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DICHOTOMIES FOR LINEAR DIFFERENCE EQUATIONS AND HOMOCLINIC ORBITS OF NONINVERTIBLE MAPS

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Abstract
In this article we first show that conditions given by Hale and Lin, Steinlein and Walther, and Sander, which ensure the presence of chaotic dynamics near a homoclinic orbit of a noninvertible map, are equivalent to the exponential dichotomy of the variational equation along the homoclinic orbit. Next we study the notion of generalized exponential dichotomy, which arises from Steinlein and Walther's notion of hyperbolicity. Finally we correct a slight mistake in our article "Exponential dichotomy for noninvertible linear difference equations", which appeared in volume 27 of Journal of Difference Equations and Applications.

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1. Introduction

First, we give the definition of exponential dichotomy for a linear difference equation

$$x(k+1) = A(k)x(k), \quad k \in \mathbb{Z}, \quad x(k) \in \mathbb{R}^n, \tag{1}$$

where $\mathbb{Z}$ is the set of integers. Recall that the transition matrix $\Phi(k, m)$ of (1) is given by $\Phi(m, m) = I$, the identity matrix and $\Phi(k, m) = A(k-1)\cdots A(m)$ for $k > m$, moreover, given a linear map $L : \mathbb{R}^2 \to \mathbb{R}^m$, and $\mathcal{R}L$ and $\mathcal{N}L$ denote the range and the kernel of L, respectively.

Definition 1. *A linear difference equation* (1) *has an exponential dichotomy on an interval $J \subseteq \mathbb{Z}$ if the transition matrix $\Phi(k, m)$ of* (1) *satisfies the following conditions: there is a projection matrix function $P(k)$ of constant rank such that*

$$\Phi(k, m)P(m) = P(k)\Phi(k, m)$$

for $k \geq m$ in J and there exist constants $K \geq 1$ and $\alpha > 0$ such that for $k \geq m$ in J,

$$|\Phi(k, m)v| \leq Ke^{-\alpha(k-m)}|v|, \quad v \in \mathcal{R}P(m),$$

$$|\Phi(k, m)v| \geq K^{-1}e^{\alpha(k-m)}|v|, \quad v \in \mathcal{N}P(m).$$

The invariance of the projection implies the subspaces are invariant and the last of the above inequalities implies that $\Phi(k, m)$ is one to one on the null-space. As the dimension of $\mathcal{N}P(k)$ is constant, we conclude that $\Phi(k, m) : \mathcal{N}P(m) \to \mathcal{N}P(k)$ is an isomorphism for any $k \geq m$.

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Here is a common situation where such a system (1) arises. Let f be a C^1 map of $\mathbb{R}^n$ into itself and let the sequence x_k of points satisfy $x_{k+1} = f(x_k)$ for $k \in \mathbb{Z}$. Then, the *variational equation* along this solution is

$$y(k+1) = Df(x_k)y(k), \quad k \in \mathbb{Z}. \tag{2}$$

Note that the transition matrix for this equation is given by

$$\Phi(k,m) = Df^{k-m}(x_m), \quad k \geq m.$$

We prove this by induction on k. It clearly holds for $k = m$. Assuming $\Phi(k,m) = Df^{k-m}(x_m)$ for some $k \geq m$, it follows by the chain rule that

$$\begin{aligned}\Phi(k+1,m) &= \Phi(k+1,k)\Phi(k,m) = Df(x_k)Df^{k-m}(x_m)\\ &= Df(f^{k-m}(x_m))Df^{k-m}(x_m) = Df^{k-m+1}(x_m).\end{aligned}$$

So it holds for $k+1$ and the induction proof is complete.

We summarize the contents of the paper. Hale and Lin [5], Steinlein and Walther [12], and Sander [11] have given conditions (equivalent in finite dimensions) which ensure that the dynamics in the neighborhood of a homoclinic orbit x_k of a noninvertible map f in $\mathbb{R}^n$ are chaotic. In Section 2, we show that these conditions are equivalent to the fact that the difference equation (2) has an exponential dichotomy on $\mathbb{Z}$. As far as we know, this is not an observation that has been made previously.

In Section 3 we study the notion of *generalized exponential dichotomy*, which is suggested by the definition of hyperbolicity given by Steinlein and Walther in [12]. This is weaker than exponential dichotomy in the respect that the null-space of the projection is not invariant. We derive some basic properties of such dichotomies and find conditions under which generalized exponential dichotomy implies exponential dichotomy. The projections in such dichotomies are not unique; for intervals containing $\mathbb{Z}_+$, we find all possible projections, and for $\mathbb{Z}_-$, we give a partial result.

Finally, in Section 4, we correct a slight mistake in [1, Theorem 4.3], where it is said that the exponential dichotomy of a linear difference system (1) on $k \geq m$ can be extended to $k \geq 0$ without altering the projection on $k \geq m$ if a certain condition is satisfied. We show by example that this does not hold and give the correct condition.

2. Homoclinic orbits of non-invertible maps: A reinterpretation of results of Hale and Lin, Steinlein and Walther, and Sander

Let f be a C^1 map of $\mathbb{R}^n$ into itself. A fixed point q of f is said to be *hyperbolic* if the eigenvalues of $Df(q)$ lie off the unit circle. This includes the possibility that one of the eigenvalues is 0. A sequence $\{x_k\}_{k=-\infty}^{\infty}$ of points in $\mathbb{R}^n$ is said to be a *homoclinic* orbit to q if $x_{k+1} = f(x_k)$ for all $k \in \mathbb{Z}$ and $x_k \to q$ as $k \to \pm\infty$. This includes the possibility of what we may call a *snapback homoclinic orbit*, that is, that $x_k = q$ for large $k > 0$. In the case that q is a repelling fixed point, that is, all the eigenvalues of $Df(q)$ lie outside the unit circle, such an orbit is known as a *snapback repeller*.

In the case when f is a diffeomorphism, $\{x_k\}_{k=-\infty}^{\infty}$ is a *homoclinic* orbit to q if, and only if, x_0 lies in the neighborhood of the *stable* manifold W^s and *unstable* manifold W^u of q. Furthermore, if this intersection is transversal, f exhibits chaotic dynamics in the neighbourhood of the homoclinic orbit. Also, it was shown in

[8] (see also [3]) that *a homoclinic orbit is transversal if, and only if,* (2) *has an exponential dichotomy on* $\mathbb{Z}$.

Now, in the case when f is not invertible, it is still true that $\{x_k\}_{k=-\infty}^{\infty}$ is a *homoclinic* orbit to q if, and only if, x_0 lies in the intersection of the stable and unstable manifolds of q. However, as shown by examples in Sander [11], even when this intersection is transversal, there may not be chaotic dynamics in the neighborbood of the homoclinic orbit.

Question: What condition must a homoclinic orbit of a non-invertible map satisfy in order that there is chaotic dynamics in its neighborhood? This question has been answered by Hale and Lin [5], Steinlein and Walther [12] and Sander [11], as we detail below.

2.1. **Hale and Lin.** This problem was first treated by Hale and Lin [5] in the context of nonlinear semi-flows. In terms of the notation used in this paper, their condition was (see page 232 in [5]) *that for sufficiently large negative* m *and large positive* k *such that* $x_m \in W^u_{loc}$ *and* $x_k \in W^s_{loc}$, f^{k-m} *sends a disc in* W^u_{loc} *containing* x_m *diffeomorphically onto its image which is transverse to* W^s_{loc} *at* x_k (here, $W^s_{loc} = W^s(q, U)$ is a *local stable manifold of* q where U is a small neighborhood of q (see p.231 in [5]). Similarly, W^u_{loc} is a local unstable manifold). Now, $Df^{k-m}(x_m)$ sends the tangent space $T_{x_m}W^u_{loc}$ at x_m to the subspace $Df^{k-m}(x_m)(T_{x_m}W^u_{loc})$. So, Hale and Lin's condition is equivalent to saying that $Df^{k-m}(x_m)$ *is one to one on* $T_{x_m}W^u_{loc}$ *and* $Df^{k-m}(x_m)(T_{x_m}W^u_{loc})$ *is transverse to* $T_{x_k}W^s_{loc}$.

