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In search of hidden symmetries

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Abstract. This paper exemplifies the importance of finding hidden symmetries of differential equations that are models of physical phenomena. The hidden symmetries (Lie symmetries) may be determined by either linking together different equations for certain values of their parameters or transforming the original model into another equivalent system of equations that may have more symmetries. Therefore, hidden symmetries may help to solve the original model or yield its hidden properties, e.g. linearity and conservation laws. Moreover Noether symmetries are shown to be preserved by going from classical to quantum mechanics, namely from Lagrangian systems to the corresponding time-dependent Schrödinger equation.

1. A gentle introduction

In order to make the paper somewhat more accessible the following preliminaries may be useful.

A symmetry S is a transformation that leaves *something* unchanged. A symmetry may belong to a set of symmetries of the same item, and if all the symmetries in the set can be composed so that the set contains (a) all the possible compositions of all the symmetries, (b) the symmetry corresponding to no transformation, (c) the inverse symmetry of each symmetry and if in addition for any three symmetries S_1, S_2, S_3 in the set the composition of S_1 and S_2 that is then composed with S_3 yields the same symmetry given by composing S_1 with the composition of S_2 and S_3 (associativity property), then the set is called a group. As an example, this short video may be watched that shows the group of symmetries of an equilateral triangle at https://www.youtube.com/watch?v=DO_pHSZ12nU (Learnifyable, (Abstract algebra 1) Symmetries of an equilateral triangle). Moreover a very interesting and educational talk by Marcus du Sautoy on the subject of symmetries may be watched here: "https://m.youtube.com/watch?v=xMtGjBmU_EE" (Talks at Google, Finding moonshine | Marcus du Sautoy | Talks at Google).

The rate of change is what differentiation actually is/means and since the basic Newtonian law of classical physics is given by equalling force $\mathbf{F}$ with the rate of change (with respect to time t) of linear momentum (namely, mass times velocity=$M\mathbf{v}(t)$ i.e. $\mathbf{F} = \frac{\mathrm{d}M\mathbf{v}(t)}{\mathrm{d}t}$, then calculus must be used in order to *understand* even a simple physical model that is a differential equation. Consequently a differential equation is an equation that has a function as its solution (quite different from the case of a polynomial equation, e.g. $x^2 - 1 = 0$ where the solutions are numbers). The order of a differential equations is the order of the highest derivative of the unknown function that appears in the differential equation. If the unknown function (aka dependent variable) depends on one single variable (aka independent variable) then one is dealing with an ordinary differential equation, e.g. the equation of motion of a

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one-dimensional free particle of mass M, namely $M\frac{d^2s(t)}{\mathrm{d}t^2} = 0$. If the unknown function depends on more than one variable, then one is dealing with a partial differential equation, e.g. the wave equation $\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} = 0$. *An ideal resource accessible to anyone* can be found at the webpages that PLUS has dedicated to differential equations at https://plus.maths.org/content/teacher-package-differential-equations (PLUS, Explore: differential equations).

What is the symmetry of a differential equation? It is a transformation of the dependent and independent variables that leaves the differential equation unchanged, and it is called a Lie symmetry. The Lie symmetries of a differential equation constitute a Lie symmetry group.

Sophus Lie's monumental work on transformation groups [1]-[3] provided systematic techniques for obtaining exact solutions of differential equations [4]. Many books have been dedicated to this subject and its generalisations, e.g. [5]-[9]. The Lie group method is indeed the most powerful tool for finding the general solution of ordinary differential equations. Any known integration technique can be shown to be a particular case of a general integration method based on the derivation of the continuous group of symmetries admitted by the differential equation, symmetries that can be easily derived by a straightforward although lengthy procedure. As computer algebra software becomes widely used, the integration of systems of ordinary differential equations by means of Lie group analysis is becoming easier to perform.

2. Prelude: which symmetries?

In January 2001 the first Whiteman Prize for notable exposition on the history of mathematics was awarded to Thomas Hawkins by the American Mathematical Society (AMA). In the citation published in the *Notices of AMS* **48** 416–417 (2001), it states that Thomas Hawkins "*has written extensively on the history of Lie groups. In particular he has traced their origins to [Lie's] work in the 1870s on differential equations (omissis) the idée fixe guiding Lie's work was the development of a Galois theory of differential equations*". Also Hawkins [10] had established "*the nature and extent of Jacobi's influence upon Lie*". "Given the fact that the Jacobi Identity is fundamental to the theory of Lie groups, Carl Jacobi's influence upon Sophus Lie will come as no surprise. But the bald fact that he inherited the Identity from Jacobi fails to convey fully or accurately the historical dimension of the impact of Jacobi's work on partial differential equations" [11] (see figure 1).

