}$$

is the "Andrade time" that represents the transient response timescale. Efroimsky (2012a, 2012b) introduced the non-dimensional ratio:

$$\zeta = \frac{\tau_A}{\tau_M}, \tag{6}$$

which allows to cast the Andrade creep law in the so called "$\alpha - \zeta$ parametrization"

$$J_a(t) = \frac{1}{\mu}\left(1 + \zeta^{-\alpha}\left(\frac{t}{\tau_M}\right)^{\alpha} + \frac{t}{\tau_M}\right), \quad t \geq 0, \tag{7}$$

with $J_a(t)$ reducing to the Maxwellian creep compliance in the limit $\zeta \to \infty$ (i.e., for an infinite Andrade time τ_A).

To reduce the number of free parameters from four to three, Castillo-Rogez et al. (2011) proposed to define the β-factor as

$$\beta = \frac{\mu^{\alpha-1}}{\eta^{\alpha}}, \tag{8}$$

which, by Equation 5, is equivalent to assuming $\tau_A = \tau_M$, or $\zeta = 1$. As remarked out by Castillo-Rogez et al. (2011) in their study about the tidal history of Iapetus, the constraint $\zeta \approx 1$ should be regarded as a first-order approximation, awaiting for laboratory data of sufficient quality. Nowadays, the approximation $\zeta \approx 1$ is generally considered as outdated, with plausible values of ζ ranging between 10^{-2} and 1, although values as high as 10^5 have been shown to be consistent with the tidal response of the Earth (e.g., Walterová et al., 2023; Amorim & Gudkova, 2023).

In Figure 1a, the Andrade creep compliance $J_a(t)$ is shown as a function of the normalized time t/τ_M, for different values of $\alpha = 1/n$, where n is an integer, and $\zeta = 1$ is assumed. The original Andrade result ($n = 3$, or $\alpha = 1/3$) is depicted by a green curve. The time scale characterizing the transient phase (i.e., the time required to reach a constant creep rate) decreases with increasing n. For $\alpha = 1$, no transient occurs after the elastic step at $t = 0$, and the response is Maxwellian. In this case, the transient term in Equation 7 duplicates the steady-state term, hence the response is that of a Maxwell body with Newtonian viscosity $\eta/2$. Figure 1b depicts $J_a(t)$ for different values of parameter ζ, keeping the fractional exponent fixed to $\alpha = 1/3$. As expected from its definition, the value of ζ controls the relative importance of the steady-state and transient terms in Equation 7, with the response more closely following that of a Maxwell body for large ζ values.

The Laplace transform of the Andrade creep law (Equation 7) can be obtained by elementary methods and it reads

$$\tilde{J}_a(s) = \frac{1}{\mu\, s}\left(1 + \frac{\Gamma(1+\alpha)}{(\zeta\tau_M s)^{\alpha}} + \frac{1}{s\tau_M}\right), \tag{9}$$

where s is the complex Laplace variable and $\Gamma(x)$ is Euler's gamma function.

2.2. Relaxation Modulus

To fully characterize the rheological behavior of the Andrade model, it is also desirable to obtain the relaxation modulus $G_a(t)$, which physically represents the stress per unit strain in a relaxation experiment (e.g., Christensen, 1982). In the time domain, the relaxation modulus $G(t)$ of a general linear visco-elastic material is related with the creep compliance $J(t)$ through $J(t) * G(t) = t$, where $*$ denotes time-convolution (e.g., Mainardi, 2022). In the Laplace domain, this relation reads $\tilde{J}(s)\,\tilde{G}(s) = 1/s^2$, hence $s\,\tilde{G}_a(s) = 1/(s\tilde{J}_a(s))$. Therefore, using Equation 9, the Laplace transformed relaxation modulus is easily obtained:

$$\tilde{G}_a(s) = \mu\tau_M \frac{(s\tau_M)^{\alpha-1}}{(s\tau_M)^{\alpha} + \zeta^{-\alpha}\Gamma(1+\alpha) + (s\tau_M)^{\alpha-1}}. \tag{10}$$

Evaluating the Laplace inverse of Equation 10 analytically is not straightforward. As far as we know, to date $G_a(t)$ has never been obtained neither in terms of elementary nor of transcendental functions. This is also true for rheological laws that, similar to Andrade's, are expressed by creep laws containing a fractional power of time t, like the modified Lomnitz or the Jeffrey-Lomnitz laws (see Ranalli, 1995; Walterová et al., 2023). We note that