2.2. **Steinlein and Walther.** In [12], Steinlein and Walther consider a map f defined on a Banach space L. Their condition (again changing the notation slightly) is (see page 339 in [12]) *that there exists a natural number* n_0 *such that for all integers* $m < -n_0$ *and* $k > n_0$, $x_m \in W^u_{loc}$ *and* $x_k \in W^s_{loc}$, $Df^{n-m}(x_m)(T_{x_m}W^u_{loc})$ *is closed with* $L = Df^{n-m}(x_m)(T_{x_m}W^u_{loc}) \oplus T_{x_k}W^s_{loc}$ *and* $Df^{n-m}(x_m)$ *is injective on* $T_{x_m}W^u_{loc}$. Note when L is finite-dimensional, the condition $Df^{n-m}(x_m)(T_{x_m}W^u_{loc})$ is closed and is not needed.

2.3. **Sander.** In [11], the condition (see Theorem 4.1 in [11]) given by Sander (again changing the notation slightly) is *that there exists a sufficiently large* N *such that for all* $m < -N$ *and* $k > N$, $x_m \in W^u_{loc}$ *and* $x_k \in W^s_{loc}$, $Df^{k-m}_{x_m}$ *is injective on* $T_{x_m}W^u_{loc}$ *and maps* $T_{x_m}W^u_{loc}$ *to a subspace transversal to* $T_{x_k}W^s_{loc}$.

These three conditions are essentially the same, and under this condition, the three authors mentioned above are able to prove the presence of chaotic dynamics in the neighborhood of the homoclinic orbit.

Let us interpret this condition (we use Sander's version) in terms of the properties of the linear difference system (2). As we saw in the Introduction, the transition matrix for this equation is given by

$$\Phi(k, m) = Df^{k-m}(x_m), \quad k \geq m.$$

Now, the equation

$$y(k+1) = Df(q)y(k), \quad k \in \mathbb{Z}$$

has an exponential dichotomy on $\mathbb{Z}$ with projection of rank r (say). Since $Df(x_k) \to Df(q)$ as $k \to \pm\infty$, it follows from the roughness theorem (see, for example, Theorems 5.2 and 5.4 in [1]) that if N is sufficiently large, (2) has dichotomies on $k \geq N$ and $k \leq -N$ with projections $P(k)$ and $Q(k)$, respectively, both projections with rank r. Now, it follows from arguments as in the diffeomorphism case (see Proposition 3.3 in [9]) that for $m < -N$, we have $T_{x_m}W^u_{loc} = \mathcal{N}Q(m)$ and for $k > N$, we have $T_{x_k}W^s_{loc} = \mathcal{R}P(k)$. Then, Sander's condition says exactly that $\Phi(k,m)$ *is injective on* $\mathcal{N}Q(m)$ *and maps* $\mathcal{N}Q(m)$ *onto a subspace transversal to* $\mathcal{R}P(k)$. In the theorem below, we show that Sander's condition is equivalent to the exponential dichotomy of (2) on $\mathbb{Z}$.

Theorem 1. *Suppose* (1) *has exponential dichotomies on* $[N_1,\infty)$ *and* $(-\infty,-N_2]$, *where* $N_1 \geq 0$, $N_2 \geq 0$, *with respective projections* $P(k)$ *and* $Q(k)$ *of the same rank* r. *Then, the following four conditions are equivalent:*

(i) (1) *has no bounded solution* $x(k)$ *such that* $x(k) \neq 0$ *for at least one* k;

(ii) *for all* $m < -N_2$ *and* $\ell > N_1$, $\Phi(\ell,m)$ *is injective on* $\mathcal{N}Q(m)$ *and maps* $\mathcal{N}Q(m)$ *onto a subspace transversal to* $\mathcal{R}P(\ell)$;

(iii) *there exist* $m < -N_2$ *and* $\ell > N_1$ *such that* $\Phi(\ell,m)$ *is injective on* $\mathcal{N}Q(m)$ *and maps* $\mathcal{N}Q(m)$ *onto a subspace transversal to* $\mathcal{R}P(\ell)$;

(iv) (1) *has an exponential dichotomy on* $\mathbb{Z}$.

Proof. To start we show (i) implies (ii). Note if $x(k)$ is a bounded solution of (1), then $x(m) \in \mathcal{N}Q(m)$ and $x(\ell) = \Phi(\ell,m)x(m) \in \mathcal{R}P(\ell)$ for any $m < -N_2$ and $\ell > N_1$. This solution is not identically zero if, and only if, $x(m) \neq 0$. Hence, (i) is equivalent to saying that if there exists $\xi \in \mathcal{N}Q(m)$ such that $\Phi(\ell,m)\xi \in \mathcal{R}P(\ell)$, then $\xi = 0$. This, in turn, is equivalent to saying that $\Phi(\ell,m)$ maps $\mathcal{N}Q(m)$ one to one onto a subspace transversal to $\mathcal{R}P(\ell)$, which is (ii).

It is clear that (ii) implies (iii).

Next we show (iii) implies (iv). To begin, since $\Phi(\ell,m)$ is injective on $\mathcal{N}Q(m)$, it follows from Theorem 4.4 (i) in [1] that the exponential dichotomy on $k \leq -N_2$ can be extended to $k \leq \ell$ with a possibly new projection $\tilde{Q}(k)$. However, $\mathcal{N}\tilde{Q}(m) = \mathcal{N}Q(m)$ must hold. Then, $\mathcal{N}\tilde{Q}(\ell) = \Phi(\ell,m)(\mathcal{N}\tilde{Q}(m)) = \Phi(\ell,m)(\mathcal{N}Q(m))$ is transversal to $\mathcal{R}P(\ell)$. The conclusion that (1) has an exponential dichotomy on $\mathbb{Z}$ now follows from [1, Corollary 3.3].

Finally, it follows from [1, Corollary 3.3] again that (iv) implies (i). □

It follows from this theorem that, in finite dimensions, *Sander's condition (and, hence, the other two) is equivalent to the condition that the variational equation* (2) *has an exponential dichotomy on* $\mathbb{Z}$. This is consistent with the result for diffeomorphisms.

Note that in Definition 4.2.2 in [6], Kalkbrenner defines a homoclinic orbit x_k to a hyperbolic fixed point q to be transversal if (2) has an exponential dichotomy on $\mathbb{Z}$. However, he adds the additional condition that the projection $P(k)$ associated with the dichotomy has the property that for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|P(k) - Q| < \varepsilon$ when $|x_k - q| < \delta$, where Q is the projection associated with the hyperbolicity of q.

3. A more general definition of exponential dichotomy

In Definition 1.1 in [12], Steinlein and Walther consider a C^1 map $f : U \to L$, where U is an open set in a Banach space L, and say that *a forward invariant set* $I \subseteq U$ *is hyperbolic (the notation has been changed slightly) if there are constants* $K \geq 1$ *and* $\alpha > 0$ *and projections* $P(x)$, *varying continuously with respect to* $x \in I$, *such that*

$$Df(x)(\mathcal{R}P(x)) \subseteq \mathcal{R}P(f(x)), \quad \mathcal{R}P(f(x)) + Df(x)(\mathcal{N}P(x)) = L, \tag{3}$$

and for all $n \geq 1$*,*

$$\begin{aligned} &|Df^n(x)v| \leq Ke^{-\alpha n}|v|, \quad v \in \mathcal{R}P(x), \\ &|(I - P(f^n(x)))Df^n(x)v| \geq K^{-1}e^{\alpha n}|v|, \quad v \in \mathcal{N}P(x). \end{aligned} \tag{4}$$

In the case of a homoclinic orbit, Steinlein and Walther show that under their condition (as described in the previous section), the invariant set consisting of the homoclinic orbit together with the fixed point form a hyperbolic set according to the definition above. For such hyperbolic sets, they show that shadowing holds and hence deduce the presence of chaos in the set.