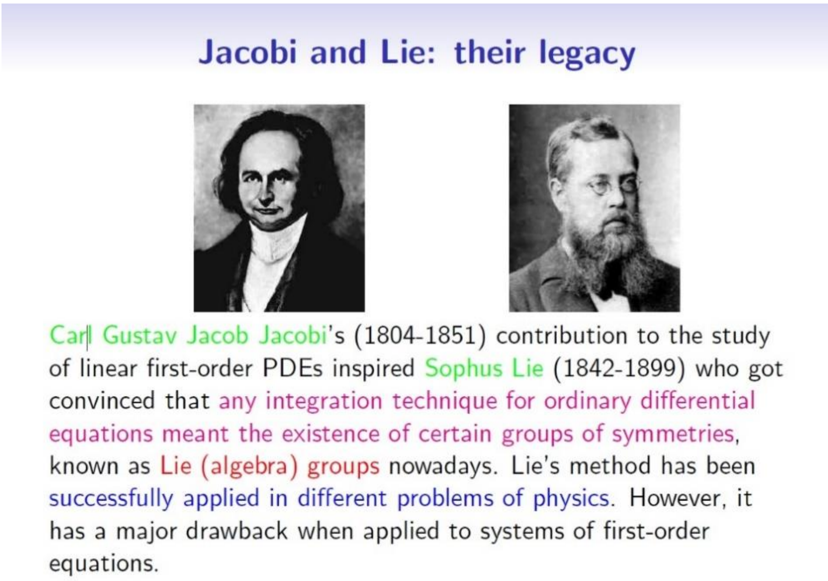

Figure 1. The legacy of Carl Jacobi and Sophus Lie.

There exists an infinite number of Lie point symmetries admitted by a first-order ordinary differential equation, e.g. $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y)$. The generator of a Lie point symmetry is characterised by two functions of the independent and dependent variables, i.e. $\xi(x,y),\ \eta(x,y)$ that satisfy the determining equation, namely a single linear first-order partial differential equation. Consequently there are infinite solutions and there is no general method that provides at least one solution. However, if one solution (a Lie symmetry) is found then a simple formula by Lie provides a link between an integrating factor (aka Euler multiplier) μ of the first-order ordinary differential equation and the symmetry, i.e. $\mu = 1/(\eta - f\xi)$. An example is shown by considering an equation and its multiplier as given by Leonhard Euler in 1772 (figure 2).

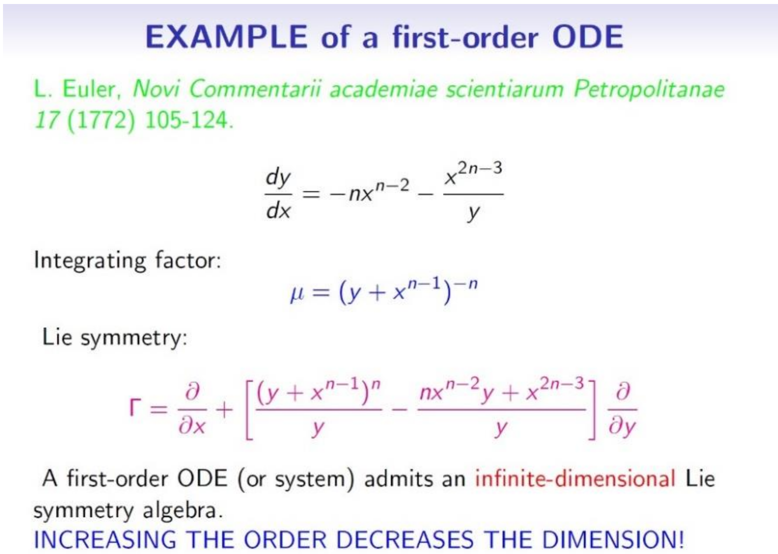

Figure 2. Euler's example of a first-order ordinary differential equation (ODE).

In the case of ordinary differential equations of order greater than one the determining equation splits into several partial differential equations and a finite number of Lie point symmetries can be determined if they exist. In particular a linear second-order ordinary differential equation admits eight Lie point symmetries.