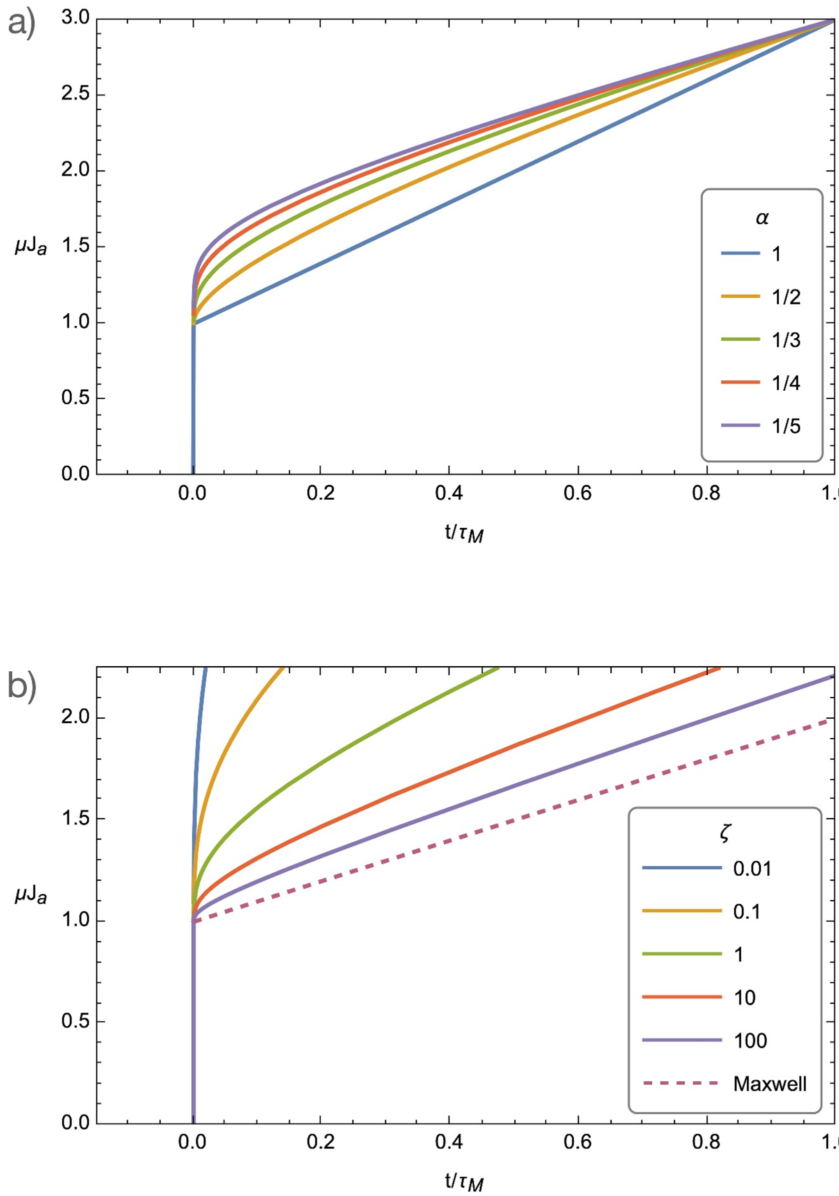

Figure 1. Andrade creep compliance $J_a(t)$, according to Equation 7. In panel (a), $J_a(t)$ is shown for $\zeta = 1$ and $\alpha = 1/n$ ($n = 1, \ldots, 5$), while in panel (b) α is set to 1/3 and different values of parameter $\zeta = \tau_A/\tau_M$ are considered. The Maxwell response is attained for $\zeta \to \infty$.

for the Jeffrey-Lomnitz rheological law Mainardi and Spada (2012) have recovered the relaxation modulus by solving numerically $J(t) * G(t) = t$, which they have cast in the form of an integral Volterra equation of the second kind. However, they did not attempt to obtain a closed-form solution of this equation.

With the purpose of performing the Laplace inversion of Equation 10 to retrieve $G_a(t)$, we rely upon the result expressed by Equation E53 of the treatise of Mainardi (2022) and on symbolic manipulation. The approach is based on a suitable partial fraction decomposition of Equation 10, along the lines followed by Mainardi et al. (1995) to solve the generalized Basset problem that concerns the motion of a particle within a viscous fluid, which shows some formal analogies with the problem of concern here. The analysis has been assisted by the *Mathematica*© symbolic manipulator (Wolfram Research, Inc, 2024), through an iterated application of the function `InverseLaplaceTransform`.

Assuming that α takes the particular form of a proper fraction, that is, $0 < \alpha = p/q \le 1$, where $p,q \in \mathbb{N}$, we have found that it is possible to express the relaxation modulus as a finite sum, with:

$$G_a(t) = \frac{\mu}{\left(\frac{t}{\tau_M}\right)^{1-\frac{1}{q}}} \sum_{k=1}^{q} \left(\frac{E_{\frac{1}{q},\frac{1}{q}}\left(r_k \left(\frac{t}{\tau_M}\right)^{\frac{1}{q}}\right)}{q\, r_k^{q-1} + \frac{q-p}{\zeta^{p/q}} \Gamma\left(\frac{p}{q}+1\right) r_k^{q-p-1}} \right), \quad t \ge 0, \tag{11}$$

where

$$E_{\alpha,\beta}(z) \equiv \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \text{with} \quad \alpha,\beta \in \mathbb{C},\ \mathrm{Re}(\alpha) > 0,\ z \in \mathbb{C}, \tag{12}$$

is the two-parameter Mittag-Leffler transcendental function (Mittag-Leffler, 1903), whose properties and fields of application are now well known (Gorenflo et al., 2020; Mainardi, 2020) and where the r_k's in Equation 11 are the (distinct) roots of the algebraic equation $R_{p,q}(x) = 0$, where

$$R_{p,q}(x) \equiv x^q + \frac{1}{\zeta^{p/q}} \Gamma\left(\frac{p}{q}+1\right) x^{q-p} + 1 \tag{13}$$

is a degree q polynomial with positive coefficients, whose derivative appears at the denominator of Equation 11, evaluated at $z = r_k$. Note that although some of the r_k's are complex, the properties of the Mittag-Leffler function ensure that in Equation 11 the relaxation modulus $G_a(t)$ is a real function, as we have also verified by symbolic manipulation. Note also that, for $\zeta \gg 1$ (i.e., for $\tau_A \gg \tau_M$), the roots of polynomial (Equation 13) can be approximated by $r_k \approx e^{i(2k-1)\pi/q}$ for $k = 1, \ldots, q$.

We mention that Mainardi and Spada (2011) already noted that, for the Andrade law in the fundamental form given by Equation 1, the relaxation modulus can be expressed in terms of the classical (one index) Mittag-Leffler transcendental function defined as $E_\alpha(z) = E_{\alpha,1}(z)$. Therefore, the result expressed by Equation 11 can be considered as an extension of the previous findings of Mainardi and Spada (2011) to the case of the generalized Andrade model expressed by Equation 3.

The Andrade relaxation modulus $G_a(t)$ is shown for $\zeta = 1$ in Figure 2a as a function of the normalized time t/τ_M, in the special case $\alpha = 1/n$ with $n = 1, \ldots, 5$ whereas, for the same n values, a plot of the polynomial $R_{1,n}(x)$ is shown in Figure 3. These figures include the original Andrade result ($n = 3$) and the purely Maxwellian response ($n = 1$) as special cases. Figure 2b shows the relaxation modulus for $\alpha = 1/3$ and different values of ζ. It is evident that, as expected, the relaxation modulus converges to that of a Maxwell solid as $\zeta \to \infty$.

In Figure 2, the dashed green curve corresponding to $n = 3$ and $\zeta = 1$ has been obtained by performing analytically an alternative partial fraction decomposition of Equation 10, which leads to the form:

$$\begin{aligned} G_a(t) = {} & \mu \sum_{k=1}^{3} a_k e^{s_k t} \times \\ & \times \left\{ \left(\frac{1}{\tau_M} + s_k\right) \left[\frac{1}{\tau_M} + s_k - \gamma s_k^{\frac{2}{3}} P\left(-\frac{2}{3}, s_k t\right)\right] + \gamma^2 s_k^{\frac{4}{3}} P\left(-\frac{4}{3}, s_k t\right) \right\}, \\ & t \ge 0, \end{aligned} \tag{14}$$

where

$$P(\nu, x) \equiv \frac{1}{\Gamma(\nu)} \int_0^x z^{\nu-1} e^{-z}\, dz \tag{15}$$