In Steinlein and Walther's definition, if we consider $x \in I$ and define $x_k = f^k(x)$ for $f \geq 0$, then the transition matrix of (2) is given by $\Phi(k,m) = Df^{k-m}(x_m)$. We also define $A(k) = Df(x_k)$ and $P(k) = P(x_k)$. Then, taking $x = x_k$ in (3), we see that

$$A(k)(\mathcal{R}P(k)) \subseteq \mathcal{R}P(k+1), \quad \mathcal{R}P(k+1) + A(k)(\mathcal{N}P(k)) = L.$$

Taking $n = k - m$ and $x = x_m$, the first inequality in (4) implies that

$$|\Phi(k,m)v| \leq Ke^{-\alpha(k-m)}|v|, \quad v \in \mathcal{R}P(m), \ k \geq m.$$

Similarly, the second inequality in (4) implies that

$$|(I - P(k))\Phi(k,m)v| \geq K^{-1}e^{\alpha(k-m)}|v|, \quad v \in \mathcal{N}P(m), \ k \geq m.$$

So, Steinlein and Walther's definition implies the following more general definition of exponential dichotomy.

Definition 2. *A linear difference equation* (1) *has a generalized exponential dichotomy on an interval* $J \subseteq \mathbb{Z}$ *if the transition matrix* $\Phi(k,m)$ *of* (1) *satisfies the following conditions: there is a bounded projection function* $P(k)$ *of constant rank* r *such that*

$$A(k)(\mathcal{R}P(k)) \subseteq \mathcal{R}P(k+1), \quad \mathcal{R}P(k+1) + A(k)(\mathcal{N}P(k)) = \mathbb{R}^n,$$

for $k, k+1 \in J$ *and there exist constants* $K \geq 1$ *and* $\alpha > 0$ *such that for* $k \geq m$ *in* J*,*

$$|\Phi(k,m)x| \leq Ke^{-\alpha(k-m)}|x|, \quad x \in \mathcal{R}P(m) \tag{5}$$

and

$$|(I - P(k))\Phi(k,m)x| \geq K^{-1}e^{\alpha(k-m)}|x|, \quad x \in \mathcal{N}P(m). \tag{6}$$

This definition is more general than Definition 1 because it is not assumed that $\mathcal{N}P(k)$ is invariant under $A(k)$.

Remark 1. *Note we do not need to assume* $P(k)$ *has constant rank. To see this, let* $P(k)$ *have rank* r*.* (6) *implies that* $(I - P(k+1))A(k)x \neq 0$ *if* $x \in \mathcal{N}P(k)$*,* $x \neq 0$*. This implies* $A(k)$ *is one to one on* $\mathcal{N}P(k)$ *and so the dimension of* $A(k)(\mathcal{N}P(k))$ *is* $n - r$*. Let* $x \in \mathcal{R}P(k+1) \cap A(k)(\mathcal{N}P(k))$*. Then,* $x = A(k)y$*, where* $y \in \mathcal{N}P(k)$ *and* $(I - P(k+1))A(k)y = 0$*. As we just saw, this implies* $y = 0$ *and hence* $x = 0$*. Therefore,* $\mathcal{R}P(k+1) \oplus A(k)(\mathcal{N}P(k)) = \mathbb{R}^n$*, and so the dimension of* $\mathcal{R}P(k+1)$ *is* r*.*

3.1. **Basic properties of generalized exponential dichotomies.** Let us denote by $W_+(m)$ and $W_-(m)$ the subspaces of initial conditions at $k=m$ for which there exists a solution of (1) bounded in the future and in the past, respectively, namely,

$$\begin{aligned} &\xi \in W_+(m) \text{ if, and only if, } \sup_{k\geq m} |\Phi(k,m)\xi| < \infty,\\ &\xi \in W_-(m) \text{ if, and only if, there exists a solution } y(k) \text{ of (1)}\\ &\qquad\text{such that } y(m)=\xi \text{ and } \sup_{k\leq m}|y(k)|<\infty. \end{aligned} \tag{7}$$

Notice that for a fixed $\xi \in W_-(m)$, a priori $y(k)$ might not be uniquely defined.

Proposition 1. *Assume that* (1) *admits a generalized exponential dichotomy in* J *with projection* $P(k)$*. Then,*

(i) $(I-P(k))\Phi(k,m)$ *is one to one on* $\mathcal{N}P(m)$*, for* $k \geq m$*.*
(ii) $\mathcal{R}P(m) = \Phi(k,m)^{-1}(\mathcal{R}P(k))$ *for* $k\geq m$*.*
(iii) $\mathcal{N}(\Phi(k,m)) \subseteq \mathcal{R}P(m)$ *for* $k \geq m$*.*
(iv) *if* J *contains* $\mathbb{Z}_+$*, for all* $m\in J$ *we have* $\mathcal{R}P(m)=W_+(m)$*, so that* $\mathcal{R}P(m)$ *is uniquely defined.*
(v) *If* J *contains* $\mathbb{Z}_-$*,* $\mathcal{R}P(m)\cap W_-(m)=\{0\}$ *for all* $m\in J$*.*
(vi) *If* (1) *has a generalized exponential dichotomy with projection* $P(k)$ *and we make the transformation* $x=T(k)y$*, where* $|T(k)|$ *and* $|T^{-1}(k)|$ *are bounded by* N*, then the transformed equation*

$$y(k+1) = B(k)y = T^{-1}(k+1)A(k)T(k)y$$

has a generalized exponential dichotomy with projection $Q(k)=T^{-1}(k)P(k)T(k)$ *and the same exponent.*

Proof. (i) This follows immediately from $|(I-P(k))\Phi(k,m)x| \geq K^{-1}e^{\alpha(k-m)}|x|$ if $k\geq m$ and $x\in\mathcal{N}P(m)$.

(ii) To begin, we prove it for $k=m+1$. From the definition of generalized exponential dichotomy, we see that $A(m)(\mathcal{R}P(m))\subseteq\mathcal{R}P(m+1)$ when $m, m+1\in J$. Hence, $\mathcal{R}P(m)\subseteq A(m)^{-1}(\mathcal{R}P(m+1))$. Suppose $x\in A(m)^{-1}(\mathcal{R}P(m+1))$ and write $x=u+v$, where $u\in\mathcal{R}P(m)$, $v\in\mathcal{N}P(m)$. Then,

$$A(m)v = A(m)x - A(m)u \in \mathcal{R}P(m+1)$$

and hence $(I-P(m+1))A(m)v=0$. However then $v=0$ by (i) with $k=m+1$. So, $x=u\in\mathcal{R}P(m)$. Hence, $A(m)^{-1}(\mathcal{R}P(m+1))\subseteq\mathcal{R}P(m)$, and so they are equal. This proves (ii) for $k=m+1$. Iterating the argument using

$$\Phi(k+1,m)^{-1}(\mathcal{R}P(k+1)) = \Phi(k,m)^{-1}(A(k)^{-1}(\mathcal{R}P(k+1))),$$

we find that $\Phi(k,m)^{-1}(\mathcal{R}P(k))=\mathcal{R}P(m)$ when $k>m$, and so (ii) is proved.

(iii) Suppose $\Phi(k,m)x=0$. Then, $x\in\Phi(k,m)^{-1}(\mathcal{R}P(k))=\mathcal{R}P(m)$.

(iv) The proof that $\mathcal{R}P(m)\subseteq W_+(m)$ goes as in [1, Lemma 3.1]. To prove that $W_+(m)\subseteq\mathcal{R}P(m)$, suppose that $\xi\in W_+(m)$ but that $(I-P(m))\xi\neq 0$. Then, since $\Phi(k,m)P(m)\xi\in\mathcal{R}P(k)$, so that $(I-P(k))\Phi(k,m)P(m)\xi=0$, we have

$$\begin{aligned} |(I-P(k))\Phi(k,m)\xi| &= |(I-P(k))\Phi(k,m)(I-P(m))\xi|\\ &\geq L^{-1}e^{\alpha(k-m)}|(I-P(m))\xi| \to \infty \end{aligned}$$

as $k\to\infty$, contradicting $\sup_{k\geq m}|\Phi(k,m)\xi|<\infty$.