3. An example: how to turn a butterfly into a tornado and vice versa

The motion of a heavy rigid body about a fixed point is one of the most famous problems of classical mechanics [12]. In 1750 [13] Euler derived the equations of motion, which now bear his name and described what is nowadays known as the Euler–Poinsot case because of the geometrical description given by Poinsot about a hundred years later [14]. It was Jacobi [15] who integrated this case by using the elliptic functions which he had developed (along with Adrien-Marie Legendre, Niels Abel and Carl Friedrich Gauss) and mastered [16]. More than two hundred years later in 1963 a paper was published [17] where a system of three ordinary differential equations was presented. The author considered a hydrodynamical system developed by Lord Rayleigh [18] and reduced it by applying a double Fourier series as in [19]. Thus, he obtained what nowadays is the famous Lorenz system [20]. Three parameters are part of the Lorenz system. For particular values of those parameters the Lorenz system can be integrated in closed form by means of Jacobi elliptic functions [21]. This system is called the Lorenz integrable system. In [22] Lie group analysis was applied to a third-order differential equation equivalent to the Lorenz integrable system and a two-dimensional Lie symmetry algebra was obtained, which was then used to integrate the Lorenz integrable system in terms of Jacobi elliptic functions. In [23] it was shown that the same Lie symmetry algebra is admitted by a third-order differential equation which is equivalent to the Euler equations of a torque-free rigid body moving about a fixed point. Then a transformation is easily derived by which the Lorenz integrable system becomes the Euler equations of a torque-free rigid body moving about a fixed point. Thus, it can be stated that “the Lorenz integrable system moves á la Poinsot." If the same transformation is applied to the Lorenz system with any value of parameters then the Euler equations of a rigid body moving about a fixed point and subjected to a torsion depending on time and angular velocity are obtained. The numerical solution of this system yields a three-dimensional picture which resembles a *tornado,* the cross-section of which has a butterfly shape. By means of this transformation Lorenz’s butterfly becomes a tornado (see figure 3).

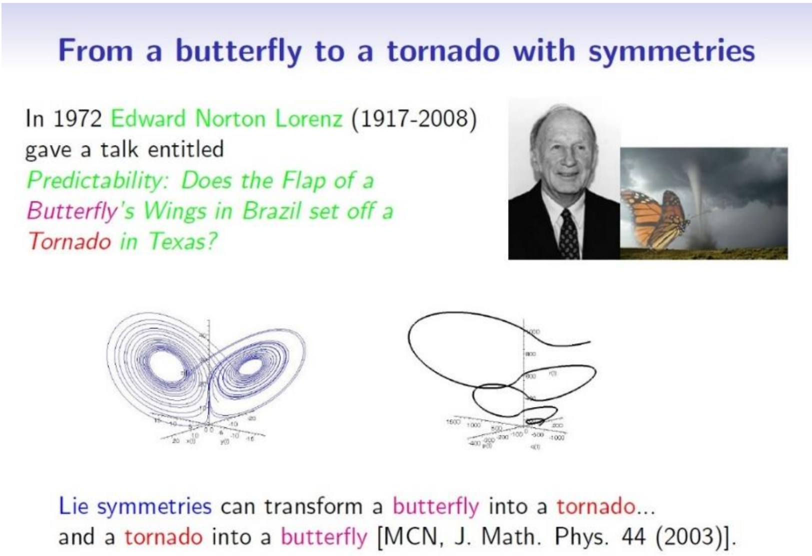

Figure 3. Transformation of a butterfly into a tornado using symmetry.