is the normalized lower incomplete gamma function, we have defined the constants

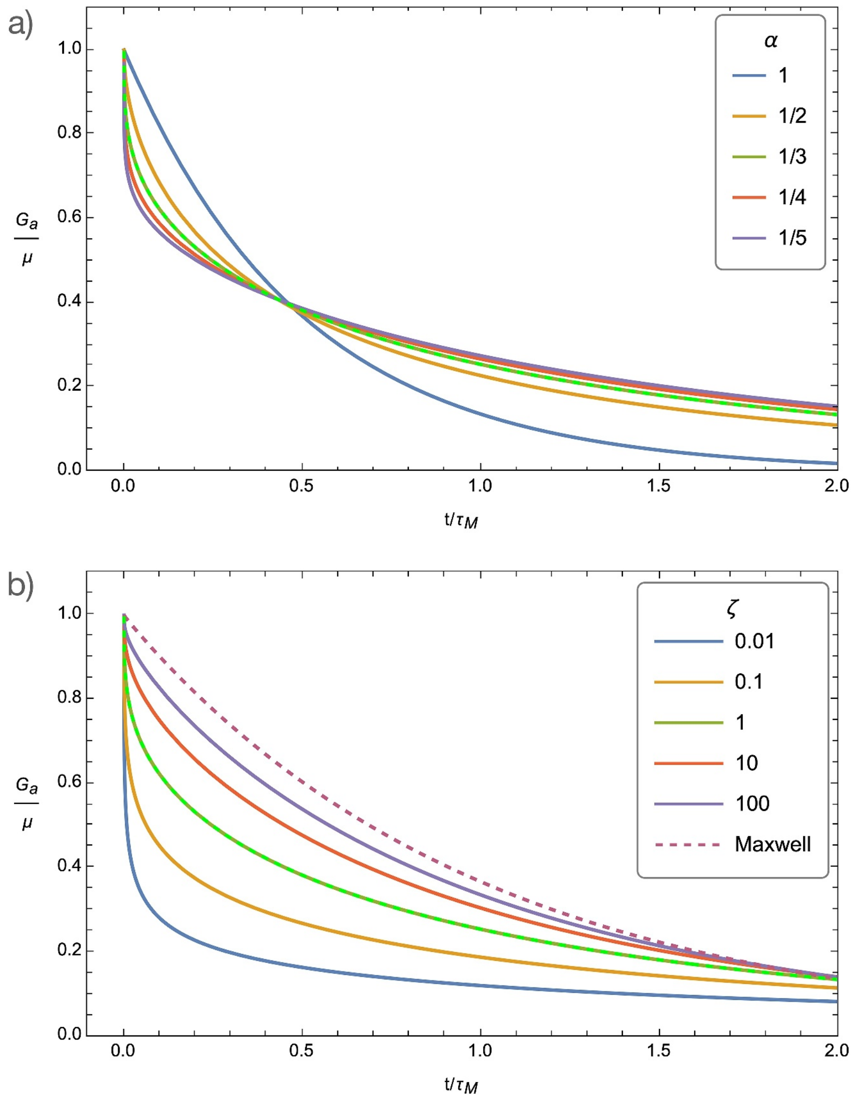

Figure 2. Relaxation modulus $G_a(t)$ for an Andrade body, according to Equation 11. In panel (a), $\zeta = 1$ and $\alpha = 1/n$ are assumed ($n = 1, \ldots, 5$). The green curve corresponds to $n = 3$, with the dashed one showing Equation 14. For $n = 1$ the decay is exponential, since $E_{1,1}(-x) = \mathrm{e}^{-x}$. In panel (b), the modulus is shown for $\alpha = 1/3$, using different values of the ratio $\zeta = \tau_A/\tau_M$. The Maxwell case (dashed purple) corresponds to $\zeta \to \infty$.

$$\gamma = \frac{\Gamma(\alpha + 1)}{\tau_M^{\frac{1}{3}}} \tag{16}$$

and

$$a_k = \prod_{\substack{j=1 \\ j \neq k}}^{3} \left(\frac{1}{s_k - s_j} \right), \tag{17}$$

and where the s_k's ($k = 1, 2, 3$) are the (distinct) roots of the cubic equation

$$s^3 + \left(\frac{3}{\tau_M} + \gamma^3 \right) s^2 + \frac{3}{\tau_M^2} s + \frac{1}{\tau_M^3} = 0. \tag{18}$$

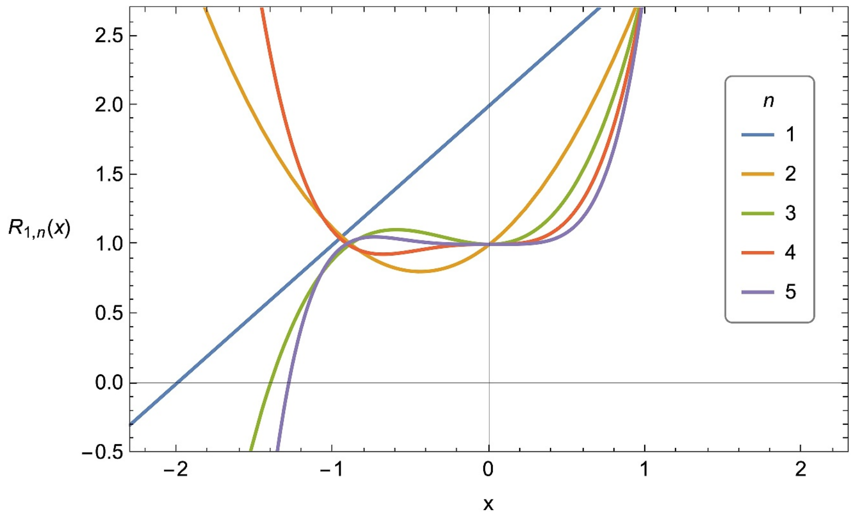

Figure 3. Plot of the $R_{1,n}(x)$ polynomial defined by Equation 13, for some n values and $\zeta = 1$. One real and negative root is only found for odd values of n, as we have also verified for $n > 5$. For $n = 3$, the value suggested by the original work of Andrade (1910), the three roots of the polynomial are $r_1 \approx -1.401$ and $r_{2,3} \approx (0.254 \pm i\ 0.805)$.

Figure 2 visually shows that, for $n = 3$, Equations 11 and 14 are providing largely consistent results. Indeed, inspection of the numerical values indicates a very close match, to within 0.1%. After some experiments using the *Mathematica*© symbolic manipulator (Wolfram Research, Inc, 2024), we have directly verified that Equation 14 is about twice computationally less expensive than Equation 11. The details of the analytical derivation that we have followed to obtain Equation 14 shall be reported elsewhere.

3. Love Numbers of an Andrade Planet

In this section, we discuss three different forms of the Love numbers for a uniform planet with Andrade rheology, namely the Laplace domain (Section 3.1), the frequency domain (Section 3.2) and the time domain Love numbers (Section 3.3), respectively. Along with well established results, we present some new analytical findings for the Love numbers in the time domain, which derive from the general expressions of the relaxation modulus $G_a(t)$ obtained in Section 2.

3.1. Love Numbers in the Laplace Domain

The classical results of Love (1909), valid for an elastic Earth, can be straightforwardly extended to anelastic bodies by means of the Correspondence Principle of linear visco-elasticity, substituting the elastic shear modulus μ with a suitable complex modulus $\tilde{\mu}(s)$ (see e.g., Biot, 1954; Christensen, 1982). This approach is at the basis of the visco-elastic normal modes method (Peltier, 1974) and it is widely employed in glacial isostatic adjustment studies (see e.g., Spada et al., 2011).