(v) When $J=\mathbb{Z}$ or $\mathbb{Z}_-$, we prove that $\mathcal{R}P(m)\cap W_-(m)=\{0\}$. Let $\xi\in\mathcal{R}P(m)\cap W_-(m)$. Then, there exists a solution $y(k)$ of (1) with $\|y\|_\infty = \sup_{k\leq m}|y(k)|<\infty$

such that $y(m) = \xi$. Since by (ii), $\Phi(m,k)^{-1}(\mathcal{R}P(m)) = \mathcal{R}P(k)$, it follows that $y(k) \in \mathcal{R}P(k) \cap W_{-}(k)$ for $k \leq m$. Then, for $k \leq m$,

$$|\xi| = |y(m)| \leq Ke^{-\alpha(m-k)}|y(k)| \leq Ke^{-\alpha(m-k)}\|y\|_{\infty} \to 0 \quad \text{as} \quad k \to -\infty.$$

So, $\xi = 0$.

(vi) The proof is straightforward using the fact that $\Phi(k,m) = T(k)\Psi(k,m)T^{-1}(m)$, where $\Psi(k,m)$ is the transition matrix for $y(k+1) = B(k)y$. □

Next we note the following lemma, which has a similar proof to that of Lemma 5.3 in [1].

Lemma 1. *Let $J \subseteq \mathbb{Z}$ be an interval in $\mathbb{Z}$ and assume* (1) *has a generalized exponential dichotomy on J with constants K and α and projection $P(k)$. Define*

$$\tilde{A}(k) = \begin{cases} \mathcal{A} = e^{-\alpha}P(a) + e^{\alpha}(I - P(a)) & k < a = \inf J, \\ A(k) & k \in J \\ \mathcal{B} = e^{-\alpha}P(b) + e^{\alpha}(I - P(b)) & k \geq b = \sup J. \end{cases}$$

Then, $x(k+1) = \tilde{A}(k)x(k)$ has a generalized exponential dichotomy on $\mathbb{Z}$ with the same constant K, exponent α and projection

$$\tilde{P}(k) = \begin{cases} P(a) & \text{if } k \leq a \\ P(k) & \text{if } a \leq k \leq b \\ P(b) & \text{if } k \geq b. \end{cases}$$

3.2. **When generalized exponential dichotomy implies exponential dichotomy.** In this subsection, we derive conditions under which generalized exponential dichotomy implies exponential dichotomy. We begin with a lemma.

Lemma 2. *The block triangular system*

$$x(k+1) = A(k)x(k) = \begin{pmatrix} A_{11}(k) & A_{12}(k) \\ 0 & A_{22}(k) \end{pmatrix} x(k), \tag{8}$$

where $A_{11}(k)$ is $r \times r$, has a generalized exponential dichotomy on an interval J with projection $P_r = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ and exponent α if, and only if, the corresponding block diagonal system has an exponential dichotomy on J with projection P_r and the same exponent α.

Proof. First suppose (8) has a generalized exponential dichotomy on an interval J with projection P_r. Then, there exist positive constants K and α such that if $\Phi(k,m)$ is the transition matrix for (8), for $k \geq m$,

$$|\Phi(k,m)x| \leq Ke^{-\alpha(k-m)}|x|, \quad x \in \mathcal{R}P_r$$

and

$$|(I - P_r)\Phi(m,k)x| \geq K^{-1}e^{\alpha(k-m)}|x|, \quad x \in \mathcal{N}P_r.$$

Note that

$$\Phi(k,m) = \begin{pmatrix} \Phi_{11}(k,m) & \Phi_{12}(k,m) \\ 0 & \Phi_{22}(k,m) \end{pmatrix},$$

where $\Phi_{ii}(k,m)$ is the transition matrix corresponding to $A_{ii}(k)$. The above inequalities mean that for $k \geq m$,

$$|\Phi_{11}(k,m)x_1| \leq Ke^{-\alpha(k-m)}|x_1|, \quad x_1 \in \mathbb{R}^r$$

and
$$|\Phi_{22}(k,m)x_2| \geq K^{-1}e^{\alpha(k-m)}|x_2|, \quad x_2 \in \mathbb{R}^{n-r}.$$
This means the diagonal system corresponding to (8) has an exponential dichotomy with projection P_r and the same exponent α. The converse is proved by reversing the argument. □

Proposition 2. *Generalized exponential dichotomy implies exponential dichotomy if $|A(k)|$ is bounded.*

Proof. To begin $P(k)$ is bounded and has rank r, there exists an invertible matrix function $T(k)$ with $|T(k)|, |T^{-1}(k)| \leq N$ such that $P(k) = T(k)P_rT^{-1}(k)$, where $P_r = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ (see Lemma 1 on page 39 in [4]). Then the transformation $x = T(k)y$ takes (1) into
$$y(k+1) = T^{-1}(k+1)A(k)T(k)y(k) = B(k)y(k). \tag{9}$$
By Proposition 1 (vi) with $Q(k) = P_r$, it follows that (9) has a generalized exponential dichotomy with projection $Q(k) = P_r$. Next, it follows from $B(k)(\mathcal{R}P_r) \subseteq \mathcal{R}P_r$ that $(I - P_r)B(k)P_r = 0$, and so
$$B(k) = \begin{pmatrix} B_{11}(k) & B_{12}(k) \\ 0 & B_{22}(k) \end{pmatrix},$$
where $B_{11}(k)$ is $r \times r$, etc.. Then, it follows from Lemma 2 that the diagonal system corresponding to (9) has an exponential dichotomy with projection P_r. So by [2, Theorem 3.1], since $B_{12}(k)$ is bounded because $A(k)$ is, (9) has an exponential dichotomy and, hence, also the original system (1). □

Examples. Here are examples to show that boundedness is necessary in all cases $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{Z}_-$. In these examples we, use the system
$$x(k+1) = x(k)/2 + \gamma^k y(k), \quad y(k+1) = 2y(k), \tag{10}$$
where $\gamma > 0$. Note that, by Lemma 2, (10) has a generalized exponential dichotomy with $P(k) = P_1$ since the diagonal system has an exponential dichotomy with $P(k) = P_1$. The solution of (10) starting from $(x(0), y(0))$ is
(11)
$$\begin{pmatrix} x(k) \\ y(k) \end{pmatrix} = \Phi(k,0)\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{cases} \begin{pmatrix} (x(0) - c_0y(0))2^{-k} + c_0y(0)(2\gamma)^k \\ y(0)2^k \end{pmatrix} & \gamma \neq 1/4 \\ \begin{pmatrix} x(0)2^{-k} + ky(0)2^{-k+1} \\ y(0)2^k \end{pmatrix} & \gamma = 1/4, \end{cases}$$
where $c_0 = 2/(4\gamma - 1)$. For system (10), we see that $W_+(m) = \mathrm{span}(e_1)$ for all m and that
$$W_-(m) = \begin{cases} \{c_0\gamma^m\eta, \eta) : \eta \in \mathbb{R}\} & (\gamma \geq 1/2) \\ \{0\} & 0 < \gamma < 1/2. \end{cases} \tag{12}$$
Note that this shows that if a system has a generalized exponential dichotomy, it does not follow that dim $W_-(k) = n - r$, where r is the rank of the projection, as in the case of exponential dichotomy.

Example for $\mathbb{Z}_+$: Consider (10). If there is an exponential dichotomy with projection $Q(k)$ (say), then by Proposition 1 (iv), the range of $Q(k)$ must be $W_+(k)$

and so is spanned by e_1 and, therefore, $Q(k) = \begin{pmatrix} 1 & b(k) \\ 0 & 0 \end{pmatrix}$. Assume $\gamma > 1$. We can take the null-space of $Q(0)$ as the span of $(c_0, 1)$. Then, $\mathcal{N}Q(k) = \Phi(k,0)(\mathcal{N}Q(0))$ is spanned by $z(k) = (c_0\gamma^k, 1)$. Since $Q(k)z(k) = 0$, it follows that $b(k) = -c_0\gamma^k$. Hence, $Q(k)$ is unbounded, so there is no exponential dichotomy when $\gamma > 1$.