4. Linearisation of superintegrable systems by means of hidden Lie symmetries

A major drawback of Lie’s method is that it is useless when applied to equations of first order, e.g. Hamiltonian equations, because they admit an infinite number of symmetries, and there is no systematic way to find even one Lie symmetry. One may try to derive it by making an ansatz on the form of its generator but when successful (rarely) it is just a lucky guess. However, in [24] it was remarked that any system of first-order equations could be transformed into an equivalent system where at least one of the equations is of second order. Then the admitted Lie symmetry algebra is no longer infinite dimensional and nontrivial symmetries of the original system could be retrieved. Consequently in [24] hidden symmetries of the Kepler problem were determined by this method. Also in [25] the well-known linearisation of the Kepler problem as well as the linearity of generalisations of the Kepler problem with and without drag were determined by means of hidden symmetries. Such hidden symmetries are more general than those considered, e.g. in [26], which are just *symmetries of the Hamiltonian, in the sense that they are canonical transformations where both positions and momenta change, and that leave the Hamiltonian function unchanged*. In [27] it was shown that a two-dimensional superintegrable system [28] such that the corresponding Hamilton-Jacobi equation does not admit the separation of variables in any coordinates, can be transformed into a linear third-order equation by means of hidden symmetries. In [29] several examples of classical superintegrable systems in two-dimensional Euclidean space [30], [31] were shown to possess hidden symmetries leading to their linearisation and it was conjectured that all classical superintegrable systems in two-dimensional spaces have hidden symmetries that make them linearisable. In [32] nineteen classical superintegrable systems in two-dimensional non-Euclidean spaces [33]-[35] were shown to possess hidden symmetries leading to linearity. In [36] maximally superintegrable Hamiltonian systems in three-dimensional Euclidean space [37], [38] were also linearised by means of their hidden symmetries and it was conjectured that three-dimensional minimally superintegrable systems may be similarly linearisable. In [39] minimally superintegrable Hamiltonian systems in three-dimensional Euclidean space [37] were shown to possess hidden symmetries leading to their linearisation. In [40] all fifteen nonlinear minimally superintegrable systems in the presence of a magnetic field in three-dimensional Euclidean space (see [41] and references therein) were shown to be intrinsically linear by determining their hidden Lie symmetries.

5. Is mechanics a branch of mathematical analysis?

Lagrangian mechanics is based on the Lagrangian function, which has a profound physical meaning, although Lagrange himself was hinting that Mechanics shall be a branch of Analysis (figure 4). However, in the years 1842-1845 Jacobi introduced the properties of its last multiplier, which was supposed to be a generalisation of Euler multipliers, and then Jacobi also described how to obtain a Lagrangian from it in the case of a second-order ordinary differential equation. Consequently any such equation possesses an infinite number of Lagrangians (figure 5).

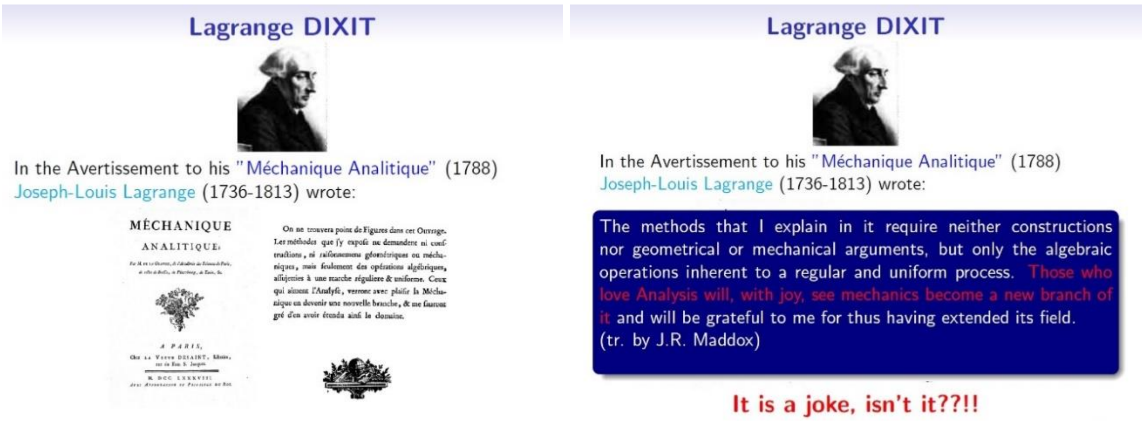

Figure 4. Lagrange's introduction to his Méchanique Analitique.

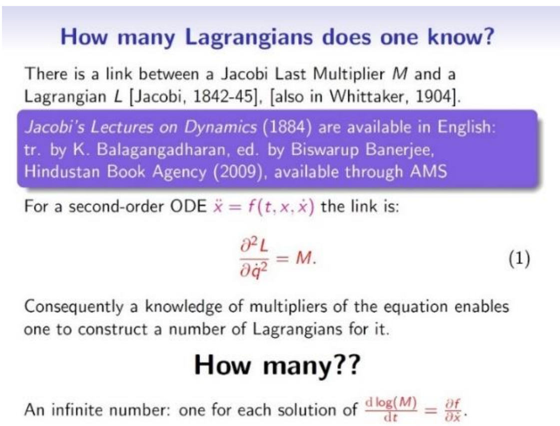

Figure 5. How many Lagrangians does one know?