For the sake of our purposes here, the general form of the Laplace transformed Love numbers for a homogeneous, incompressible, visco-elastic planet subject to an impulsive forcing is:

$$\frac{\tilde{L}(s)}{L_f} = \frac{1}{1 + c\left(\frac{\tilde{\mu}(s)}{\mu}\right)}, \tag{19}$$

which is valid for any type of Love number, either corresponding to tidal or loading perturbations (see Table 1), the dependence upon the harmonic degree ℓ is implicit, $L_f = \lim_{s\to 0}\tilde{L}(s)$ is the fluid Love number, $\tilde{\mu}(s)$ is the complex shear modulus and c is the positive dimensionless constant

$$c = \frac{2\ell^2 + 4\ell + 3}{\ell}\mu', \tag{20}$$

Table 1
Expressions of the Fluid Limits $L_f = L_f(\ell)$ in Equation 19 for the Love Numbers h_ℓ, k_ℓ and l_ℓ, Corresponding to Tidal and Surface Loading Boundary Conditions, as a Function of the Harmonic Degree ℓ

Love number	L_f Tidal forcing	L_f Surface loading
Potential, k_ℓ	$\frac{3}{2(\ell-1)}$	-1
Vertical, h_ℓ	$1+\frac{3}{2(\ell-1)}$	$-\frac{2\ell+1}{3}$
Horizontal, l_ℓ	$\frac{3}{2\ell(\ell-1)}$	$-\frac{1}{\ell}$

Note. These expressions are based upon Wu and Peltier (1982).

where we have introduced the normalized elastic shear modulus

$$\mu' = \frac{\mu}{\rho g_a a}, \tag{21}$$

where $\mu = \lim_{s\to+\infty} \tilde{\mu}(s)$ is the elastic shear modulus, ρ is density, a is the planetary radius, $g_a = \frac{4}{3}\pi G\rho a$ is gravity at the surface, G is the gravitational constant and, for the fundamental mode with $\ell = 2$, the ℓ-dependent factor in Equation 20 takes the well known value of $\frac{19}{2}$.

Table 2 lists numerical values for the bulk properties of some terrestrial bodies, along with the corresponding μ' and c values obtained at degree $\ell = 2$ by Equations 20 and 21, respectively. For the Earth, $\mu' \approx 0.4$, about twice the value $\mu' \approx \frac{1}{5}$ estimated by Love (1911) in his seminal work. Hence in Equation 20, at degree $\ell = 2$, the c constant can be approximated as

$$c \approx 4\left(\frac{\mu}{\mu_e}\right)\left(\frac{\rho_e}{\rho}\right)^2\left(\frac{a_e}{a}\right)^2, \tag{22}$$

where a_e, ρ_e and μ_e are the Earth's radius, average density and bulk shear rigidity, respectively, as given in Table 2.

According to Equation 19, in order to evaluate the Love numbers explicitly, the expression for the Andrade complex shear modulus $\tilde{\mu}_a(s)$ is necessary. The modulus is related to the transformed creep compliance and relaxation modulus by $\tilde{\mu}_a(s) = 1/s\tilde{J}_a(s) = s\,\tilde{G}_a(s)$ (e.g., Mainardi, 2022). Using, in particular, the second of these identities and recalling Equation 10, the expression of the complex shear modulus for the Andrade rheology turns out to be

$$\frac{\tilde{\mu}_a(s)}{\mu} = \frac{(s\tau_M)^\alpha}{(s\tau_M)^\alpha + \zeta^{-\alpha}\Gamma(1+\alpha) + (s\tau_M)^{\alpha-1}}, \tag{23}$$

and is easily verified that, as expected, in the limit $\zeta \to \infty$ the complex shear modulus (Equation 23) reduces to that of a 1-D Maxwell body, that is,

$$\lim_{\zeta\to\infty} \tilde{\mu}_a(s) = \frac{\mu\, s}{s + 1/\tau_M} \equiv \tilde{\mu}_m(s). \tag{24}$$

The substitution of Equation 23 into Equation 19 yields, after some algebra,

$$\frac{\tilde{L}(s)}{L_f} = 1 - \frac{c(s\tau_M)^\alpha}{(1+c)(s\tau_M)^\alpha + \zeta^{-\alpha}\Gamma(1+\alpha) + (s\tau_M)^{\alpha-1}}, \tag{25}$$

which represents the Laplace transformed Love number for a uniform Andrade planet subject to an impulsive forcing.

In Figure 4, we show the landscape of the complex-valued function $F(z) = \tilde{L}(z)/L_f$ in the Argand-Gauss plane, where variable z is defined as $z \equiv s\tau_M = x + iy$, with $i = \sqrt{-1}$. Here we have assumed the traditional Andrade power law with exponent $\alpha = 1/3$ and $\zeta = 1$, and we have set $c = 4$, a value representative of the Earth. Function Re($F(z)$), shown in Figure 4a, is continuous in the whole z plane and symmetric with respect to the x axis, where the extrema are attained for $x < 0$ (Figure 4c, solid line). Function Im($F(z)$) is anti-symmetric with respect to the x-axis and vanishes for $x \geq 0$ (see Figure 4b); furthermore, it shows a jump discontinuity along the real negative axis (Figure 4d, solid lines). In Figures 4c and 4d, dashed lines show numerical results for the ratio between the h_2 tidal LN and its fluid limit for a

Table 2
Numerical Values of the μ' and c Constants (at Degree $\ell = 2$) for Some Terrestrial Bodies in the Solar System

Planet	a km	ρ kg/m^3	μ 10^{11} Pa	μ' –	c $\ell = 2$
Mercury	2,439.7	5,427	0.75	1.53	14.55
Venus	6,051.8	5,243	1.45	0.52	4.90
Earth	6,370.9	5,514	1.46	0.42	4.02
Moon	1,737.1	3,348	0.67	7.09	67.35
Mars	3,389.5	3,918	1.05	2.13	20.24

Note. Average radii (a), densities (ρ) and elastic shear moduli (μ) are from Table 1 of Zhang (1992).