Remark 2. *The example above can be used for $\mathbb{Z}$ also since exponential dichotomy on $\mathbb{Z}$ implies exponential dichotomy on $\mathbb{Z}_+$.*

Example for $\mathbb{Z}_-$: As noted above, (10) has a generalized exponential dichotomy with $P(k) = P_1$. A solution $x(k)$ is bounded in $k \le m$ if, and only if, $x(m) \in W_-(m)$ so that when $\gamma \ge 1/2$, $x(m) = (c_0\gamma^m y(m), y(m))$, where we have used (12). So if there is an exponential dichotomy with projection $Q(k)$ (say), the null-space of $Q(k)$ must be the span of $z(k) = (c_0\gamma^k, 1)$. Then, we may take the range of $Q(0)$ as the span of e_1. Then, by Proposition 1 (ii), $\mathcal{R}Q(k) = \Phi(k,0)^{-1}(\mathcal{R}Q(0))$, which is the span of e_1. Thus, $Q(k) = \begin{pmatrix} 1 & b(k) \\ 0 & 0 \end{pmatrix}$ and $Q(k)z(k) = 0$, so that $b(k) = -c_0\gamma^k$. If $\gamma < 1$, we see that $Q(k)$ is unbounded. So there is no exponential dichotomy if $1/2 \le \gamma < 1$. Now, when $0 < \gamma < 1/2$, it follows from (12) that $W_-(m) = \{0\}$. If (10) has an exponential dichotomy with projection $Q(m)$, it also admits a generalized exponential dichotomy with the same projection. Then, Proposition 3.12 below implies that $\mathcal{N}Q(m)$, which must be $W_-(m)$, has dimension 1. This is a contradiction. So there is no exponential dichotomy if $0 < \gamma < 1$.

In the next proposition, we show that generalized exponential dichotomy implies exponential dichotomy if, and only if, there is a bounded invariant projection with the same range as the original projection.

Proposition 3. *Assume* (1) *admits a generalized exponential dichotomy in J with projections $P(k)$ of rank r. Then,* (1) *admits an exponential dichotomy in J if, and only if, there is a bounded invariant projection $Q(k)$ with $\mathcal{R}Q(k) = \mathcal{R}P(k)$ for all k.*

Proof. To start we prove the sufficiency. As we saw in the proof of Proposition 2, there is a bounded invertible transformation $x = T(k)y$ with $T^{-1}(k)$ also bounded such that

$$y(k+1) = T^{-1}(k+1)A(k)T(k)y(k) = B(k)y(k) = \begin{pmatrix} A_{11}(k) & A_{12}(k) \\ 0 & A_{22}(k) \end{pmatrix} y(k) \tag{13}$$

has a generalized exponential dichotomy with projection $T^{-1}(k)P(k)T(k) = P_r = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$. Then, by Lemma 2, the diagonal system

$$y(k+1) = \begin{pmatrix} A_{11}(k) & 0 \\ 0 & A_{22}(k) \end{pmatrix} y(k) \tag{14}$$

has an exponential dichotomy with projection $P_r = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$.

Now suppose (1) has a bounded invariant projection $Q(k)$ with the same range as $P(k)$. Thus, $A(k)Q(k) = Q(k+1)A(k)$, which implies that $B(k)\tilde{Q}(k) = \tilde{Q}(k+1)B(k)$, where $\tilde{Q}(k) = T^{-1}(k)Q(k)T(k)$ has the same range as P_r. This means

$$\tilde{Q}(k) = \begin{pmatrix} I_r & q(k) \\ 0 & 0 \end{pmatrix},$$

where $q(k)$ is bounded and, by invariance, satisfies

$$q(k+1)A_{22}(k) - A_{11}(k)q(k) + A_{12}(k) = 0.$$

Define

$$S(k) = \begin{pmatrix} I_r & q(k) \\ 0 & I_{n-r} \end{pmatrix}.$$

Then, we see that

$$S^{-1}(k+1)\begin{pmatrix} A_{11}(k) & 0 \\ 0 & A_{22}(k) \end{pmatrix} S(k) = \begin{pmatrix} A_{11}(k) & A_{12}(k) \\ 0 & A_{22}(k) \end{pmatrix}.$$

It follows by Proposition 1 (vi) that (13) has an exponential dichotomy with projection $S^{-1}(k)P_r S(k) = \tilde{Q}(k)$. In turn, it follows that (1) has an exponential dichotomy with projection $T(k)\tilde{Q}(k)T^{-1}(k) = Q(k)$.

Now we prove the necessity. Suppose (1) has a generalized exponential dichotomy on J with projection $P(k)$ and also an exponential dichotomy on J with projection $Q(k)$. When $J = \mathbb{Z}_+$, from Proposition 1 (iv), it follows that $\mathcal{R}P(k) = W_+(k) = \mathcal{R}Q(k)$, for any $k \in \mathbb{Z}_+$, proving the result on $\mathbb{Z}_+$. Now suppose $J = \mathbb{Z}$. We know that in this case $Q(k)$ is uniquely defined and that, because of the previous part, $\mathcal{R}Q(k) = \mathcal{R}P(k)$ for all $k \geq 0$. Now, suppose $k < 0$. From Proposition 1 (ii) and using the similar property for exponential dichotomy, we then have

$$\mathcal{R}P(k) = \Phi(0,k)^{-1}(\mathcal{R}P(0)) = \Phi(0,k)^{-1}(\mathcal{R}Q(0)) = \mathcal{R}Q(k)$$

and, hence, $\mathcal{R}P(k) = \mathcal{R}Q(k)$ for all $k \in \mathbb{Z}$. Finally, suppose $J = \mathbb{Z}_-$. Then, the rank of $Q(k)$ equals the rank r of $P(k)$. Also, $\mathcal{N}Q(k) = W_-(k)$ for all $k \leq 0$. It follows from Proposition 1 (v) that $\mathcal{R}P(k) \oplus W_-(k) = \mathbb{R}^n$. So we can take the range of $Q(k)$ to be the range of $P(k)$. Then, $Q(k)$ is a bounded invariant projection with $\mathcal{R}Q(k) = \mathcal{R}P(k)$. □

Remark 3. *It follows from this proposition that if a system on $\mathbb{Z}_-$ has a generalized exponential dichotomy with projection $P(k)$ such that $\mathcal{N}P(k) = W_-(k)$ for $k \leq 0$, then the system has an exponential dichotomy with this projection.*

3.3. **Changing the projection in a generalized exponential dichotomy.** In this subsection, for systems on $\mathbb{Z}$ and $\mathbb{Z}_+$ with a generalized exponential dichotomy, we characterize all possible projections. For systems on $\mathbb{Z}_-$, we can only give partial results in this direction.

Proposition 4. *Suppose system* (1) *has a generalized exponential dichotomy on J with projection $P(k)$ of rank r and exponent α. Then, if $Q(k)$ is a family of bounded projections such that $\mathcal{R}Q(k) = \mathcal{R}P(k)$,* (1) *also has a generalized exponential dichotomy with projection $Q(k)$ and the same exponent α.*

Proof. To begin we show $A(k)$ is one to one on $\mathcal{N}Q(k)$. Let $A(k)x = 0$ where $x \in \mathcal{N}Q(k)$. $x \in \mathcal{N}(A(k))$, and so by Proposition 1 (iii), $x \in \mathcal{R}P(k) = \mathcal{R}Q(k)$. Hence, $x \in \mathcal{R}Q(k) \cap \mathcal{N}Q(k)$ and so $x = 0$. This implies that dim $A(k)(\mathcal{N}Q(k)) = n - r$. Next, let

$$x \in \mathcal{R}Q(k+1) \cap A(k)(\mathcal{N}Q(k)) = \mathcal{R}P(k+1) \cap A(k)(\mathcal{N}Q(k)).$$

Then, $x = A(k)y$, where $y \in \mathcal{N}Q(k)$ and $(I - P(k+1))A(k)y = 0$. So,

$$\begin{aligned} 0 &= (I - P(k+1))A(k)P(k)y + (I - P(k+1))A(k)(I - P(k))y \\ &= (I - P(k+1))A(k)(I - P(k))y. \end{aligned}$$

Hence, by Proposition 1 (i), $(I-P(k))y = 0$ so that $y \in \mathcal{R}P(k) = \mathcal{R}Q(k)$. Thus, $y = 0$ and hence $x = 0$. Therefore, $\mathcal{R}Q(k+1) \cap A(k)(\mathcal{N}Q(k)) = \{0\}$ and we conclude

$$\mathcal{R}Q(k+1) \oplus A(k)(\mathcal{N}Q(k)) = \mathbb{R}^n.$$

Let the constant in the generalized exponential dichotomy of (1) be K. Then, the fact that

$$|\Phi(k,m)x| \le Ke^{-\alpha(k-m)}|x|$$

for any $x \in \mathcal{R}Q(m)$ is clear, since $\mathcal{R}Q(m) = \mathcal{R}P(m)$.