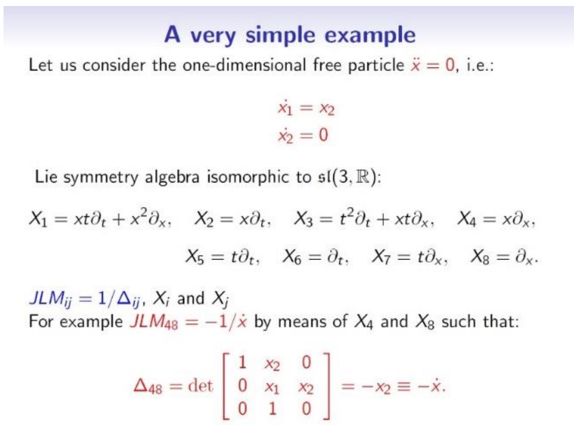

Figure 6. A very simple example.

In 1874 Lie determined a method to find the Jacobi last multiplier by means of his symmetries. A simple example is provided below (figure 6).

Ten Lagrangians (figure 7) can be determined in this way including the physical Lagrangian.

Ten Lagrangians

Ten different JLM and consequently as many Lagrangians:

$$M_{13} = -\frac{1}{(t\dot{x}-x)^3} \Rightarrow L_{1,3} = -\frac{1}{2t^2(t\dot{x}-x)} + \frac{\mathrm{d}g}{\mathrm{d}t}(t,x)$$

$$M_{15} = -\frac{1}{\dot{x}(t\dot{x}-x)^2} \Rightarrow L_{1,5} = \frac{\dot{x}}{x^2}\left(\log(t\dot{x}-x) - \log(\dot{x})\right)$$

$$M_{16} = \frac{1}{\dot{x}^2(t\dot{x}-x)} \Rightarrow L_{1,6} = \left(\frac{t\dot{x}}{x^2} - \frac{1}{x}\right)\left(\log(\dot{x}) - \log(t\dot{x}-x)\right)$$

$$M_{17} = -\frac{1}{(t\dot{x}-x)^2} \Rightarrow L_{1,7} = -\frac{1}{t^2}\log(t\dot{x}-x)$$

$$M_{18} = \frac{1}{\dot{x}(t\dot{x}-x)} \Rightarrow L_{1,8} = -\frac{\dot{x}}{x}\log(\dot{x}) - \left(\frac{1}{t} - \frac{\dot{x}}{x}\right)\log(t\dot{x}-x) + \frac{1}{t}(1+\log(x))$$

$$M_{62} = \frac{1}{\dot{x}^3} \Rightarrow L_{6,2} = \frac{1}{2\dot{x}}$$

$$M_{28} = \frac{1}{\dot{x}^2} \Rightarrow L_{2,8} = -\log(\dot{x})$$

$$M_{38} = \frac{1}{t\dot{x}-x} \Rightarrow L_{3,8} = \left(\frac{\dot{x}}{t} - \frac{x}{t^2}\right)\left(\log(t\dot{x}-x) - 1\right)$$

$$M_{48} = -\frac{1}{\dot{x}} \Rightarrow L_{4,8} = \dot{x}(1 - \log(\dot{x}))$$

$$M_{87} = 1 \Rightarrow L_{8,7} = \tfrac{1}{2}\dot{x}^2$$

FINALLY, THE TRUE LAGRANGIAN

Figure 7. The ten Lagrangians.

Therefore, how can the physical Lagrangian be distinguished from the other nine (figure 8)?

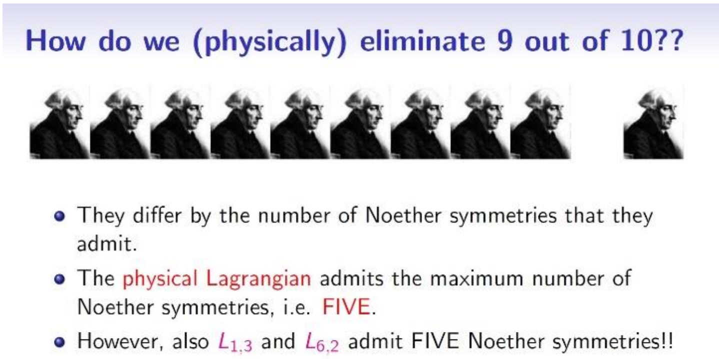

Figure 8. How to find the physical Lagrangian from the other nine.

Emmy Noether and Erwin Schrödinger may help: wait until the end.