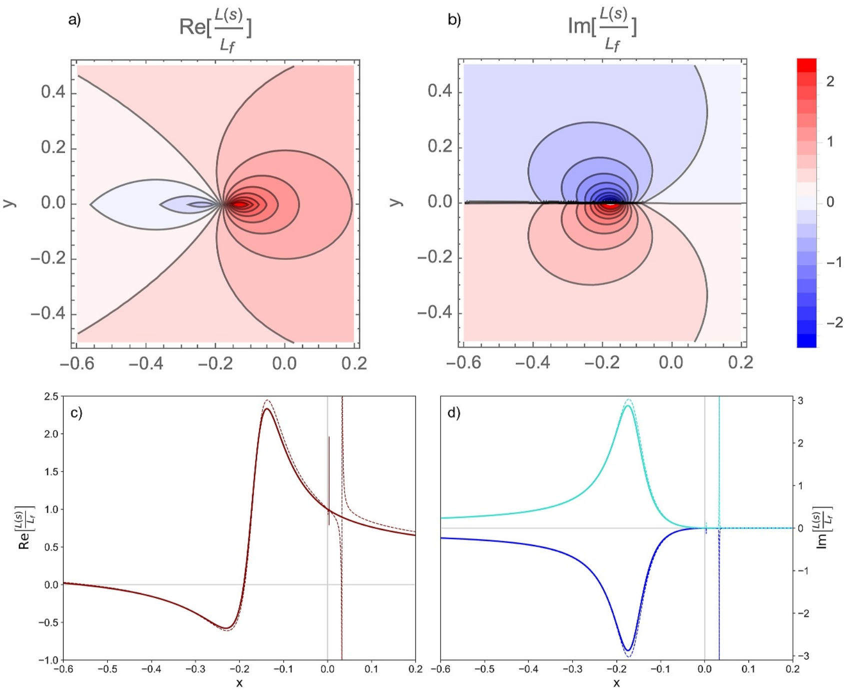

Figure 4. Contour plots of the real part (a) and of the imaginary part (b) of $F(z) = \tilde{L}(z)/L_f$, for an Andrade sphere with $\alpha = \frac{1}{3}$, $c = 4$ and $\zeta = 1$. In panel (c), the real part is shown along the real axis while in panel (d), the imaginary part is plotted along two axes just above and just below the x-axis, defined by $y = \pm 10^{-4}$. In panels (c, d), dashed lines show numerical results for the ratio between the h_2 tidal LN of a compressible sphere and its fluid limit.

compressible sphere, obtained by substituting the complex shear modulus in Equation 23 in the analytical expressions published by Love (1911) and assuming a Lamé first parameter $\lambda = \mu$, which corresponds to a Poisson ratio $\nu = \frac{1}{4}$. It is readily seen that, on the real negative axis, the differences between the Love numbers spectra in the compressible and incompressible cases are very small, of the order of a few percent. Conversely, on the real positive axis the compressible spectrum shows singularities related to the Rayleigh-Taylor instabilities, which for a layered Earth have been discussed by Hanyk et al. (1999) and Vermeersen and Mitrovica (2000) in the framework of the visco-elastic normal modes theory of Peltier (1974).

3.2. Love Numbers in the Frequency Domain

In studies of the planetary tidal response, a natural representation of the forcing potential can be obtained as a sum of Fourier components whose angular frequencies are appropriate combinations of Delaunay arguments (see, e.g., Williams & Boggs, 2015). As discussed for example, by Melini et al. (2022), in that context the response to a periodic forcing of frequency ω is described by complex-valued Love numbers, which can be evaluated by sampling the Laplace transformed LNs for an impulsive load along the imaginary axis of the complex plane (i.e., for $s = i\omega$). Hence, from Equation 25 the expression for the complex-valued LNs corresponding to the periodic forcing of a uniform Andrade sphere at frequency ω is readily obtained:

$$\frac{\tilde{L}(\omega)}{L_f} = 1 - \frac{c}{(1+c) + (i\zeta\omega\tau_M)^{-\alpha}\Gamma(1+\alpha) + (i\omega\tau_M)^{-1}}. \tag{26}$$

Along the lines of Bierson and Nimmo (2016), it is now convenient to define

$$A(\omega) \equiv \Gamma(\alpha+1)(\omega\tau_A)^{-\alpha} \tag{27}$$

and

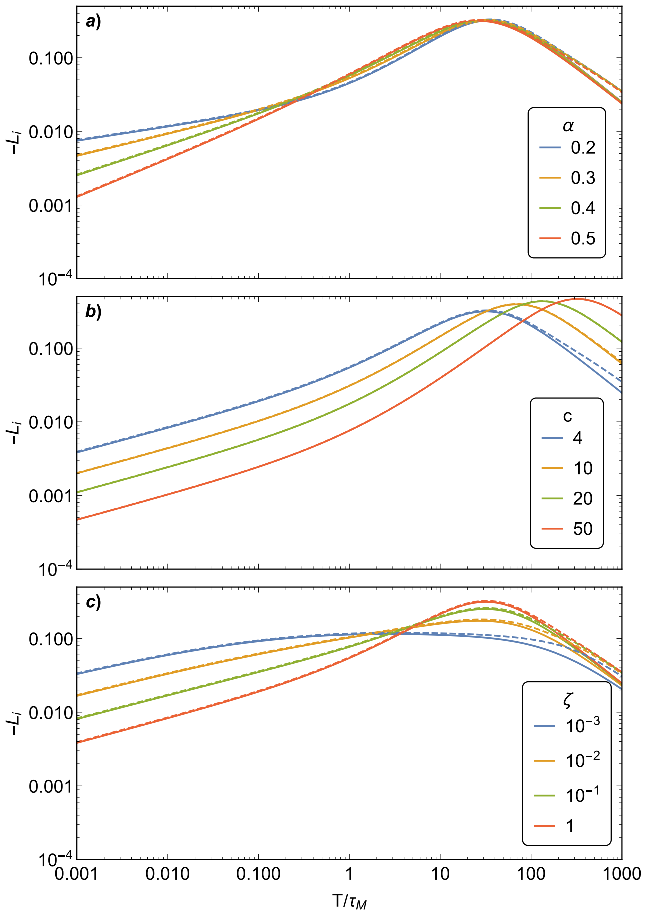

Figure 5. Imaginary part of the LN L_i as a function of the normalized period T/τ_M for different values of α, assuming $c = 4$ and $\zeta = 1$ (a); for different values of c, assuming $\alpha = 1/3$ and $\zeta = 1$ (b); for different values of ζ, assuming $\alpha = 1/3$ and $c = 4$ (c). Dashed lines show the imaginary parts of the ratio between the h_2 tidal LN and its fluid limit for a compressible homogeneous sphere, assuming an Earth's average Poisson ratio $\nu_e = 0.35$.

$$\xi \equiv \frac{\alpha\pi}{2}, \tag{28}$$

in such a way that Equation 26 can be written in the form

$$\frac{\tilde{L}(\omega)}{L_f} = 1 - c\,\frac{[(1+c) + A\cos\xi] + i\left[A\sin\xi + (\omega\tau_M)^{-1}\right]}{[(1+c) + A\cos\xi]^2 + \left[A\sin\xi + (\omega\tau_M)^{-1}\right]^2}. \tag{29}$$

in terms of A and ξ, the real and imaginary parts of $\tilde{L}(\omega)/L_f$ are then given by

$$L_r(\omega) = 1 - c\,\frac{(1+c) + A\cos\xi}{[(1+c) + A\cos\xi]^2 + \left[A\sin\xi + (\omega\tau_M)^{-1}\right]^2} \tag{30}$$

and

$$L_i(\omega) = -c\,\frac{A\sin\xi + (\omega\tau_M)^{-1}}{[(1+c) + A\cos\xi]^2 + \left[A\sin\xi + (\omega\tau_M)^{-1}\right]^2}, \tag{31}$$

respectively. For a periodic tidal forcing, L_r characterizes the purely elastic deformation, while L_i expresses the frequency dependence of tidal energy dissipation in the planet (see, e.g., Remus et al., 2012). We note that, since it is generally assumed that $L_i \ll L_r$, one has $|\tilde{L}(\omega)| \approx L_r L_f$.