Next we make a remark here that

$$|(I-P(k))x| \ge |I-Q(k)|^{-1}|(I-Q(k))x| \ge N^{-1}|(I-Q(k))x|,$$

where $|I-Q(k)| \le N$. This follows from

$$|(I-Q(k))x| = |(I-Q(k))(I-P(k))x| \le |I-Q(k)|\,|(I-P(k)x|.$$

where we have used $Q(k)P(k) = P(k)$. Similarly, we also have

$$|(I-Q(k))x| \ge |I-P(k)|^{-1}|(I-P(k))x| \ge N^{-1}|(I-P(k))x|,$$

if we also assume $|I-P(k)| \le N$. Then, if $x \in \mathcal{N}Q(m)$ and $k \ge m$,

$$\begin{aligned}
|(I-Q(k))\Phi(k,m)x| &\ge N^{-1}|(I-P(k))\Phi(k,m)x| \\
&= N^{-1}|(I-P(k))\Phi(k,m)(I-P(m))x| \\
&\qquad \text{since } \Phi(k,m)(\mathcal{R}P(m)) \subseteq \mathcal{R}P(k) \\
&\ge N^{-1}K^{-1}e^{\alpha(k-m)}|(I-P(m)x| \\
&\ge N^{-2}K^{-1}e^{\alpha(k-m)}|(I-Q(m))x| \\
&= N^{-2}K^{-1}e^{\alpha(k-m)}|x|. \qquad \square
\end{aligned}$$

Remark 4. *In view of Proposition 1 (iv), when $J = \mathbb{Z}$ or $\mathbb{Z}_+$, Proposition 4 gives all the possible projections with respect to which* (1) *has a generalized exponential dichotomy.*

Now we consider systems on $\mathbb{Z}_-$. For $J = \mathbb{Z}$ or $\mathbb{Z}_+$, we know that the range of the projection in a generalized exponential dichotomy must be $W_+(k)$. For $J = \mathbb{Z}_-$, this is clearly not true but at least we can show that the rank of the projection is well-defined.

Proposition 5. *Assume* (1) *admits a generalized exponential dichotomy in $\mathbb{Z}_-$ with projections $P(k)$ and $Q(k)$. Then, $P(k)$ and $Q(k)$ have the same rank.*

Proof. Let K and α be the constant and exponent for the generalized exponential dichotomy when the projection is $P(k)$. To start we show if m is large negative

$$\mathcal{N}P(m) \cap \mathcal{R}Q(m) = \{0\}.$$

Let $\xi \in \mathcal{N}P(m) \cap \mathcal{R}Q(m)$ with $\xi \ne 0$. Then, for $m \le 0$,

$$|\Phi(0,m)\xi| \le Ke^{\alpha m}|\xi|$$

and

$$|I-P(0)|\,|\Phi(0,m)\xi| \ge |(I-P(0))\Phi(0,m)\xi| \ge K^{-1}e^{-\alpha m}|\xi|.$$

It follows that

$$K^{-1}e^{-\alpha m} \le |I-P(0)|\,Ke^{\alpha m},$$

which clearly does not hold if m is large negative. So $\xi = 0$ and the assertion is proved. By symmetry, we also have for m large negative

$$\mathcal{R}P(m) \cap \mathcal{N}Q(m) = \{0\}.$$

Now we need a lemma.

Lemma 3. *Let P and Q be projections such that $\mathcal{N}P \cap \mathcal{R}Q = \{0\}$ and $\mathcal{R}P \cap \mathcal{N}Q = \{0\}$. Then, P and Q have the same rank.*

Proof. Define $J = PQ + (I-P)(I-Q)$. Then, $PJ = JQ$. Suppose $J\xi = 0$. Then

$$PQ\xi = (I-P)(I-Q)\xi = 0.$$

It follows that $Q\xi = (I-P)Q\xi = (I-P)\xi$. Since $\mathcal{N}P \cap \mathcal{R}Q = \{0\}$, $Q\xi = (I-P)\xi = 0$. Hence, $\xi \in \mathcal{R}P \cap \mathcal{N}Q$ and therefore $\xi = 0$. So J is invertible and, hence, $Q = J^{-1}PJ$. The lemma follows. □

It follows from the lemma that $P(m)$ and $Q(m)$ have the same rank for large negative m and hence the same rank for all m. □

Remark 5. (1) *may have a generalized exponential dichotomy on $\mathbb{Z}_-$ with projections $P(k)$ and $Q(k)$, but the exponents may be different if $\mathcal{R}P(k) \neq \mathcal{R}Q(k)$ (note we saw in Proposition 4 that if $\mathcal{R}P(k) = \mathcal{R}Q(k)$, the exponents are the same). Consider* (10) *with $\frac{1}{4} < \gamma < \frac{1}{2}$. We know this system has a generalized exponential dichotomy with projection P_1 and exponent $\alpha = \log 2$. Let $Q(k)$ be the projection with range spanned by $v(k) = \begin{pmatrix} c_0\gamma^k \\ 1 \end{pmatrix}$, where $c_0 = \frac{2}{4\gamma-1}$, and null-space spanned by $e_2 = \binom{0}{1}$. Then, $Q(k) = \begin{pmatrix} 1 & 0 \\ c_0^{-1}\gamma^{-k} & 0 \end{pmatrix}$ is bounded on $\mathbb{Z}_-$.*

We show that (10) *has a generalized exponential dichotomy on $\mathbb{Z}_-$ with projection $Q(k)$ and that $|\log(2\gamma)|$ is the largest possible exponent. To begin we see that $\mathcal{R}Q(k)$ is invariant under $A(k)$ and that $\mathcal{R}Q(k+1) + A(k)(\mathcal{N}Q(k)) = \mathbb{R}^2$. To prove the exponential estimates, we observe first that*

(15)
$$|\Phi(k,m)v(m)| = 2^{k-m}|v(k)| = 2^{k-m}\frac{\sqrt{c_0^2\gamma^{2k}+1}}{\sqrt{c_0^2\gamma^{2m}+1}}|v(m)| = (2\gamma)^{k-m}\frac{\sqrt{c_0^2+\gamma^{-2k}}}{\sqrt{c_0^2+\gamma^{-2m}}}|v(m)|.$$

Now since $0 < \gamma < 1$, we have $0 \leq \gamma^{-2m} < \gamma^{-2k} \leq 1$ for $m < k \leq 0$, and hence

$$|\Phi(k,m)v(m)| \leq (2\gamma)^{k-m}\frac{\sqrt{c_0^2+1}}{\sqrt{c_0^2}}|v(m)| = \sqrt{\frac{c_0^2+1}{c_0^2}}e^{-|\log(2\gamma)|(k-m)}|v(m)|,$$

noting that $\log(2\gamma) < 0$ since $0 < 2\gamma < 1$. Now, suppose that $K \geq 1$ and $\delta > 0$ exist such that $|\Phi(k,m)v(m)| \leq Ke^{-\delta(k-m)}|v(m)|$. Then, from (15) *we have*

$$(2\gamma)^{k-m}\frac{\sqrt{c_0^2+\gamma^{-2k}}}{\sqrt{c_0^2+\gamma^{-2m}}}|v(m)| \leq Ke^{-\delta(k-m)}|v(m)|$$

that is,

$$\frac{\sqrt{c_0^2+\gamma^{-2k}}}{\sqrt{c_0^2+\gamma^{-2m}}} \leq Ke^{-(\delta+\log(2\gamma))(k-m)} \quad \forall m \leq k \leq 0.$$