6. Quantisation through the conservation of Noether symmetries

In [42]-[44] a procedure which obviates the constraint imposed by the conflict between consistent quantisation and the invariance of the Hamiltonian description under nonlinear canonical transformation was proposed. This new quantisation scheme preserves the Noether point symmetries of the underlying Lagrangian in order to construct the Schrödinger equation (figure 9) and can be summarised as follows:

1) Find the Lie symmetries of the Lagrange equations.
2) Among them find the Noether symmetries (this may require searching for the Lagrangian yielding the maximum possible number of Noether symmetries).
3) Construct the Schrödinger equation admitting these Noether symmetries as Lie symmetries without adding any other symmetries apart from the two symmetries that are present in any linear homogeneous partial differential equation.

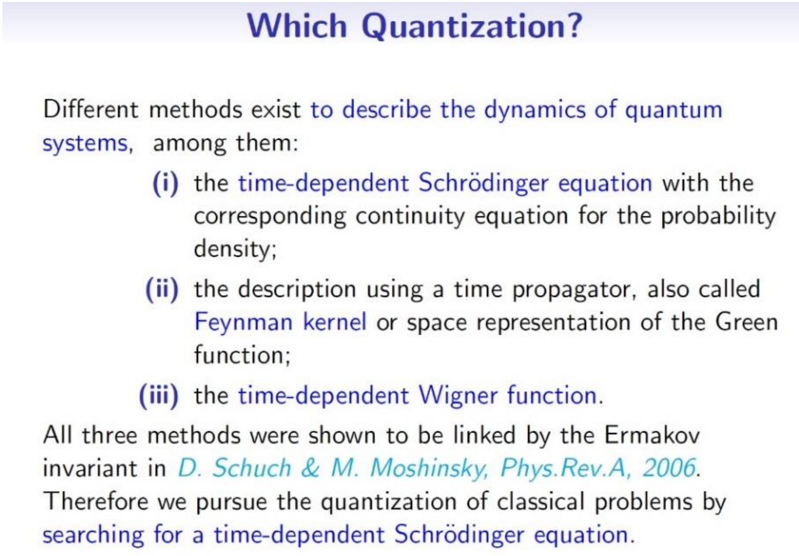

Figure 9. Which quantisation?

This method has been successfully applied to various classical problems, e.g. the dynamics of a charged particle in a uniform magnetic field and a non-isochronous Calogero's goldfish system [44], two nonlinear equations somewhat related to the Riemann problem [45], a Liénard I nonlinear oscillator [46], a family of Liénard II nonlinear oscillators [47], N planar rotors and an isochronous Calogero's goldfish system [48], the motion of a free particle and that of an harmonic oscillator on a double cone [49].

However, the method as stated may be misleading. What happens if this method is applied to the three different Lagrangians of the one-dimensional free particle (see figure 10)?

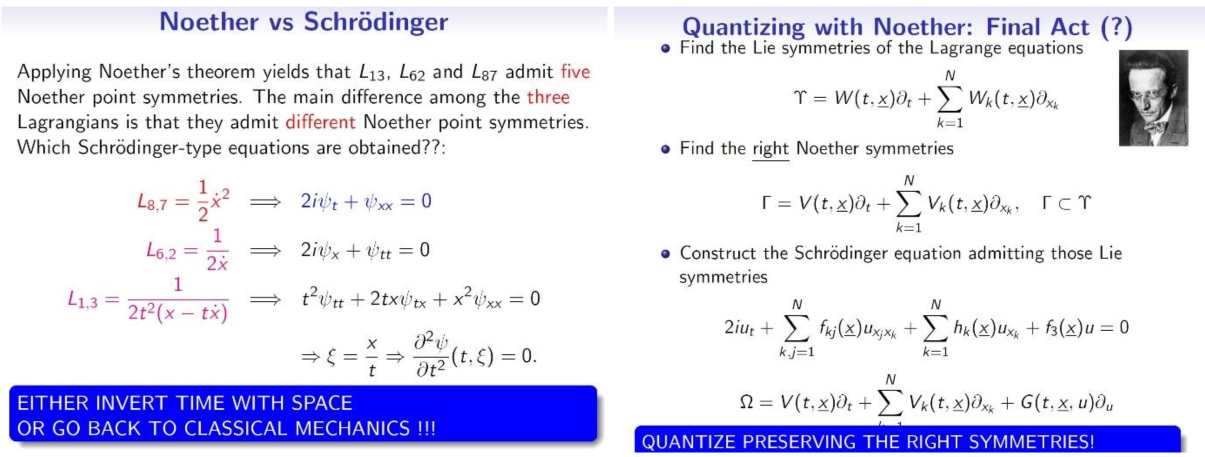

Figure 10. Noether versus Schrödinger.

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