Figure 5 shows the imaginary part L_i as a function of the normalized tidal period $T/\tau_M = 2\pi/(\omega\tau_M)$ for different combinations of α, c and ζ. If we consider $\zeta = 1$ and $c = 4$ (Figure 5a), appropriate for the degree $\ell = 2$ forcing of an Earth-like planet, dissipation is sensitive to α for tidal periods up to $\approx 0.1\,\tau_M$, with peak dissipation occurring at periods of the order $\approx 10\,\tau_M$, almost independently of the value of α. If we consider $\alpha = 1/3$ while keeping $\zeta = 1$ (Figure 5b), dissipation is strongly sensitive to c, which is directly related to the model parameters (see Equation 20), with peak dissipation occurring for T in the range between $10\,\tau_M$ and $10^2\,\tau_M$. Finally, setting $\alpha = 1/3$ and $c = 4$ (Figure 5c), dissipation turns out to be generally dependent upon ζ, with the notable exception of tidal periods $T \approx 3\tau_M$ for which all the considered values of ζ result in approximately the same L_i. A peak in dissipation occurs at period $T \approx 30\,\tau_M$ and progressively vanishes as ζ decreases. Numerical results for a compressible Kelvin sphere with an Earth-like Poisson ratio $\nu_e = 0.35$, represented by dashed curves in Figure 5, show that in all the experiments performed incompressibility represents a very good approximation for tidal periods up to at least $\approx 10\,\tau_M$. Figure 6 shows L_i as a function of parameters α and c for $T/\tau_M = 0.01$ and 0.1, assuming $\zeta = 1$. It is evident that in both cases there is a marked sensitivity of the dissipation from the model parameters. However, the same value of L_i can be obtained with different combinations of c and α, so that a direct observation of L_i would not allow to infer the model parameters without additional constraints.

It is now of interest to consider the asymptotic behavior of L_i for tidal periods that are either much larger or much smaller than the characteristic timescales of the relaxation. First, we note that in the high frequency limit, that is, for $\omega\tau_M \gg 1$ and $\omega\tau_A \gg 1$, it is possible to approximate Equation 31 as

$$L_i(\omega) \approx -c\,\frac{\Gamma(\alpha+1)\sin\xi}{(1+c)^2}(\omega\tau_M\zeta)^{-\alpha}, \tag{32}$$

which corresponds to the power law scaling of dissipation with low tidal periods $L_i \sim T^{\alpha}$ evident in Figure 5. For an Earth-like planet with average Maxwell viscosity 10^{21} Pa · s we have $\tau_M \approx 1.3$ kyr, hence Equation 32 would be

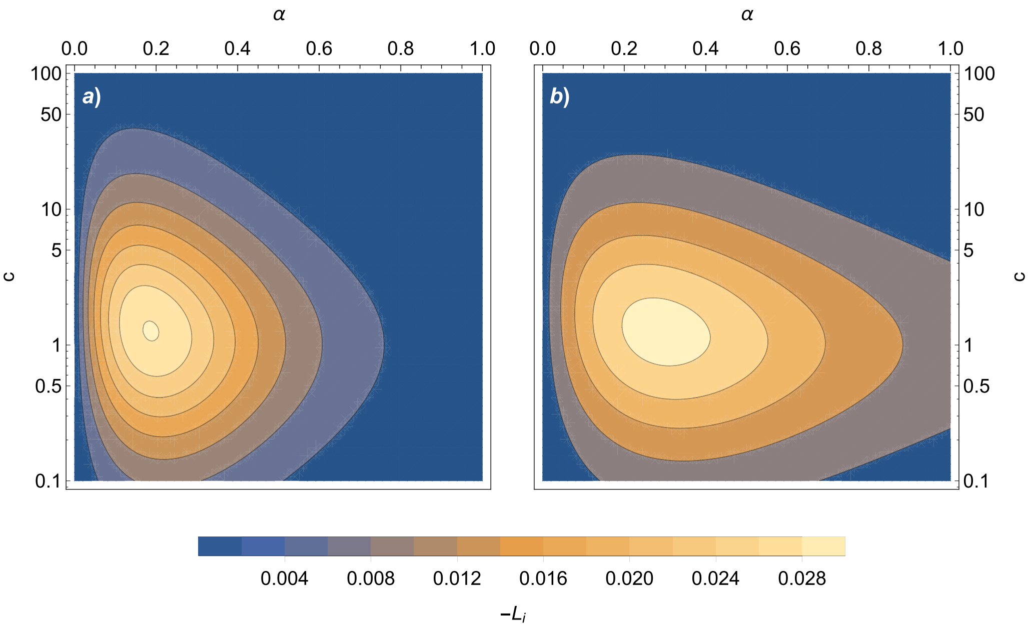

Figure 6. Imaginary part of the LN L_i as a function of α and c for $T/\tau_M = 0.01$ (a) and for $T/\tau_M = 0.1$ (b). In both frames, $\zeta = \tau_A/\tau_M = 1$ has been assumed.

valid for the monthly tide even for low values of the ζ ratio. In the low frequency limit $\omega\tau_M \ll 1$, Equation 31 instead simplifies as

$$L_i(\omega) \approx -c\,\omega\tau_M, \tag{33}$$

with the tidal response being only determined by the c constant and by the Maxwell time τ_M. We remark, however, that for a realistic description of the response at low tidal frequencies the relaxation of the elastic component of the deformation shall be taken into account, for instance through the Sundberg and Cooper (2010) model. It is easily verified that Equations 32 and 33 are equivalent to the "high-frequency band" and "low-frequency band" approximations given in Appendix C.2 of Efroimsky (2012a), respectively.