Taking $k = 0$ and letting $m \to -\infty$, we see that $\delta + \log(2\gamma)$ must be ≤ 0. Next, as $e_2 = v(m) - c_0\gamma^m e_1$, we have

$$(I - Q(k))\Phi(k,m)e_2 = (I - Q(k))\Phi(k,m)[v(m) - c_0\gamma^m e_1]$$

$$= -c_0\gamma^m (I - Q(k))\Phi(k,m)e_1 = -c_0\gamma^m (I - Q(k))2^{m-k}e_1$$

$$= -c_0(2\gamma)^m 2^{-k}[-c_0^{-1}\gamma^{-k}]e_2 = (2\gamma)^{m-k}e_2$$

and then

$$|(I - Q(k))\Phi(k,m)e_2| = (2\gamma)^{m-k}|e_2| = e^{|\log(2\gamma)|(k-m)}|e_2|.$$

We conclude that equation (10) *has a generalized exponential dichotomy with projections $Q(k)$ and exponent $|\log(2\gamma)|$, and this is the largest possible exponent.*

Note that $\log 2 - |\log(2\gamma)| = \log(4\gamma) > 0$ *since $4\gamma > 1$. So when the projection is $Q(k)$, the exponent cannot be taken as $\alpha = \log 2$.*

In the case of a system on $\mathbb{Z}_-$ with a generalized exponential dichotomy, we have not been able to classify all the possible projections. However, if the system has an exponential dichotomy, we can find all the projections with respect to which the system has a generalized exponential dichotomy.

Proposition 6. *Assume* (1) *admits an exponential dichotomy in $\mathbb{Z}_-$ with projections $P(k)$ and exponent α.*

Let V be any complement of $W_-(0) = \mathcal{N}P(0)$. Then, (1) *admits a generalized exponential dichotomy with exponent α with respect to any bounded projection $Q(k)$ such that*

$$\mathcal{R}Q(k) = \Phi(0,k)^{-1}(V), \quad k \leq 0.$$

Conversely, if (1) *has a generalized exponential dichotomy with projection $Q(k)$, then $\mathcal{R}Q(0)$ is a complement of $W_-(0) = \mathcal{N}P(0)$ and the exponent in the generalized exponential dichotomy can be taken as α.*

Proof. Let V be any complement of $W_-(0) = \mathcal{N}P(0)$. Then, from Proposition 3.4 (ii) in [1] we see that there is a family of invariant projections $\tilde{P}(k)$ with respect to which (1) admits an exponential dichotomy with exponent α and

$$\mathcal{N}\tilde{P}(0) = \mathcal{N}P(0), \qquad \mathcal{R}\tilde{P}(0) = V.$$

Then by invariance we see that

$$\mathcal{R}\tilde{P}(k) = \Phi(0,k)^{-1}(\mathcal{R}\tilde{P}(0)) = \Phi(0,k)^{-1}(V) \qquad \text{for any } k \leq 0.$$

Then, applying Proposition 4, we see that (1) admits a generalized exponential dichotomy with respect to any family of bounded projections $Q(k)$ having the same range as $\tilde{P}(k)$ and with exponent α. Thus, the proof of the first part is concluded.

Conversely, suppose (1) has a generalized exponential dichotomy with projection $Q(k)$. Then, we know from Proposition 5 that $Q(k)$ has the same rank as $P(k)$. Next, by Proposition 1 (v), $\mathcal{R}Q(k) \cap \mathcal{N}P(k) = \mathcal{R}Q(k) \cap W_-(k) = \{0\}$ so that $\mathcal{R}Q(k) \oplus \mathcal{N}P(k) = \mathbb{R}^n$. In particular, $\mathcal{R}Q(0)$ is a complement of $W_-(0) = \mathcal{N}P(0)$. Let $\tilde{P}(k)$ be the projection with the same range as $Q(k)$ and null-space $\mathcal{N}P(k)$. Then, $\tilde{P}(k)$ is invariant and (1) has an exponential dichotomy with respect to $\tilde{P}(k)$ and with exponent α. It it follows from Proposition 4 that the exponent for $Q(k)$ can also be taken as α. □

4. Correction of an error

In [1] we state the following theorems:

Theorem 2. *Suppose* (1) *has an exponential dichotomy on $k \geq m$ with projection $P(k)$ of rank r, where $m \geq 1$. Then, the exponential dichotomy can be extended to $k \geq 0$ with $P(k)$ unchanged for $k \geq m$ if*

$$\dim \left(\Phi(m,0)^{-1}(\mathcal{R}P(m))\right) = r. \tag{16}$$

Conversely, if (1) *has an exponential dichotomy on $k \geq 0$ with projection of rank r, then* (16) *holds.*

Remark 6. *Let $W_+(0)$ be as in* (7)*. Since*

$$\sup_{k\geq 0} |\Phi(k,0)\xi| < \infty \iff \Phi(m,0)\xi \in \mathcal{R}P(m) \iff \xi \in (\Phi(m,0)^{-1}(\mathcal{R}P(m))),$$

it follows that $W_+(0) = (\Phi(m,0)^{-1}(\mathcal{R}P(m)))$. Hence, (16) *is equivalent to dim $W_+(0) = r$.*

Theorem 3. *For $m < 0$, suppose* (1) *has an exponential dichotomy on $k \leq m$ with projection $P(k)$ of rank r. Then,*

(i) (1) *has an exponential dichotomy on $k \leq 0$ with projection of rank r if, and only if, $\Phi(0,m)$ is one to one on $\mathcal{N}P(m)$;*

(ii) (1) *has an exponential dichotomy on $k \leq 0$ with projection $P(k)$ for $k \leq m$ if, and only if, $\Phi(0,m)$ is one to one on $\mathcal{N}P(m)$ and $\mathcal{N}(\Phi(0,m)) \subseteq \mathcal{R}P(m)$.*

Remark 7. *Let $W_-(0)$ be as in* (7)*. We claim that $W_-(0) = \Phi(0,m)(\mathcal{N}P(m))$. This follows from the facts that $y(k)$ is a bounded solution on $k \leq 0$ if, and only if, $y(m) \in \mathcal{N}P(m)$ and $y(0) = \Phi(0,m)y(m)$. Since dim $\mathcal{N}P(m) = n - r$, it follows that $\Phi(0,m)$ being one to one on $\mathcal{N}P(m)$ is equivalent to* $\dim W_-(0) = n - r$.

To begin we show by an example, that the part of the statement of Theorem 2 saying that the projection can be taken unaltered for $k \geq m$ is not true in general. We use the fact that it follows from invariance that if (1) has an exponential dichotomy on $k \geq 0$, then $\mathcal{N}P(1) = \Phi(1,0)(\mathcal{N}P(0)) \subseteq \mathcal{R}(\Phi(1,0))$.

Example 1. *Now consider system* (1)*, where*

$$A(0) = \begin{pmatrix} 0 & 1 \\ 0 & 1/2 \end{pmatrix}, \quad A(k) = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad k \geq 1,$$

which has an exponential dichotomy on $k \geq 1$ with projection

$$P(k) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad k \geq 1$$

of rank $r = 1$. We have $\Phi(1,0) = A(0)$ and

$$\mathcal{R}(A(0)) = \mathrm{span}\{2e_1 + e_2\},\ \mathcal{R}P(0) = \mathrm{span}\{e_2\},\ A(0)^{-1}(\mathcal{R}P(0)) = \mathrm{span}\{e_1\},$$

where $e_1 = \binom{1}{0}$, $e_2 = \binom{0}{1}$. Since dim $A(0)^{-1}(\mathcal{R}P(0)) = 1 = r$, it follows that the dichotomy can be extended to $k \geq 0$. However, $\mathcal{N}P(1) = \mathrm{span}\{e_2\}$ is not a subset of $\mathcal{R}(A(0))$ and so the dichotomy cannot be extended without changing $\mathcal{N}P(1)$.