3.3. Love Numbers in the Time Domain

To describe the time-evolution of the Love numbers, instead of the impulsive solution in Equation 25 it is more meaningful and physically intuitive to consider a load imposed at time $t = 0$ and held in place thereafter, as it is customarily done in glacial isostatic adjustment studies (e.g., Spada et al., 2011). Since the Laplace transform of a Heaviside step function $H(t)$ is $1/s$, the corresponding Love number is $\tilde{L}^H(s) = \tilde{L}(s)/s$. After multiplication of Equation 25 by $1/s$, it is possible to write the Heaviside Love number in the computationally convenient form:

$$\frac{\tilde{L}^H(s)}{L_f} = \frac{1}{s} - \frac{c\tau_M}{(1+c)(s\tau_M) + \zeta^{-\alpha}\Gamma(1+\alpha)(s\tau_M)^{1-\alpha} + 1}, \tag{34}$$

and it is easily shown that, in the limit $\zeta \to \infty$, $\tilde{L}^H(s)$ reduces to the Laplace transform for the Love number of a uniform Maxwell sphere, that is,

$$\lim_{\zeta\to\infty} \frac{\tilde{L}^H(s)}{L_f} = \frac{1 + s\tau_M}{(1+c)(s\tau_M) + 1} = \frac{1}{s}\frac{1}{1 + c\left(\frac{\tilde{\mu}_m(s)}{\mu}\right)}, \tag{35}$$

where $\tilde{\mu}_m(s)$, defined in Equation 24, is the complex shear modulus appropriate for the 1-D Maxwell rheological body (e.g., Mainardi & Spada, 2011).

Assuming that $\alpha = p/q \leq 1$, where p and q are integers, the r.h.s. of Equation 34 can be Laplace-inverted in closed-form following the same approach adopted to invert $\tilde{G}_a(s)$ in Section 2. Indeed, with the aid of *Mathematica*© (Wolfram Research, Inc, 2024), we have verified that the time-domain Heaviside Love number can be cast in the form:

$$\frac{L^H(t)}{L_f} = 1 - \Phi_{p,q}(t), \quad t \geq 0, \tag{36}$$

where we have defined

$$\Phi_{p,q}(t) = \frac{c}{\left(\frac{t}{\tau_M}\right)^{1-\frac{1}{q}}} \sum_{k=1}^{q} \left(\frac{E_{\frac{1}{q},\frac{1}{q}}\left(\left(\frac{t}{\tau_M}\right)^{\frac{1}{q}} z_k\right)}{\frac{q-p}{\zeta^{p/q}}\Gamma\left(1+\frac{p}{q}\right) z_k^{q-p-1} + q(1+c) z_k^{q-1}} \right), \tag{37}$$

and where the z_k's are the (distinct) roots of the algebraic equation $F_{p,q}(x) = 0$, with $F_{p,q}(x) = R_{p,q}(x) + cx^q$, with polynomial $R_{p,q}(x)$ defined by Equation 13. Note that the z_k's are themselves depending upon parameters c and ζ. Since the properties of the Mittag-Leffler function ensure that $\lim_{t\to+\infty} \Phi_{p,q}(t) = 0$, in Equation 36 condition $\lim_{t\to+\infty} L^H(t) = L_f$ is met, as it is expected for consistency. Furthermore, we have verified that $\lim_{t\to 0^+} \Phi_{p,q}(t) = c/(1+c)$, corresponding to the instantaneous elastic response $L^H(0^+) = L_f/(1+c)$.

To corroborate the mathematical result given by Equations 36 and 37, we have obtained independent numerical results by the ALMA[3] planetary Love numbers calculator (Melini et al., 2022). ALMA[3] is a code that implements the Post-Widder Laplace inversion technique (Post, 1930; Widder, 1934) for spherically symmetric models with general incompressible linear visco-elastic rheology (see also Spada, 2008; Spada & Boschi, 2006). Figure 7, obtained using a uniform model whose parameters are listed in the caption, considers various combinations of the α and ζ parameters. The match between the analytical (solid curves) and numerical results (dotted) is very satisfactory, with a relative error never exceeding the 0.1% level. Note that the Love numbers are characterized by short timescale (elastic) and long timescale (fluid) asymptotes that are not influenced by the value of parameter α. However, the transition from elastic to fluid regimes is controlled by the value of α, with the response becoming slower for decreasing α.

To study the sensitivity of the Love numbers to the model parameters, in Figure 8 we consider the tidal Love number $k_2^H(t)$, a quantity of fundamental importance in planetary studies since it characterizes the tidal response of the body. Figure 8a shows the Love number $k_2^H(t)$, obtained from Equation 36 with $\alpha = 1/3$ and $\zeta = 1$ as a function of time, for various values of the normalized shear modulus μ' characterizing distinct hypothetical planetary models. Since $\mu' \propto \mu/(\rho\, a)^2$, a small μ' may correspond to a low-rigidity planet (small μ) or to a body with with large a and/or ρ (hence a large gravity at the surface). Conversely, a large μ' value may correspond to an elastically stiff planet (large μ) and/or to a small-radius and a low-density body. We note that for an Earth-like planet, $\mu' \approx 0.4$ (see Table 2). From Figure 8a it is apparent that the value of μ' has a strong influence on the evolution of $k_2^H(t)$. In particular, the elastic response $k_2^H(0)$ increases with decreasing μ', which is expected since the initial deformation is large for a low-rigidity body. Furthermore, the transition to the fluid limit $k_2^H(\infty) = 3/2$ is characterized by a time scale that increases with increasing μ', since stiff bodies relax slowly. However, Figure 8 shows that, as expected, the fluid limit attained for $t \to \infty$ is not dependent upon μ'.

In Figure 8b, the $k_2^H(t)$ Love number is shown for an Earth-like planet, assuming $\alpha = 1/3$, $c = 4$ and varying ζ in the range between 10^{-2} and 10^2. As expected, the choice of ζ does not affect the elastic and fluid limits, which are controlled by the μ' parameter, but it affects significantly the transition between the two regimes. Indeed, as we have discussed in Section 2, the values of ζ control the relative importance of the transient and steady-state terms, with the former becoming negligible for increasing values of ζ. The dashed curve in Figure 8b shows the $k_2^H(t)$ Heaviside Love number for a Maxwell rheology. As ζ increases, the response of the Andrade rheology approaches the Maxwell curve, with a close match for $\zeta = 10^2$. We have also verified analytically that, by virtue of the

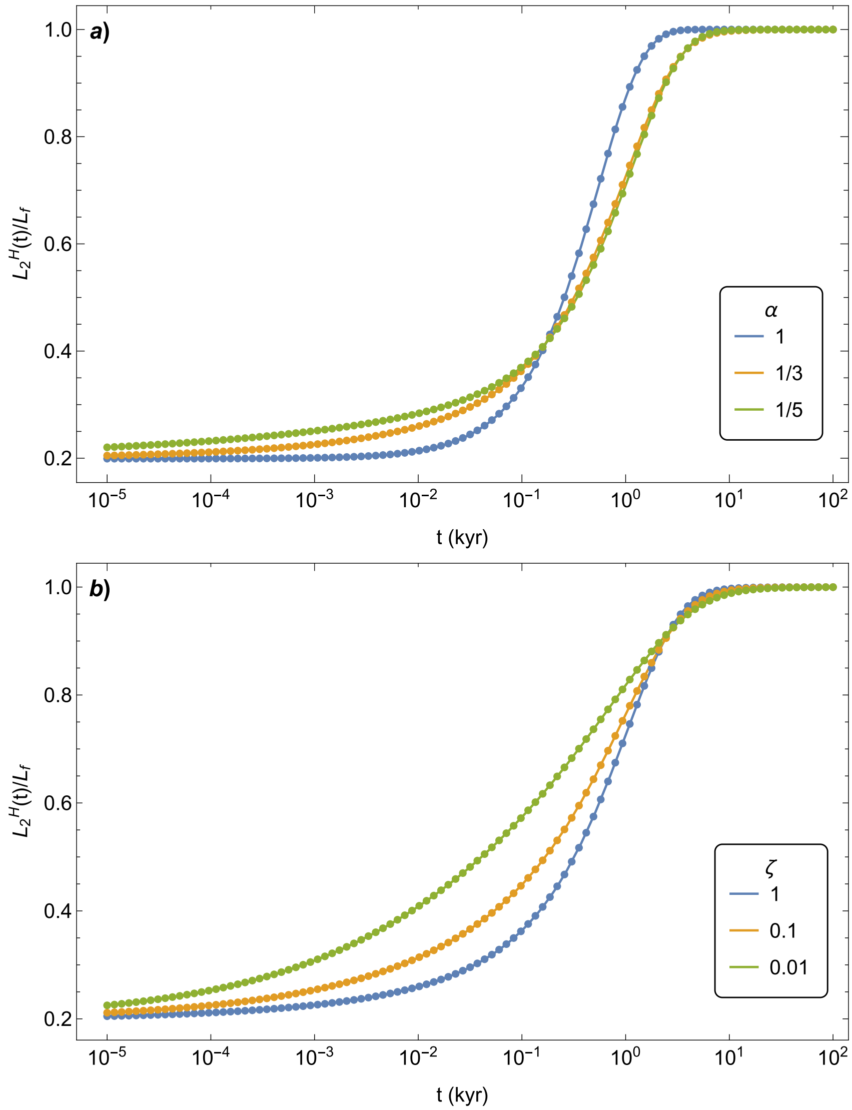

Figure 7. Comparison between Love numbers obtained analytically (solid curves) and numerically by the ALMA^3 code (dotted). A homogeneous Andrade planet is assumed, with radius $a = 6371$ km, elastic shear modulus $\mu = 1.46 \times 10^{11}$ Pa, Newtonian viscosity $\eta = 10^{21}$ Pa $\cdot$ s and density $\rho = 5514$ kg $\cdot$ m^{-3}. Curves in panels (a, b) are obtained by setting $\zeta = 1$ and $\alpha = 1/3$, respectively. All numerical experiments with ALMA^3 have been carried out in a multi-precision environment using 128 digits.

properties of the Mittag-Leffler functions, the expression (Equation 36) for $L^H(t)$ reduces to that of a Maxwell rheology for $\zeta \to \infty$, although the proof is rather cumbersome and therefore is not reported here.

4. Conclusions

Despite its simple structure, in planetary science the homogeneous Kelvin model is still having an important role. Similarly, Andrade rheology is often employed to study the tidal response of planets since its linear constitutive law provides a simple way to extend the Maxwell rheology in order to account for transient creep. Here we have studied various aspects of the response of a Kelvin sphere with Andrade rheology adopting the "$\alpha - \zeta$ parametrization," and mainly using analytical methods. Our conclusions can be summarized as follows. (a) A previously unknown expression has been found for the relaxation modulus of an Andrade 1-D rheological element, which is expressed in terms of the higher Mittag-Leffler transcendental function $E_{\alpha,\beta}(z)$. For $\alpha = 1/3$, an equivalent but computationally more efficient expression has been also established, which involves the normalized lower incomplete gamma function $P(\nu, z)$. (b) Taking advantage of the analytical solutions obtained in the 1-D case, for

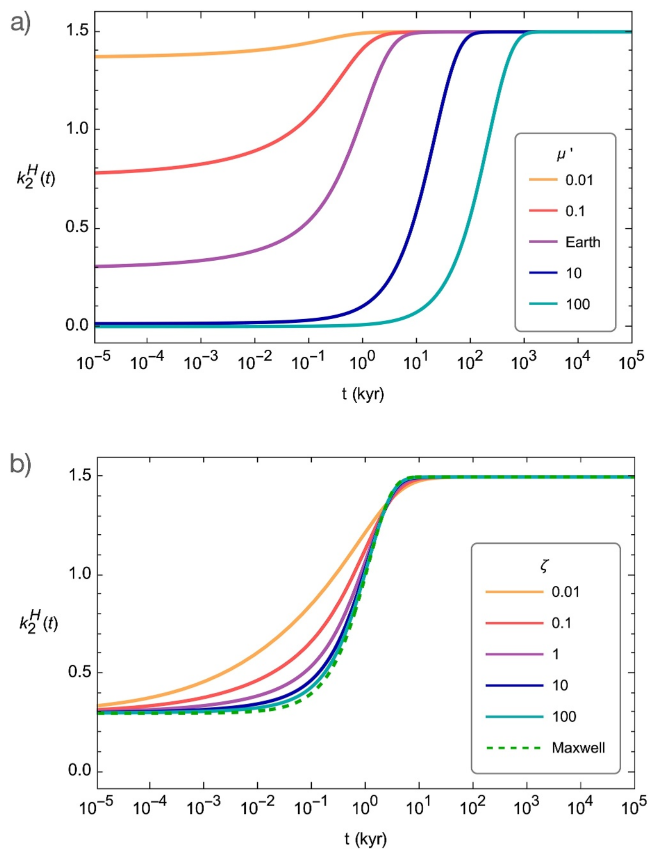

Figure 8. Tidal Heaviside Love number $k_2^H(t)$ for an uniform Earth-like planet with viscosity $\eta = 10^{21}$ Pa · s and $\alpha = 1/3$. In panel (a), $\zeta = \tau_A/\tau_M = 1$, and results for different values of the normalized shear modulus μ' are shown. In panel (b) we have used $c = 4$, corresponding to $\mu' \approx 0.4$, and curves for different values of parameter ζ are shown. The dashed curve shows the Maxwell response, attained for values of $\zeta \gg 1$.

the Kelvin sphere we have established new closed-form—but not elementary—expressions for the tidal and surface loading Love numbers in the Laplace domain, in the frequency domain and lastly in the time domain. In these three representations, we have discussed the role of the Kelvin model parameters in the assessment of the planetary response in various time and frequency regimes. The formulas we have established are useful to analyze the deformation and gravity field variations of planets with an unknown or poorly constrained structure, to obtain estimates of the response to tidal or surface forcings.

Data Availability Statement

This work is purely theoretical and makes no use of models, data, or analysis software. All plots shown in figures correspond to analytical expressions that are explicitly given in the text, along with the numerical values of parameters used to draw each specific plot. Numerical results used for the benchmark shown in Figure 7 have been obtained using the ALMA^3 Love numbers calculator available from https://github.com/danielemelini/ALMA3.

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