The purpose of this section is to correct the statement of Theorem 2 as follows.

Theorem 4. *Suppose* (1) *has an exponential dichotomy on $k \geq m$ with projection $P(k)$ of rank r, where $m \geq 1$. Then, the exponential dichotomy can be extended to $k \geq 0$ if, and only if,* (16) *holds. Moreover, the projection $P(k)$, for $k \geq m$, can be taken unchanged if, and only if, in addition,*

$$\mathcal{N}P(m) \subseteq \mathcal{R}(\Phi(m,0)). \tag{17}$$

Proof. To begin we prove the necessity of the conditions. Suppose the dichotomy can be extended to $k \geq 0$ and let $\tilde{P}(k)$, $k \geq 0$, be the new projection. A priori we only know that $\mathcal{R}\tilde{P}(k) = \mathcal{R}P(k)$ for all $k \geq m$. Then, as proved in [1, Remark 1-iv)], we have

$$\Phi(m,0)^{-1}(\mathcal{R}P(m)) = \Phi(m,0)^{-1}(\mathcal{R}\tilde{P}(m)) = \mathcal{R}\tilde{P}(0)$$

and hence $\dim \Phi(m,0)^{-1}(\mathcal{R}P(m)) = r$. Furthermore, suppose $\tilde{P}(k) = P(k)$ for any $k \geq m$. Then, $\mathcal{N}\tilde{P}(m) = \mathcal{N}P(m)$ and so $\Phi(m,0) : \mathcal{N}\tilde{P}(0) \to \mathcal{N}P(m)$ is an isomorphism. Hence, it follows immediately that $\mathcal{N}P(m) \subseteq \mathcal{R}(\Phi(m,0))$.

Now we prove the sufficiency by induction on m. To start we prove the case $m = 1$. Suppose that (1) has an exponential dichotomy on $k \geq 1$ with projection of rank r and that (16) holds with $m = 1$.

Let V be an $(n-r)$-dimensional linear subspace such that

$$\mathbb{R}^n = A(0)^{-1}(\mathcal{R}P(1)) \oplus V.$$

Since $\mathcal{N}(A(0)) \subseteq A(0)^{-1}(\mathcal{R}P(1))$, we have $\mathcal{N}(A(0)) \cap V = \{0\}$ and, hence, $A(0) : V \to W := A(0)(V)$ is one to one and onto. In particular, $\dim W = n - r$. Next we prove that $W \cap \mathcal{R}P(1) = \{0\}$. Indeed, suppose that $x \in W \cap \mathcal{R}P(1)$. Then, there exists $v \in V$ such that $x = A(0)v$. Hence, $v \in A(0)^{-1}(\{x\}) \subseteq A(0)^{-1}(\mathcal{R}P(1))$. Then, $v = 0$, and so $x = 0$, and the claim is proved. It follows that

$$\mathbb{R}^n = \mathcal{R}P(1) \oplus W.$$

Let $\tilde{P}(1)$ be the projection such that

$$\mathcal{R}\tilde{P}(1) = \mathcal{R}P(1), \quad \text{and} \quad \mathcal{N}\tilde{P}(1) = W.$$

Then, it follows from Lemma 4.1 in [1] with $U = A(0)^{-1}(\mathcal{R}P(1))$ that the dichotomy can be extended to $k \geq 0$ with projection $\tilde{P}(1)$ at $k = 1$ and with $\tilde{P}(0)$, and the projection with range U and null-space V, as the projection at $k = 0$.

Finally, suppose that $\mathcal{N}P(1)$ is a subset of $\mathcal{R}(A(0))$. Then, we take V as an $(n-r)$-dimensional subspace such that $A(0)(V) = \mathcal{N}P(1)$ and $V \cap \mathcal{N}(A(0)) = \{0\}$. If $x \in A(0)^{-1}(\mathcal{R}P(1)) \cap V$, $A(0)x \in \mathcal{R}P(1) \cap \mathcal{N}P(1)$ so that $A(0)x = 0$ and, hence, $x = 0$ since $A(0)$ is one to one on V. Thus, $A(0)^{-1}(\mathcal{R}P(1)) \cap V = \{0\}$ and so $\mathbb{R}^n = A(0)^{-1}(\mathcal{R}P(1)) \oplus V$. Thus, we may take $W = \mathcal{N}P(1)$ in the argument above, and it follows from Lemma 4.1 in [1] with $U = A(0)^{-1}(\mathcal{R}P(1))$ that the dichotomy can be extended to $k \geq 0$ with projection $P(1)$ at $k = 1$ and with $\tilde{P}(0)$, and the projection with range U and null-space V, as the projection at $k = 0$.

To continue the induction proof, we assume the theorem is true for $m-1 \geq 1$ and prove it for m. So we are assuming (1) has a dichotomy on $k \geq m$ with projection $P(k)$ such that (16) holds. It follows as in the proof of Theorem 4.3 in [1] that $\dim\, \Phi(m,1)^{-1}(\mathcal{R}P(m)) = r$. By the induction hypothesis, (1) has a dichotomy on $k \geq 1$ with some projection $\bar{P}(k)$. Then it follows again as in the proof of Theorem 4.3 in [1] that $\dim\, A(0)^{-1}(\mathcal{R}\bar{P}(1)) = r$ and so, by the $m = 1$ case, the dichotomy can be extended to $k \geq 0$ with possibly another change of projection. We continue to denote this projection as $\bar{P}(k)$.

Suppose now that $\mathcal{N}P(m) \subseteq \mathcal{R}(\Phi(m,0))$ holds. Note that for $k \geq 0$,

$$\mathcal{R}\bar{P}(k) = W_+(k) = \left\{\xi : \sup_{\ell \geq k} |\Phi(\ell,m)\xi| < \infty\right\}.$$

In particular,

$$\mathbb{R}^n = W_+(m-1) \oplus \mathcal{N}\bar{P}(m-1), \tag{18}$$

where $\mathcal{N}\bar{P}(m-1) = \Phi(m-1,0)(\mathcal{N}\bar{P}(0))$. Since $\Phi(m,0) = \Phi(m,m-1)\Phi(m-1,0)$, it follows that $\mathcal{N}P(m) \subseteq \mathcal{R}(\Phi(m,m-1))$. So by the $m = 1$ case, the dichotomy can be extended to $k \geq m-1$ with projection $\tilde{P}(k)$ such that $\tilde{P}(k) = P(k)$ for $k \geq m$ with $\mathcal{N}\tilde{P}(m-1)$ as any V such that

$$\mathbb{R}^n = \Phi(m,m-1)^{-1}(\mathcal{R}P(m)) \oplus V = W_+(m-1) \oplus V.$$

It follows from (18) that we may take $V = \mathcal{N}\tilde{P}(m-1) = \Phi(m-1,0)(\mathcal{N}\bar{P}(0))$. We have now extended the dichotomy to $k \geq m-1$ with the projection $\tilde{P}(k)$ with $\tilde{P}(k) = P(k)$ for $k \geq m$ and with $\tilde{P}(m-1)$ as the projection with range $W_+(m-1)$ and null-space $V \subseteq \mathcal{R}(\Phi(m-1,0))$. By the induction hypothesis, we can extend the dichotomy to $k \geq 0$ without changing $\tilde{P}(k)$ for $k \geq m-1$ and, hence, without changing $P(k)$ for $k \geq m$. The proof is complete. □

Remark 8. *According to Theorem 3 above, if* (1) *has an exponential dichotomy on $k \leq m < 0$, then the exponential dichotomy can be extended to $k \leq 0$ if and only if, $\Phi(0,m)$ is one to one on $\mathcal{N}P(m)$. Of course, this condition is equivalent to*

$$\dim \Phi(0,m)(\mathcal{N}P(m)) = n - r$$

since $\dim \mathcal{N}P(m) = n - r$. *Moreover, it is also proved that the projections can be taken so that they are unaltered for $k \leq m$ if, and only if,*

$$\mathcal{N}(\Phi(0,m)) \subseteq \mathcal{R}P(m)$$

which is the counterpart of (17) *when $k \leq 0$.*

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